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STURM–LIOUVILLE AND FOCAL HIGHER ORDER BVPs WITH SINGULARITIES IN PHASE VARIABLES

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Abstract. The paper deals with the existence of solutions for singular higher order differential equations with Sturm–Liouville or $(p, n-p)$ right focal boundary conditions or the $(n-p, p)$ left focal boundary conditions. Right-hand sides of differential equations may be singular in the zero values of all their phase variables. The proofs are based on the regularization and sequential techniques.

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1. Introduction

Let T be a positive constant, $J = [0, T]$ and $\mathbb{R}_- = (-\infty, 0)$, $\mathbb{R}_+ = (0, \infty)$, $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$.

We will consider two types of singular boundary value problems for nth order differential equations, where $n \geq 2$. The first one is the singular *Sturm–Liouville* boundary value problem (BVP for short)

$$-x^n(t) = f(t, x(t), \dots, x^{(n-1)}(t)), \tag{1.1}$$

$$\begin{gathered} x^{(i)}(0) = 0, \ \ 0 \leq i \leq n-3, \\ \alpha x^{(n-2)}(0) - \beta x^{(n-1)}(0) = 0, \\ \gamma x^{(n-2)}(T) + \delta x^{(n-1)}(T) = 0, \end{gathered} \tag{1.2}$$

where α, $\gamma > 0$, β, $\delta \geq 0$ and f satisfies the local Carathéodory conditions on $J \times D$ ($f \in Car(J \times D)$) with

$$D = \mathbb{R}_+^{n-1} \times \mathbb{R}_0.$$

The second one is the singular $(p, n-p)$ *right focal* BVP

$$(-1)^{n-p} x^{(n)}(t) = f(t, x(t), \dots, x^{(n-1)}(t)), \tag{1.3}$$

$$x^{(i)}(0) = 0, \ \ 0 \leq i \leq p-1, \quad x^{(i)}(T) = 0, \ \ p \leq i \leq n-1, \tag{1.4}$$

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where $p \in \mathbb{N}$ is fixed, $1 \le p \le n-1$, and $f \in Car(J \times X)$ with

$$X = \begin{cases} \underbrace{\mathbb{R}_+^{p+1} \times \mathbb{R}_- \times \mathbb{R}_+ \times \mathbb{R}_- \times \cdots \times \mathbb{R}_+}_{n} & \text{if } n-p \text{ is odd} \\ \underbrace{\mathbb{R}_+^{p+1} \times \mathbb{R}_- \times \mathbb{R}_+ \times \mathbb{R}_- \times \cdots \times \mathbb{R}_-}_{n} & \text{if } n-p \text{ is even} \end{cases}.$$

In both cases the function $f(t, x_0, \dots, x_{n-1})$ may be singular at the points $x_i = 0$, $0 \le i \le n-1$, of all its phase variables $x_0, \dots, x_{n-1}$.

The aim of this paper is to give conditions for the existence of solutions to problems (1.1), (1.2) and (1.3), (1.4).

Definition 1.1. By *a solution of BVP* (1.1), (1.2) we understand a function $x \in AC^{n-1}(J)$ which is positive on $(0, T]$, satisfies conditions (1.2) and for a.e. $t \in J$ fulfils (1.1).

Similarly, by *a solution of BVP* (1.3), (1.4) we understand a function $x \in AC^{n-1}(J)$ which is positive on $(0, T]$, satisfies conditions (1.4) and for a.e. $t \in J$ fulfils (1.3).

From now on, $\|x\| = \max\{|x(t)| : \ 0 \le t \le T\}$, $\|x\|_L = \int_0^T |x(t)|\,dt$ and $\|x\|_\infty = \operatorname{ess\,max}\,\{|x(t)| : \ 0 \le t \le T\}$ stand for the norm in $C^0(J)$, $L_1(J)$ and $L_\infty(J)$, respectively. For a subset Ω of a Banach space, $cl(\Omega)$ and $\partial\Omega$ stand for the closure and the boundary of Ω, respectively. Finally, for any measurable set $\mathcal{M}$, $\mu(\mathcal{M})$ denotes the Lebesgue measure of $\mathcal{M}$.

The fact that a BVP is singular means that the right-hand side f of the considered differential equation does not fulfil the Carathéodory conditions on a domain where we seek for solutions, i.e. on $J \times cl(D)$ if we work with equation (1.1) or on $J \times cl(X)$ if we study equation (1.3). In singular problems the Carathéodory conditions can be broken in the time variable t or in the phase variables or in the both types of variables. The first type of singularities where f need not be integrable on J for fixed phase variables was studied by many authors. For BVPs of n-th order differential equations such problems were considered for the first time by Kiguradze in [16]. The second type of singularities, i.e., the case where f is unbounded in some values of its phase variables $x_0, x_1, \dots, x_{n-1}$ for fixed $t \in J$ was mainly solved for BVPs of second order differential equations, but during the last decade papers dealing with higher order BVPs having singularities in phase variables have appeared as well. We can refer to the papers [2]–[7], [9]–[14] and [19]–[23]. Some of them (see [3]–[7]) concern higher order singular Sturm–Liouville or right focal BVPs.

In this paper we extend results in the cited papers to the case of a general Carathéodory right-hand side f which may depend on higher derivatives up to order $n-1$ and which may have singularities in all its phase variables. The proofs are based on the construction of a proper sequence of regular problems and in limiting processes. The correctness of such processes is warrantable by the Lebesgue dominated convergence theorem in the case of problem (1.3), (1.4).

As regards problem (1.1), (1.2) note that conditions (1.2) imply that for any solution x of this problem its derivative $x^{(n-1)}$ is a sign-changing function on J. Therefore this derivative goes through the singularity of f somewhere inside J, which makes it impossible to find a Lebesgue integrable majorant function for any auxiliary sequence of regular functions $\{f_m\}$ relevant to problem (1.1), (1.2). This implies that in this case instead of the Lebesgue theorem the Vitali convergence theorem is used.

The proofs of the existence results for the auxiliary regular BVPs considered in Section 3 are based on the Nonlinear Fredholm Alternative (see e.g. [17], Theorem 4 or [21], p. 25) which we formulate in the form convenient for the application to the problems mentioned above. In particular, we consider the differential equation

$$x^{(n)}(t) + \sum_{i=0}^{n-1} a_i(t) x^{(i)}(t) = h\big(t, x(t), \dots, x^{(n-1)}(t)\big) \tag{1.5}$$

and the corresponding homogeneous equation

$$x^{(n)}(t) + \sum_{i=0}^{n-1} a_i(t) x^{(i)}(t) = 0, \tag{1.6}$$

where $a_i \in L_1(J)$, $0 \le i \le n-1$, $h \in Car(J \times \mathbb{R}^n)$. Further we deal with the boundary conditions

$$\mathcal{L}_j(x) = r_j, \quad 1 \le j \le n, \tag{1.7}$$

with $r_j \in \mathbb{R}$ and continuous linear functionals $\mathcal{L}_j : C^{n-1}(J) \to \mathbb{R}$, $1 \le j \le n$.

Definition 1.2. By *a solution of BVP* (1.5), (1.7) we understand a function $x \in AC^{n-1}(J)$ which satisfies conditions (1.7) and for a.e. $t \in J$ fulfils (1.5).

Theorem 1.3 (Nonlinear Fredholm Alternative)**.** *Let problem* (1.6), (1.7) *has only a trivial solution and there exist a function* $g \in L_1(J)$ *such that*

$$\big|h(t, x_0, \dots, x_{n-1})\big| \le g(t) \ \text{ for a.e. } t \in J \ \text{ and all } x_0, \dots, x_{n-1} \in \mathbb{R}.$$

Then problem (1.5), (1.7) *has a solution.*

The following assumptions will be used in the study of problem (1.1), (1.2):

(H_1) $f \in Car(J \times D)$ and there exist positive constants ε, r such that

$$\varepsilon t^r \le f(t, x_0, \dots, x_{n-1})$$

for a.e. $t \in J$ and each $(x_0, \dots, x_{n-1}) \in D$;

(H_2) For a.e. $t \in J$ and for each $(x_0, \dots, x_{n-1}) \in D$,

$$f(t, x_0, \dots, x_{n-1}) \le \phi(t) + \sum_{i=0}^{n-1} q_i(t)\omega_i(|x_i|) + \sum_{i=0}^{n-1} h_i(t)|x_i|^{\alpha_i}, \tag{1.8}$$

where $\alpha_i \in (0,1)$, ϕ, $h_i \in L_1(J)$, $q_i \in L_\infty(J)$ are nonnegative, $\omega_i : \mathbb{R}_+ \to \mathbb{R}_+$ are nonincreasing, $0 \le i \le n-1$, and

$$\int_0^T \omega_{n-1}(t^{r+1})\,dt < \infty, \quad \int_0^T \omega_i(t^{n-i-1})\,dt < \infty \text{ for } 0 \le i \le n-2; \tag{1.9}$$

(H_3) For a.e. $t \in J$ and for each $(x_0, \dots, x_{n-1}) \in D$, the function f satisfies (1.8) where $\alpha_i \in (0,1)$, ϕ, h_i, $q_{n-2} \in L_1(J)$, q_j, $q_{n-1} \in L_\infty(J)$ are nonnegative, $\omega_i : \mathbb{R}_+ \to \mathbb{R}_+$ are nonincreasing, $0 \le i \le n-1$, $0 \le j \le n-3$, and[1]

$$\int_0^T \omega_{n-1}(t^{r+1})\,dt < \infty, \quad \int_0^T \omega_j(t^{n-j-2})\,dt < \infty \text{ for } 0 \le j \le n-3. \tag{1.10}$$

In the study of problem (1.3), (1.4) we will work with the assumptions:

(H_4) $f \in Car(J \times X)$ and there exist positive constants ε, r such that

$$\varepsilon(T-t)^r \le f(t, x_0, \dots, x_{n-1})$$

for a.e. $t \in J$ and each $(x_0, \dots, x_{n-1}) \in X$;

(H_5) For a.e. $t \in J$ and for each $(x_0, \dots, x_{n-1}) \in X$, the function f satisfies (1.8) where $\alpha_i \in (0,1)$, ϕ, $h_i \in L_1(J)$, $q_i \in L_\infty(J)$ are nonnegative, $\omega_i : \mathbb{R}_+ \to \mathbb{R}_+$ are nonincreasing, $0 \le i \le n-1$, and

$$\int_0^T \omega_i(t^{r+n-i})\,dt < \infty \text{ for } 0 \le i \le n-1. \tag{1.11}$$

Remark 1.4. Since $\omega_i : \mathbb{R}_+ \to \mathbb{R}_+$ in (H_2) are nonincreasing, assumption (1.9) implies that

$$\int_0^V \omega_{n-1}(t^{r+1})\,dt < \infty, \quad \int_0^V \omega_i(t^{n-i-1})\,dt < \infty, \quad 0 \le i \le n-2$$

for each $V \in \mathbb{R}_+$. The same is true for all integrals into (1.10) and (1.11).

Remark 1.5. After substituting $t = T - s$ in (1.3), (1.4), we get the singular $(n-p, p)$ *left focal* BVP

$$(-1)^p x^{(n)}(s) = \widetilde{f}\big(s, x(s), \dots, x^{(n-1)}(s)\big), \tag{1.12}$$

$$x^{(i)}(0) = 0, \quad p \le i \le n-1, \quad x^{(i)}(T) = 0, \quad 0 \le i \le p-1, \tag{1.13}$$

[1]Throughout the paper conditions and statements depending on j with $0 \le j \le n-3$ are realized only in the case where $n \ge 3$.

where p is fixed, $1 \le p \le n-1$, and $\widetilde{f} \in Car(J \times Y)$ fulfils $\widetilde{f}(s, x_0, x_1, \dots, x_{n-1}) = f(T-s, x_0, -x_1, \dots, (-1)^{n-1} x_{n-1})$. Here

$$Y = \begin{cases} \underbrace{\mathbb{R}_+ \times \mathbb{R}_- \times \mathbb{R}_+ \times \cdots \times \mathbb{R}_- \times \mathbb{R}_+^{n-p}}_{n} & \text{if } p \text{ is even,} \\ \underbrace{\mathbb{R}_+ \times \mathbb{R}_- \times \mathbb{R}_+ \times \cdots \times \mathbb{R}_+ \times \mathbb{R}_-^{n-p}}_{n} & \text{if } p \text{ is odd.} \end{cases}$$

By a *solution of BVP* (1.12), (1.13) we understand a function $x \in AC^{n-1}(J)$ which is positive on $[0, T)$, satisfies conditions (1.13) and for a.e. $s \in J$ fulfils (1.12).

The corresponding assumptions for problem (1.12), (1.13) have the form:

(H_6) $\tilde{f} \in Car(J \times Y)$ and there exist positive constants ε, r such that

$$\varepsilon s^r \le \tilde{f}(s, x_0, \dots, x_{n-1})$$

for a.e. $s \in J$ and each $(x_0, \dots, x_{n-1}) \in Y$;

(H_7) For a.e. $s \in J$ and for each $(x_0, \dots, x_{n-1}) \in Y$ the function $\widetilde{f}$ satisfies

$$\widetilde{f}(s, x_0, \dots, x_{n-1}) \le \phi(s) + \sum_{i=0}^{n-1} q_i(s) \omega_i(|x_i|) + \sum_{i=0}^{n-1} h_i(s) |x_i|^{\alpha_i},$$

where $\alpha_i \in (0, 1)$, ϕ, $h_i \in L_1(J)$, $q_i \in L_\infty(J)$ are nonnegative, $\omega_i : \mathbb{R}_+ \to \mathbb{R}_+$ are nonincreasing, $0 \le i \le n-1$, and ω_i fulfil (1.11).

2. Green's Functions and a Priori Estimates

2.1. **Problem (1.1), (1.2).** From now on, $G(t, s)$ denotes the Green's function of the BVP

$$-x''(t) = 0, \tag{2.1}$$

$$\alpha x(0) - \beta x'(0) = 0, \quad \gamma x(T) + \delta x'(T) = 0. \tag{2.2}$$

Then (see, e.g., [1])

$$G(t, s) = \begin{cases} \dfrac{1}{d} (\beta + \alpha s)(\delta + \gamma(T - t)) & \text{for } 0 \le s \le t \le T, \\ \dfrac{1}{d} (\beta + \alpha t)(\delta + \gamma(T - s)) & \text{for } 0 \le t < s \le T, \end{cases} \tag{2.3}$$

where

$$d = \alpha\gamma T + \alpha\delta + \beta\gamma > 0.$$

Let us choose positive constants ε and r and define the set

$$\mathcal{A}(r, \varepsilon) = \{x \in AC^{n-1}(J) : \ x \text{ fulfils (1.2) and (2.4)}\}$$

where

$$-x^{(n)}(t) \ge \varepsilon t^r \ \text{ for a.e. } t \in J. \tag{2.4}$$

2.1.1. *Problem* (1.1), (1.2) *with* $\min\{\beta,\delta\} = 0$. In this subsection we assume that at least one constant of β and δ appearing in (1.2) is equal to zero.

Lemma 2.1. *Let $x \in \mathcal{A}(r,\varepsilon)$ and set*

$$A = \frac{\varepsilon}{(r+1)(r+2)}\left(\frac{T}{2}\right)^{r+1}. \tag{2.5}$$

Then $x^{(n-1)}$ is decreasing on J,

$$\begin{aligned} x^{(n-1)}(t) &\geq \frac{\varepsilon}{r+1}(\xi - t)^{r+1} \quad \text{for } t \in [0,\xi], \\ x^{(n-1)}(t) &< -\frac{\varepsilon}{r+1}(t-\xi)^{r+1} \quad \text{for } t \in (\xi, T], \end{aligned} \tag{2.6}$$

where $\xi \in (0,T)$ is the unique zero of $x^{(n-1)}$,

$$x^{(n-2)}(t) \geq \begin{cases} At & \text{for } t \in \left[0, \frac{T}{2}\right] \\ A(T-t) & \text{for } t \in \left(\frac{T}{2}, T\right] \end{cases} \tag{2.7}$$

and

$$x^{(j)}(t) \geq \frac{A}{4(n-j-1)!}t^{n-j-1} \quad \text{for } t \in J, \ \ 0 \leq j \leq n-3. \tag{2.8}$$

Proof. From the equality

$$x^{(n-2)}(t) = -\int_0^T G(t,s)x^{(n)}(s)\,ds, \quad t \in J,$$

(2.3) and (2.4) we deduce that

$$\begin{aligned} x^{(n-2)}(0) &= -\int_0^T G(0,s)x^{(n)}(s)\,ds = -\frac{\beta}{d}\int_0^T (\delta + \gamma(T-s))x^{(n)}(s)\,ds \\ &\geq \frac{\varepsilon\beta\gamma}{d}\int_0^T (T-s)s^r\,ds \geq 0, \end{aligned} \tag{2.9}$$

$$\begin{aligned} x^{(n-2)}(T) &= -\int_0^T G(T,s)x^{(n)}(s)\,ds = -\frac{\delta}{d}\int_0^T (\beta + \alpha s)x^{(n)}(s)\,ds \\ &\geq \frac{\varepsilon\alpha\delta}{d}\int_0^T s^{r+1}\,ds \geq 0, \end{aligned} \tag{2.10}$$

$$x^{(n-1)}(0) = -\int_0^T \left.\frac{\partial G(t,s)}{\partial t}\right|_{t=0} x^{(n)}(s)\,ds = -\frac{\alpha}{d}\int_0^T (\delta + \gamma(T-s))x^{(n)}(s)\,ds$$

$$\geq \frac{\varepsilon\alpha\gamma}{d}\int_0^T (T-s)s^r\,ds > 0$$

and

$$x^{(n-1)}(T) = -\int_0^T \frac{\partial G(t,s)}{\partial t}\bigg|_{t=T} x^{(n)}(s)\,ds = \frac{\gamma}{d}\int_0^T (\beta+\alpha s)x^{(n)}(s)\,ds$$
$$\leq -\frac{\varepsilon\alpha\gamma}{d}\int_0^T s^{r+1}\,ds < 0.$$

Since $x^{(n-1)}$ is decreasing on J by (2.4) and $x^{(n-1)}(0) > 0$, $x^{(n-1)}(T) < 0$, we see that $x^{(n-1)}$ has a unique zero $\xi \in (0,T)$. Then

$$-x^{(n-1)}(t) = \int_t^\xi x^{(n)}(s)\,ds \leq -\varepsilon\int_t^\xi s^r\,ds = -\frac{\varepsilon}{r+1}(\xi^{r+1}-t^{r+1}),\ \ t\in[0,\xi],$$

and so

$$x^{(n-1)}(t) \geq \frac{\varepsilon}{r+1}(\xi-t)^{r+1},\ \ t\in[0,\xi],$$

since $\xi^{r+1} - t^{r+1} \geq (\xi-t)^{r+1}$ for $t\in[0,\xi]$. Analogously, using the inequality $t^{r+1}-\xi^{r+1} > (t-\xi)^{r+1}$ for $t\in(\xi,T]$, we obtain

$$x^{(n-1)}(t) = \int_\xi^t x^{(n)}(s)\,ds \leq -\varepsilon\int_\xi^t s^r\,ds = -\frac{\varepsilon}{r+1}(t^{r+1}-\xi^{r+1}) < -\frac{\varepsilon}{r+1}(t-\xi)^{r+1}$$

for $t\in(\xi,T]$. We have proved that (2.6) holds.

We are going to verify (2.7). From the assumption $\min\{\beta,\delta\}=0$, (2.9) and (2.10) we have $x^{(n-2)}(0)\geq 0$, $x^{(n-2)}(T)\geq 0$ and $\min\{x^{(n-2)}(0),x^{(n-2)}(T)\}=0$. Moreover, $x^{(n-2)}$ is concave on J, which follows from (2.4). Hence to prove (2.7) it suffices to show that

$$x^{(n-2)}\Big(\frac{T}{2}\Big) \geq \frac{AT}{2}\,, \tag{2.11}$$

where A is given by (2.5). From (2.6) it follows that

$$x^{(n-2)}(t) = x^{(n-2)}(0) + \int_0^t x^{(n-1)}(s)\,ds \geq \frac{\varepsilon}{r+1}\int_0^t (\xi-s)^{r+1}\,ds$$
$$= \frac{\varepsilon}{(r+1)(r+2)}\,(\xi^{r+2}-(\xi-t)^{r+2}),\ \ t\in[0,\xi],$$
$$-x^{(n-2)}(t) = -x^{(n-2)}(T) + \int_t^T x^{(n-1)}(s)\,ds < -\frac{\varepsilon}{r+1}\int_t^T (s-\xi)^{r+1}\,ds$$
$$= -\frac{\varepsilon}{(r+1)(r+2)}\,((T-\xi)^{r+2}-(t-\xi)^{r+2}),\ \ t\in(\xi,T],$$

and since $\xi^{r+2} - (\xi - t)^{r+2} \ge t^{r+2}$ for $t \in [0, \xi]$ and $(T - \xi)^{r+2} - (t - \xi)^{r+2} \ge (T - t)^{r+2}$ for $t \in (\xi, T]$, we have

$$x^{(n-2)}(t) \ge \frac{\varepsilon}{(r+1)(r+2)} t^{r+2} \quad \text{for } t \in [0, \xi], \tag{2.12}$$

$$x^{(n-2)}(t) > \frac{\varepsilon}{(r+1)(r+2)} (T - t)^{r+2} \quad \text{for } t \in (\xi, T]. \tag{2.13}$$

We know that $\max\{x^{(n-2)}(t) : t \in J\} = x^{(n-2)}(\xi)$. If $\xi \ge T/2$, then (2.12) gives (2.11) and if $\xi < T/2$ then from (2.13) we obtain (2.11) as well.

It remains to prove (2.8). By (2.7) and $x^{(n-3)}(0) = 0$, we have

$$x^{(n-3)}(t) = \int_0^t x^{(n-2)}(s)\, ds \ge A \int_0^t s\, ds = \frac{A}{2} t^2$$

for $t \in [0, T/2]$. Hence $x^{(n-3)}(T/2) \ge (A/2)(T/2)^2$ and since $x^{(n-3)}$ is increasing on J and $(t/2)^2 \le (T/2)^2$ for $t \in J$, we see that

$$x^{(n-3)}(t) \ge \frac{A}{4 \cdot 2!} t^2, \quad t \in J.$$

Then using the equalities

$$x^{(i)}(t) = \int_0^t x^{(i+1)}(s)\, ds, \quad t \in J, \ 0 \le i \le n - 4,$$

we can verify that inequalities (2.8) are satisfied. □

Lemma 2.2. *For $0 \le i \le n-1$ let $\widehat{\phi}$, $h_i \in L_1(J)$, $q_i \in L_\infty(J)$ be nonnegative, $\omega_i : \mathbb{R}_+ \to \mathbb{R}_+$ be nonincreasing and satisfy* (1.9) *and $\alpha_i \in (0, 1)$. Then there exists a positive constant $\widehat{M}$ such that for each $x \in \mathcal{A}(r, \varepsilon)$ satisfying*

$$-x^{(n)}(t) \le \widehat{\phi}(t) + \sum_{i=0}^{n-1} q_i(t)\omega_i(|x^{(i)}(t)|) + \sum_{i=0}^{n-1} h_i(t)|x^{(i)}(t)|^{\alpha_i} \tag{2.14}$$

for a.e. $t \in J$, the estimates

$$\|x^{(i)}\| \le \widehat{M} \quad \textit{for } 0 \le i \le n - 1 \tag{2.15}$$

are valid.

Proof. Let $x \in \mathcal{A}(r, \varepsilon)$ satisfy inequalities (2.14) a.e. on J. By Lemma 2.1, $x^{(n-1)}$ has a unique zero $\xi \in (0, T)$ and x satisfies inequalities (2.6)–(2.8) with A given by (2.5). From $x^{(n-2)}(0) = (\beta/\alpha)x^{(n-1)}(0) \ge 0$ (see (1.2)) it follows that

$$|x^{(n-2)}(t)| \le \frac{\beta}{\alpha} x^{(n-1)}(0) + \int_0^t |x^{(n-1)}(s)|\, ds \le \Big(T + \frac{\beta}{\alpha}\Big)\|x^{(n-1)}\|, \quad t \in J.$$

Hence

$$\|x^{(n-2)}\| \le \Big(T + \frac{\beta}{\alpha}\Big)\|x^{(n-1)}\|$$

and then the equalities

$$x^{(j)}(t) = \frac{1}{(n-j-3)!}\int_0^t (t-s)^{n-j-3} x^{(n-2)}(s)\,ds, \quad t \in J, \quad 0 \le j \le n-3,$$

give

$$\begin{aligned}\|x^{(j)}\| &\le \frac{T^{n-j-2}}{(n-j-3)!}\|x^{(n-2)}\| \\ &\le \frac{T^{n-j-2}}{(n-j-3)!}\Big(T+\frac{\beta}{\alpha}\Big)\|x^{(n-1)}\|, \quad 0 \le j \le n-3.\end{aligned}$$

Setting

$$V = \Big(T+\frac{\beta}{\alpha}\Big)\max\{1, V_1\}, \tag{2.16}$$

where

$$V_1 = \max\Big\{\frac{T^{n-j-2}}{(n-j-3)!} : \; 0 \le j \le n-3\Big\},$$

we see that

$$\|x^{(j)}\| \le V\|x^{(n-1)}\| \;\text{ for }\; 0 \le j \le n-2. \tag{2.17}$$

Now (2.14) yields

$$\begin{aligned}|x^{(n-1)}(t)| &= \Big|\int_\xi^t x^{(n)}(s)\,ds\Big| \\ &\le \int_0^T \Big[\widehat{\phi}(t) + \sum_{i=0}^{n-1} q_i(t)\omega_i(|x^{(i)}(t)|) + \sum_{i=0}^{n-1} h_i(t)|x^{(i)}(t)|^{\alpha_i}\Big]\,dt \\ &\le \|\widehat{\phi}\|_L + \sum_{i=0}^{n-1}\|q_i\|_\infty \int_0^T \omega_i(|x^{(i)}(t)|)\,dt + \sum_{i=0}^{n-1}\|h_i\|_L V^{\alpha_i}\|x^{(n-1)}\|^{\alpha_i}.\end{aligned} \tag{2.18}$$

Set $K = \sqrt[r+1]{\varepsilon/(r+1)}$ and $R_j = \sqrt[n-j-1]{A/(4(n-j-1)!)}$, $0 \le j \le n-3$. Since (cf. (2.6)–(2.8))

$$\begin{aligned}&\int_0^T \omega_{n-1}(|x^{(n-1)}(t)|)\,dt \\ &\le \int_0^\xi \omega_{n-1}\Big(\frac{\varepsilon}{r+1}\,(\xi-t)^{r+1}\Big)\,dt + \int_\xi^T \omega_{n-1}\Big(\frac{\varepsilon}{r+1}\,(t-\xi)^{r+1}\Big)\,dt \\ &= \frac{1}{K}\Big[\int_0^{K\xi}\omega_{n-1}(t^{r+1})\,dt + \int_0^{K(T-\xi)}\omega_{n-1}(t^{r+1})\,dt\Big] \le \frac{2}{K}\int_0^{KT}\omega_{n-1}(t^{r+1})\,dt,\end{aligned} \tag{2.19}$$

$$\int_0^T \omega_{n-2}(|x^{(n-2)}(t)|)\,dt$$
$$\le \int_0^{T/2} \omega_{n-2}(At)\,dt + \int_{T/2}^{T} \omega_{n-2}(A(T-t))\,dt = \frac{2}{A}\int_0^{(AT)/2} \omega_{n-2}(t)\,dt$$

and (for $0 \le j \le n-3$)

$$\int_0^T \omega_j(|x^{(j)}(t)|)\,dt \le \int_0^T \omega_j\Big(\frac{A}{4(n-j-1)!}t^{n-j-1}\Big)\,dt = \frac{1}{R_j}\int_0^{R_jT} \omega_j(t^{n-j-1})\,dt,$$

we deduce from (1.9) and Remark 1.4 that there is a positive constant Q independent of x such that

$$\sum_{i=0}^{n-1} \|g_i\|_\infty \int_0^T \omega_i(|x^{(i)}(t)|)\,dt \le Q.$$

Then (2.1.1) yields

$$\|x^{(n-1)}\| \le \|\widehat{\phi}\|_L + Q + \sum_{i=0}^{n-1} \|h_i\|_L V^{\alpha_i}\|x^{(n-1)}\|^{\alpha_i}. \tag{2.20}$$

Setting $z(u) = (\|\widehat{\phi}\|_L + Q)/u + \sum_{i=0}^{n-1} \|h_i\|_L V^{\alpha_i} u^{\alpha_i - 1}$ for $u \in (0,\infty)$, we have $\lim_{u\to\infty} z(u) = 0$, and so there is a positive constant P such that $z(u) < 1$ for all $u \ge P$. Then from (2.20) it follows that $\|x^{(n-1)}\| \le P$ and, by (2.17), $\|x^{(j)}\| \le VP$ for $0 \le j \le n-2$. Hence (2.15) is true with $\widehat{M} = \max\{P, VP\}$. □

2.1.2. *Problem* (1.1), (1.2) *with* $\min\{\beta,\delta\} > 0$. Throughout this subsection we assume that the constants β and δ in (1.2) are positive.

Lemma 2.3. *Let* $x \in \mathcal{A}(r,\varepsilon)$ *and set*

$$B = \frac{\varepsilon}{d}\min\Big\{\beta\gamma\int_0^T (T-s)s^r\,ds,\ \alpha\delta\int_0^T s^{r+1}\,ds\Big\} > 0. \tag{2.21}$$

Then $x^{(n-1)}$ *is decreasing on* J, *satisfies inequalities* (2.6) *where* $\xi \in (0,T)$ *is its unique zero,*

$$x^{(n-2)}(t) \ge B \ \text{ for } \ t \in J \tag{2.22}$$

and

$$x^{(j)}(t) \ge \frac{B}{(n-j-2)!}t^{n-j-2} \ \text{ for } \ t \in J,\ \ 0 \le j \le n-3. \tag{2.23}$$

Proof. The properties of $x^{(n-1)}$ follow immediately from Lemma 2.1 and its proof. Next, by (2.9) and (2.10),

$$x^{(n-2)}(0) \geq \frac{\varepsilon\beta\gamma}{d}\int_0^T (T-s)s^r\,ds > 0, \quad x^{(n-2)}(T) \geq \frac{\varepsilon\alpha\delta}{d}\int_0^T s^{r+1}\,ds > 0.$$

Since $x^{(n-2)}$ is concave on J, we see that $x^{(n-2)}(t) \geq B$ for $t \in J$. Now from the last inequality and the equalities $x^{(j)}(0) = 0$, $0 \leq j \leq n-3$ we obtain the validity of (2.23). □

Lemma 2.4. *For $0 \leq i \leq n-1$ and $0 \leq j \leq n-3$, let $\widehat{\phi}$, h_i, $q_{n-2} \in L_1(J)$, q_j, $q_{n-1} \in L_\infty(J)$ be nonnegative, $\omega_i : \mathbb{R}_+ \to \mathbb{R}_+$ be nonincreasing and satisfying* (1.10) *and $\alpha_i \in (0,1)$. Then there exists a positive constant $\widehat{V}$ such that*

$$\|x^{(i)}\| \leq \widehat{V} \ \ \textit{for} \ \ 0 \leq i \leq n-1 \tag{2.24}$$

whenever $x \in \mathcal{A}(r,\varepsilon)$ satisfies inequality (2.14) *for a.e. $t \in J$.*

Proof. Let $x \in \mathcal{A}(r,\varepsilon)$ satisfy inequality (2.14) for a.e. $t \in J$. By Lemma 2.3, inequalities (2.6), (2.22) and (2.23) are true, where $\xi \in (0,T)$ is the unique zero of $x^{(n-1)}$ and B is given by (2.21). From $x^{(n-2)}(0) = (\beta/\alpha)x^{(n-1)}(0)$ (see (1.2)) and using the same procedure as in the proof of Lemma 2.2 we see that (2.17) holds, where L is defined by (2.16).

Since $x^{(n-2)}(t) \geq B$ for $t \in J$ and $x^{(j)}(0) = 0$, $0 \leq j \leq n-3$, we have

$$x^{(j)}(t) \geq \frac{B}{(n-j-2)!}t^{n-j-2} \ \ \text{for} \ \ t \in J, \ \ 0 \leq j \leq n-3. \tag{2.25}$$

Hence

$$\omega_{n-2}\big(x^{(n-2)}(t)\big) \leq \omega_{n-2}(B), \ \ t \in J,$$

and

$$\int_0^T \omega_j(x^{(j)}(t))\,dt \leq \int_0^T \omega_j\Big(\frac{B}{(n-j-2)!}t^{n-j-2}\Big)\,dt = \frac{1}{m_j}\int_0^{m_jT} \omega_j(t^{n-j-2})\,dt$$

for $0 \leq j \leq n-3$, where $m_j = \sqrt[n-j-2]{B/(n-j-2)!}$. Then (see (2.14), (2.17) and (2.1.1))

$$\begin{aligned}|x^{(n-1)}(t)| &= \Big|\int_\xi^t x^{(n)}(s)\,ds\Big| \\ &\leq \int_0^T \Big[\widehat{\phi}(t) + \sum_{i=0}^{n-1} q_i(t)\omega_i(|x^{(i)}(t)|) + \sum_{i=0}^{n-1} h_i(t)|x^{(i)}(t)|^{\alpha_i}\Big]\,dt \\ &\leq \|\widehat{\phi}\|_L + \sum_{i=0}^{n-3}\frac{\|q_i\|_\infty}{m_i}\int_0^{m_iT}\omega_i(t^{n-i-2})\,dt + \|q_{n-2}\|_L\omega_{n-2}(B)\end{aligned}$$

$$+\frac{2\|q_{n-1}\|_\infty}{K}\int_0^{KT}\omega_{n-1}(t^{r+1})\,dt+\sum_{i=0}^{n-1}\|h_i\|_L L^{\alpha_i}\|x^{(n-1)}\|^{\alpha_i},$$

where $K=\sqrt[r+1]{\varepsilon/(r+1)}$. Consequently,

$$\|x^{(n-1)}\|\le D_*+\sum_{i=0}^{n-1}\|h_i\|_L L^{\alpha_i}\|x^{(n-1)}\|^{\alpha_i}, \tag{2.26}$$

where

$$D_*=\|\widehat{\phi}\|_L+\sum_{i=0}^{n-3}\frac{\|q_i\|_\infty}{m_i}\int_0^{m_iT}\omega_i(t^{n-i-2})\,dt$$
$$+\|q_{n-2}\|_L\omega_{n-2}(B)+\frac{2\|q_{n-1}\|_\infty}{K}\int_0^{KT}\omega_{n-1}(t^{r+1})\,dt$$

is independent of x. Since $\lim_{u\to\infty}(D_*/u+\sum_{i=0}^{n-1}\|h_i\|_L L^{\alpha_i}u^{\alpha_i-1})=0$, there is a positive constant P_* such that $D_*/u+\sum_{i=0}^{n-1}\|h_i\|_L L^{\alpha_i}u^{\alpha_i-1}<1$ for $u\in[P_*,\infty)$. Therefore (2.26) gives $\|x^{(n-1)}\|\le P_*$. Now (2.17) leads to (2.24) with $\widehat{V}=\max\{P_*,LP_*\}$. □

2.2. **Problem (1.3), (1.4).** Let us choose positive constants ε and r and define the set

$$\mathcal{B}(r,\varepsilon)=\big\{x\in AC^{n-1}(J):\ x \text{ fulfils (1.4) and (2.28)}\big\}, \tag{2.27}$$

where

$$(-1)^{n-p}x^{(n)}(t)\ge\varepsilon(T-t)^r\ \text{ for a.e. } t\in J. \tag{2.28}$$

Section 2.2 is devoted to the study of the set $\mathcal{B}(r,\varepsilon)$. The properties of $\mathcal{B}(r,\varepsilon)$ obtained here (Lemmas 2.5–2.7) will be used in the proof of Theorem 4.3.

Lemma 2.5. *There exists $c>0$ such that the inequalities*

$$\begin{aligned} x^{(i)}(t)&\ge ct^{r+n-i}\ \textit{ for } 0\le i\le p-1,\\ (-1)^{i-p}x^{(i)}(t)&\ge c(T-t)^{r+n-i}\ \textit{ for } p\le i\le n-1\end{aligned} \tag{2.29}$$

are true for $t\in J$ and each $x\in\mathcal{B}(r,\varepsilon)$.

Proof. Let us put

$$c=\frac{\varepsilon}{(r+1)(r+2)\dots(r+n)}.$$

Then, using (1.4) and integrating (2.28), we get step by step that (2.29) holds on J for $p\le i\le n-1$ and that

$$x^{(p-1)}(t)\ge c(T^{r+n-p+1}-(T-t)^{r+n-p+1})\ \text{ for } t\in J. \tag{2.30}$$

Put $r+n-p+1=\nu$ and consider the function $\varphi(t)=T^\nu-(T-t)^\nu-t^\nu$ on J. Since $\nu>2$, $\varphi(0)=\varphi(T)=0$ and φ is concave on J, we have $\varphi>0$ on $(0,T)$ and thus

$$T^{r+n-p+1}-(T-t)^{r+n-p+1}>t^{r+n-p+1}$$

holds on $(0,T)$, which together with (2.30) yields

$$x^{(p-1)}(t)\ge ct^{r+n-p+1} \quad \text{for} \quad t\in J. \tag{2.31}$$

Now, using (1.4) again and integrating (2.31), we successively obtain (2.29) for $0\le i\le p-1$ and $t\in J$. □

Lemma 2.6. *Let $\alpha_i\in(0,1)$, ϕ^*, $h_i\in L_1(J)$, $q_i\in L_\infty(J)$, $0\le i\le n-1$. Further, suppose that $\omega_i:\mathbb{R}_+\to\mathbb{R}_+$ are nonincreasing and fulfil* (1.11)*. Then there exists $r^*>0$ such that for each function $x\in\mathcal{B}(r,\varepsilon)$ satisfying*

$$(-1)^{n-p}x^{(n)}(t)\le\phi^*(t)+\sum_{i=0}^{n-1}q_i(t)\omega_i(|x^{(i)}(t)|)+\sum_{i=0}^{n-1}h_i(t)|x^{(i)}(t)|^{\alpha_i} \tag{2.32}$$

for a.e. $t\in J$, the estimate

$$\|x^{(n-1)}\|<r^* \tag{2.33}$$

is valid.

Proof. Having a function $x\in\mathcal{B}(r,\varepsilon)$ which satisfies (2.32), we put $\|x^{(n-1)}\|=\rho$. Then we integrate the inequality

$$|x^{(n-1)}(t)|\le\rho \quad \text{for} \quad t\in J,$$

and due to (1.4), we successively get

$$\|x^{(i)}\|\le\rho T^{n-i-1}, \quad 0\le i\le n-2. \tag{2.34}$$

Further, we integrate (2.32) on $[t,T]\subset J$ and in view of (2.34) we see that the inequality

$$\rho\le\|\phi^*\|_L+\sum_{i=0}^{n-1}\|q_i\|_\infty\int_0^T\omega_i(|x^{(i)}(t)|)\,dt+\sum_{i=0}^{n-1}\|h_i\|_L(\rho T^{n-i-1})^{\alpha_i} \tag{2.35}$$

holds. In order to find r^* fulfilling (2.33) we need to estimate the integrals

$$\int_0^T\omega_i(|x^{(i)}(t)|)\,dt, \quad 0\le i\le n-1.$$

To this end we distinguish two cases.

Case (α). Let $0\le i\le p-1$. Then, by Lemma 2.5, there exists $c>0$ such that

$$\int_0^T\omega_i(|x^{(i)}(t)|)\,dt\le\int_0^T\omega_i(ct^{r+n-i})\,dt=\int_0^T\omega_i((c_is)^{r+n-i})\,ds,$$

where $c_i^{r+n-i} = c$. Therefore, having in mind Remark 1.4, we conclude that

$$\int_0^T \omega_i\big(|x^{(i)}(t)|\big)\,dt \le C_i, \tag{2.36}$$

with

$$C_i = \frac{1}{c_i}\int_0^{c_i T} \omega_i(t^{r+n-i})\,dt \in \mathbb{R}_+. \tag{2.37}$$

Case (β). Let $p \le i \le n-1$. Then by virtue of Lemma 2.5 and (2.37) we get

$$\int_0^T \omega_i\big(|x^{(i)}(t)|\big)\,dt \le \int_0^T \omega_i(c(T-t)^{r+n-i})\,dt = \int_0^T \omega_i(ct^{r+n-i})\,dt = C_i,$$

i.e., (2.36) holds for $p \le i \le n-1$ as well.

After inserting (2.36) into (2.35), we obtain

$$\rho \le \|\phi^*\|_L + \sum_{i=0}^{n-1} \|q_i\|_\infty C_i + \sum_{i=0}^{n-1} \|h_i\|_L (\rho T^{n-i-1})^{\alpha_i}. \tag{2.38}$$

Now, suppose that r^* fulfilling (2.33) does not exist. Then we can find a sequence of functions $\{x_m\}$ such that $x_m \in \mathcal{B}(r,\varepsilon)$ satisfy (2.32) for $m \in \mathbb{N}$ and

$$\lim_{m\to\infty} \|x_m^{(n-1)}\| = \infty.$$

If we put $\|x_m^{(n-1)}\| = \rho_m$, then ρ_m satisfy (2.38) for $m \in \mathbb{N}$, which yields

$$1 \le \frac{1}{\rho_m}\left(A + \sum_{i=0}^{n-1} B_i \rho_m^{\alpha_i}\right), \tag{2.39}$$

where

$$A = \|\phi^*\|_L + \sum_{i=0}^{n-1} \|q_i\|_\infty C_i, \quad B_i = (T^{n-i-1})^{\alpha_i}\|h_i\|_L \ \text{ for } \ 0 \le i \le n-1.$$

Since $\alpha_i \in (0,1)$ for $0 \le i \le n-1$, inequality (2.39) implies

$$1 \le \lim_{m\to\infty} \frac{1}{\rho_m}\left(A + \sum_{i=0}^{n-1} B_i \rho_m^{\alpha_i}\right) = 0,$$

a contradiction. Therefore a positive constant r^* satisfying (2.33) must exist. $\square$

Lemma 2.7. *Suppose that* $\omega_i : \mathbb{R}_+ \to \mathbb{R}_+$, $0 \le i \le n-1$, *are nonincreasing and fulfil* (1.11). *Then for each* $\eta > 0$ *there exists* $\delta > 0$ *such that the condition*

$$|t_2 - t_1| < \delta \Longrightarrow \left|\int_{t_1}^{t_2} \omega_i\big(|x^{(i)}(t)|\big)\,dt\right| < \eta \tag{2.40}$$

holds for all $t_1,\ t_2 \in J,\ x \in \mathcal{B}(r,\varepsilon),\ 0 \le i \le n-1.$

Proof. Let us choose $t_1,\ t_2 \in J$ and $x \in \mathcal{B}(r,\varepsilon)$. Similarly to the proof of Lemma 2.6 we consider two cases.

Case (α). Let $0 \le i \le p-1$. Then by Lemma 2.5 there is a positive constant c such that if we put $c_i^{r+n-i} = c$, we have

$$\left|\int\limits_{t_1}^{t_2} \omega_i\big(|x^{(i)}(t)|\big)\,dt\right| \le \left|\int\limits_{t_1}^{t_2} \omega_i(ct^{r+n-i})\,dt\right| = \left|\int\limits_{t_1}^{t_2} \omega_i(c_i s)^{r+n-i} ds\right|.$$

Therefore we conclude that

$$\left|\int\limits_{t_1}^{t_2} \omega_i\big(|x^{(i)}(t)|\big)\,dt\right| \le \frac{1}{c_i}\left|\int\limits_{c_it_1}^{c_it_2} \omega_i(t^{r+n-i}))\,dt\right|. \tag{2.41}$$

Case (β). Let $p \le i \le n-1$. Then Lemma 2.5 yields

$$\left|\int\limits_{t_1}^{t_2} \omega_i\big(|x^{(i)}(t)|\big)\,dt\right| \le \left|\int\limits_{t_1}^{t_2} \omega_i(c(T-t)^{r+n-i})\,dt\right| = \left|\int\limits_{T-t_1}^{T-t_2} \omega_i(ct^{r+n-i})\,dt\right|,$$

and so

$$\left|\int\limits_{t_1}^{t_2} \omega_i\big(|x^{(i)}(t)|\big)\,dt\right| \le \frac{1}{c_i}\left|\int\limits_{c_i(T-t_1)}^{c_i(T-t_2)} \omega_i(t^{r+n-i}))\,dt\right| \tag{2.42}$$

with c_i given by *Case* (α).

Now, let us choose arbitrary $\eta > 0$ and $i \in \{0,\dots,p-1\}$, $j \in \{p,\dots,n-1\}$. Then, according to Remark 1.4, there exists $\delta > 0$ such that

$$|t_2 - t_1| < \delta \Longrightarrow \frac{1}{c_i}\left|\int\limits_{c_it_1}^{c_it_2} \omega_i(t^{r+n-i})\,dt\right| < \eta,$$

$$|t_2 - t_1| < \delta \Longrightarrow \frac{1}{c_j}\left|\int\limits_{c_j(T-t_1)}^{c_j(T-t_2)} \omega_j(t^{r+n-j})\,dt\right| < \eta$$

is valid for all $t_1,\ t_2 \in J$. So inequalities (2.41) and (2.42) imply that (2.40) is true for all $t_1,\ t_2 \in J,\ x \in \mathcal{B}(r,\varepsilon),\ 0 \le i \le n-1$. □

3. Auxiliary Regular BVPs

The aim of this section is to prove the existence of solutions to the auxiliary regular BVPs corresponding to singular problems (1.1), (1.2) and (1.3), (1.4). Constructions of regular BVPs are based on a priori estimates obtained in Section 2. The existence of their solutions will be proved by means of the Nonlinear Fredholm Alternative.

3.1. **Problem (1.1), (1.2).** For any positive constant S and each $m \in \mathbb{N}$, define $\varrho_{m,S} \in C^0(\mathbb{R})$, $\tau_S \in C^0(\mathbb{R})$ and $f_{m,S} \in Car(J \times \mathbb{R}^n)$ by

$$\varrho_{m,S}(u) = \begin{cases} \dfrac{1}{m} & \text{for } |u| < \dfrac{1}{m}, \\ |u| & \text{for } \dfrac{1}{m} \le |u| \le S+1, \\ S+1 & \text{for } |u| > S+1, \end{cases}$$

$$\tau_S(u) = \begin{cases} u & \text{for } |u| \le S+1, \\ (S+1)\operatorname{sgn} u & \text{for } |u| > S+1 \end{cases}$$

and

$$f_{m,S}(t, x_0, \dots, x_{n-2}, x_{n-1})$$

$$= \begin{cases} f\big(t, \varrho_{m,S}(x_0), \dots, \varrho_{m,S}(x_{n-2}), \tau_S(x_{n-1})\big) \\ \qquad \text{for } (t, x_0, \dots, x_{n-2}, x_{n-1}) \in J \times \mathbb{R}^{n-1} \times \Big(\Big(-\infty, -\dfrac{1}{m}\Big] \bigcup \Big[\dfrac{1}{m}, \infty\Big)\Big) \\ \dfrac{m}{2}\Big[f_{m,S}\Big(t, x_0, \dots, x_{n-2}, \dfrac{1}{m}\Big)\Big(x_{n-1} + \dfrac{1}{m}\Big) + \\ \quad + f_{m,S}\Big(t, x_0, \dots, x_{n-2}, -\frac{1}{m}\Big)\Big)\Big(\frac{1}{m} - x_{n-1}\Big)\Big] \\ \qquad \text{for } (t, x_0, \dots, x_{n-2}, x_{n-1}) \in J \times \mathbb{R}^{n-1} \times \Big(-\dfrac{1}{m}, \dfrac{1}{m}\Big). \end{cases}$$

Then we have under assumption (H_1) that

$$\varepsilon\, t^r \le f_{m,S}(t, x_0, \dots, x_{n-1}) \text{ for a.e. } t \in J \text{ and each } (x_0, \dots, x_{n-1}) \in \mathbb{R}^n \tag{3.1}$$

and under assumption (H_2) or (H_3) that

$$f_{m,S}(t, x_0, \dots, x_{n-1}) \le \widehat{\phi}(t) + \sum_{i=0}^{n-1} q_i(t)\omega_i(|x_i|) + \sum_{i=0}^{n-1} h_i(t)|x_i|^{\alpha_i} \tag{3.2}$$

for a.e. $t \in J$ and each $(x_0, \dots, x_{n-1}) \in \mathbb{R}_0^n$, where

$$\widehat{\phi}(t) = \phi(t) + \sum_{i=1}^{n-1} q_i(t)\omega_i(1) + \sum_{i=1}^{n-1} h_i(t) \tag{3.3}$$

since $\omega_j(\varrho_{m,S}(u)) \le \omega_j(1) + \omega_j(|u|)$, $\omega_{n-1}(|\tau_S(u)|) \le \omega_{n-1}(1) + \omega_{n-1}(|u|)$, $(\varrho_{m,S}(u))^{\alpha_j} \le 1 + |u|^{\alpha_i}$, $|\tau_S(u)|^{\alpha_{n-1}} \le 1 + |u|^{\alpha_{n-1}}$ for $u \in \mathbb{R}$ and $0 \le j \le n-2$.

3.1.1. *Problem* (1.1), (1.2) *with* $\min\{\beta, \delta\} = 0$. Let assumptions (H_1) and (H_2) be satisfied and let $\widehat{M}$ be the positive constant given by Lemma 2.2 with $\widehat{\phi}$ defined by (3.3). Consider the auxiliary family of regular differential equations

$$-x^{(n)}(t) = f_{m,\widehat{M}}(t, x(t), \dots, x^{(n-1)}(t)) \tag{3.4}$$

depending on $m \in \mathbb{N}$.

Lemma 3.1. *Let assumptions* (H_1) *and* (H_2) *be satisfied. Then, for each* $m \in \mathbb{N}$, *BVP* (3.4), (1.2) *has a solution* $x_m \in \mathcal{A}(r, \varepsilon)$ *and*

$$\|x_m^{(i)}\| \le \widehat{M} \;\; for \;\; 0 \le i \le n-1. \tag{3.5}$$

Proof. Fix $m \in \mathbb{N}$ and set

$$\begin{aligned} g_m(t) &= \sup\left\{f_{m,\widehat{M}}(t,x_0,\dots,x_{n-1}) :\ (x_0,\dots,x_{n-1}) \in \mathbb{R}^n\right\} \\ &= \sup\Big\{f(t,x_0,\dots,x_{n-1}) :\ \frac{1}{m} \le x_i \le \widehat{M}+1 \text{ for} \\ &\qquad 0 \le i \le n-2,\ \frac{1}{m} \le |x_{n-1}| \le \widehat{M}+1\Big\}. \end{aligned}$$

Since $f \in Car(J \times D)$ we see that $g_m \in L_1(J)$. Using the fact that the problem $-x^{(n)}(t) = 0$, (1.2) has only the trivial solution, the Nonlinear Fredholm Alternative guarantees the existence of a solution x_m of BVP (3.4), (1.2). Besides, (3.1) and (3.2) with $S = \widehat{M}$ give

$$\varepsilon t^r \le -x_m^{(n)}(t) \le \widehat{\phi}(t) + \sum_{i=0}^{n-1} q_i(t)\omega_i(|x_m^{(i)}(t)|) + \sum_{i=0}^{n-1} h_i(t)|x_m^{(i)}(t)|^{\alpha_i}$$

for a.e. $t \in J$. Therefore $x_m \in \mathcal{A}(r,\varepsilon)$ and $\|x_m^{(i)}\| \le \widehat{M}$ for $0 \le i \le n-1$ by Lemmas 2.1 and 2.2. $\square$

Lemma 3.2. *Let assumptions* (H_1) *and* (H_2) *be satisfied and let* x_m *be a solution of BVP* (3.4), (1.2)*,* $m \in \mathbb{N}$*. Then the sequence*

$$\left\{f_{m,\widehat{M}}(t,x_m(t),\dots,x_m^{(n-1)}(t))\right\} \subset L_1(J) \tag{3.6}$$

is uniformly absolutely continuous on J*.*

Proof. By Lemmas 2.1 and 3.1 we have (for $m \in \mathbb{N}$)

$$\begin{aligned} x_m^{(n-1)}(t) &\ge \frac{\varepsilon}{r+1}(\xi_m - t)^{r+1} \ \text{ for } t \in [0,\xi_m], \\ x_m^{(n-1)}(t) &< -\frac{\varepsilon}{r+1}(t-\xi_m)^{r+1} \ \text{ for } t \in (\xi_m, T], \end{aligned} \tag{3.7}$$

where $\xi_m \in (0,T)$ is the unique zero of $x_m^{(n-1)}$,

$$x_m^{(n-2)}(t) \ge \begin{cases} At & \text{for } t \in \left[0,\frac{T}{2}\right] \\ A(T-t) & \text{for } t \in \left(\frac{T}{2},T\right] \end{cases} \tag{3.8}$$

and

$$x_m^{(j)}(t) \ge \frac{A}{4(n-j-1)!}\, t^{n-j-1} \ \text{ for } t \in J,\ \ 0 \le j \le n-3, \tag{3.9}$$

where A is defined by (2.5). Moreover, by Lemma 3.1,

$$\|x_m^{(j)}\| \le \widehat{M} \ \text{ for } m \in \mathbb{N},\ \ 0 \le j \le n-1. \tag{3.10}$$

Since

$$0 \le f_{m,\widehat{M}}(t,x_m(t),\dots,x_m^{(n-1)}(t)) \le \widehat{\phi}(t) + \sum_{i=0}^{n-1} q_i(t)\omega_i\big(|x_m^{(i)}(t)|\big) + \sum_{i=0}^{n-1} h_i(t)\widehat{M}^{\alpha_i}$$

for a.e. $t \in J$ and each $m \in \mathbb{N}$ by (3.2), we see from the properties of the functions ϕ, q_i and h_i, $0 \le i \le n-1$, given in (H_2) that to prove the assertion of our lemma it suffices to show that the sequences

$$\big\{\omega_i\big(|x_m^{(i)}(t)|\big)\big\}, \quad 0 \le i \le n-1,$$

are uniformly absolutely continuous on J.

Let $0 \le i \le n-3$. Then

$$\omega_i\big(|x_m^{(i)}(t)|\big) \le \omega_i\Big(\frac{A}{4(n-i-1)!}\, t^{n-i-1}\Big), \quad t \in (0,T], \quad m \in \mathbb{N},$$

which follows from (3.9) since ω_i is nonincreasing on $\mathbb{R}_+$. Moreover, (1.9) implies that the functions $\omega_i(\frac{A}{4(n-i-1)!}\, t^{n-i-1})$ belong to the class $L_1(J)$. Hence $\{\omega_i(|x_m^{(i)}(t)|)\}$ is uniformly absolutely continuous on J for $0 \le i \le n-3$.

Analogously, (3.8) gives

$$\omega_{n-2}\big(|x_m^{(n-2)}(t)|\big) \le \omega_{n-2}(\varphi(t)), \quad t \in (0,T), \quad m \in \mathbb{N},$$

where

$$\varphi(t) = \begin{cases} At & \text{for } t \in \Big[0, \dfrac{T}{2}\Big], \\ A(T-t) & \text{for } t \in \Big(\dfrac{T}{2}, T\Big]. \end{cases}$$

Since $\omega_{n-2}(\varphi(t)) \in L_1(J)$ which follows from the assumption $\int_0^V \omega_{n-2}(t)\, dt < \infty$ for each $V \in \mathbb{R}_+$ (see Remark 1.4), the sequence $\{\omega_{n-2}(|x_m^{(n-2)}(t)|)\}$ is uniformly absolutely continuous on J.

It remains to verify the uniform absolute continuity on J of the sequence $\{\omega_{n-1}(|x_m^{(n-1)}(t)|)\}$. Let $\{(a_j,b_j)\}_{j\in\mathbb{J}}$ be a sequence of a mutually disjoint intervals $(a_j,b_j) \subset J$. Set

$$\mathbb{J}_m^1 = \big\{j:\ j \in \mathbb{J},\ (a_j,b_j) \subset (0,\xi_m)\big\}, \quad \mathbb{J}_m^2 = \big\{j:\ j \in \mathbb{J},\ (a_j,b_j) \subset (\xi_m,T)\big\}$$

for $m \in \mathbb{N}$ and set $\kappa = \sqrt[r+1]{\varepsilon/(r+1)}$. Then for $j \in \mathbb{J}_m^1$ and $k \in \mathbb{J}_m^2$ we have (see (3.7))

$$\int_{a_j}^{b_j} \omega_{n-1}\big(|x_m^{(n-1)}(t)|\big)\, dt \le \int_{a_j}^{b_j} \omega_{n-1}\Big(\big(\kappa(\xi_m - t)\big)^{r+1}\Big)\, dt = \frac{1}{\kappa} \int_{\kappa(\xi_m-b_j)}^{\kappa(\xi_m-a_j)} \omega_{n-1}(t^{r+1})\, dt,$$

$$\int_{a_k}^{b_k} \omega_{n-1}\big(|x_m^{(n-1)}(t)|\big)\, dt \le \int_{a_k}^{b_k} \omega_{n-1}\Big(\big(\kappa(t - \xi_m)\big)^{r+1}\Big)\, dt = \frac{1}{\kappa} \int_{\kappa(a_k-\xi_m)}^{\kappa(b_k-\xi_m)} \omega_{n-1}(t^{r+1})\, dt.$$

If $\{j_0\} = \mathbb{J} \setminus (\mathbb{J}_m^1 \cup \mathbb{J}_m^2)$, that is $a_{j_0} < \xi_m < b_{j_0}$, then

$$\int_{a_{j_0}}^{b_{j_0}} \omega_{n-1}\big(|x_m^{(n-1)}(t)|\big)\,dt$$

$$\le \int_{a_{j_0}}^{\xi_m} \omega_{n-1}\big((\kappa(\xi_m - t))^{r+1}\big)\,dt + \int_{\xi_m}^{b_{j_0}} \omega_{n-1}\big((\kappa(t-\xi_m))^{r+1}\big)\,dt$$

$$= \frac{1}{\kappa}\Bigg[\int_0^{\kappa(\xi_m - a_{j_0})} \omega_{n-1}(t^{r+1})\,dt + \int_0^{\kappa(b_{j_0}-\xi_m)} \omega_{n-1}(t^{r+1})\,dt\Bigg].$$

Hence

$$\sum_{j\in\mathbb{J}} \int_{a_j}^{b_j} \omega_{n-1}\big(|x_m^{(n-1)}(t)|\big)\,dt$$

$$\le \frac{1}{\kappa}\Bigg[\sum_{j\in\mathbb{J}_m^1} \int_{\kappa(\xi_m-b_j)}^{\kappa(\xi_m-a_j)} \omega_{n-1}(t^{r+1})\,dt + \sum_{j\in\mathbb{J}_m^2} \int_{\kappa(a_j-\xi_m)}^{\kappa(b_j-\xi_m)} \omega_{n-1}(t^{r+1})\,dt + E\Bigg],$$

where

$$E = \begin{cases} 0 & \text{if } \mathbb{J} = \mathbb{J}_m^1 \cup \mathbb{J}_m^2 \\ \displaystyle\int_0^{\kappa(\xi_m-a_{j_0})} \omega_{n-1}(t^{r+1})\,dt + \int_0^{\kappa(b_{j_0}-\xi_m)} \omega_{n-1}(t^{r+1})\,dt & \text{if } \{j_0\} = \mathbb{J} \setminus (\mathbb{J}_m^1 \cup \mathbb{J}_m^2) \end{cases}.$$

Since

$$\sum_{j\in\mathbb{J}_m^1} \big[\kappa(\xi_m - a_j) - \kappa(\xi_m - b_j)\big] + \sum_{j\in\mathbb{J}_m^2} \big[\kappa(b_j - \xi_m) - \kappa(a_j - \xi_m)\big] + c$$

$$= \kappa \sum_{j\in\mathbb{J}} (b_j - a_j),$$

where

$$c = \begin{cases} 0 & \text{if } \mathbb{J} = \mathbb{J}_m^1 \cup \mathbb{J}_m^2 \\ \kappa(b_{j_0} - a_{j_0}) & \text{if } \{j_0\} = \mathbb{J} \setminus (\mathbb{J}_m^1 \cup \mathbb{J}_m^2) \end{cases},$$

we se that

$$\sum_{j\in\mathbb{J}} \int_{a_j}^{b_j} \omega_{n-1}\big(|x_m^{(n-1)}(t)|\big)\,dt \le \int_{\mathcal{M}_1} \omega_{n-1}(t^{r+1})\,dt + \int_{\mathcal{M}_2} \omega_{n-1}(t^{r+1})\,dt$$

where $\mu(\mathcal{M}_i) \le \kappa \sum_{j\in\mathbb{J}}(b_j - a_j)$ for $i = 1, 2$. Hence $\{\omega_{n-1}(|x_m^{(n-1)}(t)|)\}$ is uniformly absolutely continuous on J, which follows from the fact that $\int_0^{\kappa T} \omega_{n-1}(t^{r+1})\,dt < \infty$ by (H_2) and Remark 1.4. $\square$

3.1.2. *Problem* (1.1), (1.2) *with* $\min\{\beta, \delta\} > 0$. Let assumptions (H_1) and (H_3) be satisfied and $\widehat{V}$ be the positive constant given in Lemma 2.4 with $\widehat{\phi}$ defined by (3.3). Consider the family of regular differential equations

$$-x^{(n)}(t) = f_{m,\widehat{V}}\big(t, x(t), \dots, x^{(n-1)}(t)\big) \tag{3.11}$$

depending on $m \in \mathbb{N}$.

Lemma 3.3. *Let assumptions* (H_1) *and* (H_3) *be satisfied. Then, for each* $m \in \mathbb{N}$*, BVP* (3.11), (1.2) *has a solution* $x_m \in \mathcal{A}(r, \varepsilon)$ *and*

$$\|x_m^{(i)}\| \le \widehat{V} \ \ \textit{for} \ \ 0 \le i \le n-1. \tag{3.12}$$

Proof. Fix $m \in \mathbb{N}$. To prove the existence of a solution x_m of BVP (3.11), (1.2) we can argue as in the proof of Lemma 3.1. The fact that $x_m \in \mathcal{A}(r, \varepsilon)$ and x_m satisfies (3.12) now follows from Lemmas 2.3 and 2.4. $\square$

Lemma 3.4. *Let assumptions* (H_1) *and* (H_3) *be satisfied and let* x_m *be a solution of BVP* (3.11), (1.2), $m \in \mathbb{N}$*. Then the sequence*

$$\big\{f_{m,\widehat{V}}\big(t, x_m(t), \dots, x_m^{(n-1)}(t)\big)\big\} \subset L_1(J)$$

is uniformly absolutely continuous on J*.*

Proof. By Lemma 2.3 the inequalities (3.7) are satisfied for each $m \in \mathbb{N}$, where ξ_m is the unique zero of $x^{(n-1)}$ and (for $m \in \mathbb{N}$)

$$x_m^{(n-2)}(t) \ge B \ \ \text{for} \ \ t \in J, \tag{3.13}$$

$$x_m^{(j)}(t) \ge \frac{B}{(n-j-2)!} t^{n-j-2} \ \ \text{for} \ \ t \in J, \ \ 0 \le j \le n-3, \tag{3.14}$$

where B is the positive constant defined by (2.21). Moreover, by Lemma 2.4,

$$\|x_m^{(j)}\| \le \widehat{V} \ \ \text{for} \ \ m \in \mathbb{N}, \ \ 0 \le j \le n-1. \tag{3.15}$$

From the inequalities (for a.e. $t \in J$ and each $m \in \mathbb{N}$)

$$0 \le f_{m,\widehat{V}}\big(t, x_m(t), \dots, x_m^{(n-1)}(t)\big) \le \widehat{\phi}(t) + \sum_{i=0}^{n-1} q_i(t)\omega_i\big(|x_m^{(i)}(t)|\big) + \sum_{i=0}^{n-1} h_i(t)\widehat{V}^{\alpha_i}$$

which follow from (3.2) and (3.15), from the properties of the function ϕ, q_j and h_j, $0 \le j \le n-1$, given in (H_3) and finally from (3.13), we see that to prove our lemma it suffices to verify that the sequences

$$\big\{\omega_{n-1}\big(|x_m^{(n-1)}(t)|\big)\big\} \tag{3.16}$$

and

$$\big\{\omega_j\big(|x_m^{(j)}(t)|\big)\big\}, \ \ 0 \le j \le n-3, \tag{3.17}$$

are uniformly absolutely continuous on J. The uniform absolute continuity on J of sequence (3.16) was proved in the proof of Lemma 3.2 and for sequences (3.17) this fact follows immediately from the inequalities (see (3.14)) $\omega_j(|x_m^{(j)}(t)|) \leq \omega_j(\frac{B}{(n-j-2)!}t^{n-j-2})$ for $t \in (0,T]$ and $m \in \mathbb{N}$ since (1.10) and Remark 1.4 imply that $\omega_j(\frac{B}{(n-j-2)!}t^{n-j-2}) \in L_1(J)$ for $0 \leq j \leq n-3$. □

3.2. **Problem (1.3), (1.4).** We present an existence principle for $(p, n-p)$ right focal BVPs which are regular. Assume that (H_4) and (H_5) are satisfied. Put

$$\phi^*(t) = \phi(t) + \sum_{i=0}^{n-1} q_i(t)\omega_i(1) + \sum_{i=0}^{n-1} h_i(t) \text{ for a.e. } t \in J.$$

Then $\phi^* \in L_1(J)$ and, by Lemma 2.6, a positive constant r^* satisfying (2.33) can be found. For $m \in \mathbb{N}$, $0 \leq i \leq n-1$, $x \in \mathbb{R}$, put

$$\rho_i = 1 + r^* T^{n-i-1} \tag{3.18}$$

and

$$\sigma_i\Big(\frac{1}{m}, x\Big) = \begin{cases} \dfrac{1}{m}\operatorname{sgn} x & \text{for } |x| < \dfrac{1}{m}, \\ x & \text{for } \dfrac{1}{m} \leq |x| \leq \rho_i, \\ \rho_i \operatorname{sgn} x & \text{for } \rho_i < |x|. \end{cases}$$

Extend f onto $J \times \mathbb{R}_0^n$ as an even function in each its phase variable x_i, $0 \leq i \leq n-1$, and for a.e. $t \in J$ and for all $(x_0, \dots, x_{n-1}) \in \mathbb{R}^n$ define auxiliary functions

$$f_m(t, x_0, \dots, x_{n-1}) = f\Big(t, \sigma_0\Big(\frac{1}{m}, x_0\Big), \dots, \sigma_{n-1}\Big(\frac{1}{m}, x_{n-1}\Big)\Big). \tag{3.19}$$

In such a way we get the family of differential equations

$$(-1)^{n-p} x^{(n)}(t) = f_m\big(t, x(t), \dots, x^{(n-1)}(t)\big) \tag{3.20}$$

depending on the parameter $m \in \mathbb{N}$. Now, we will study BVPs (3.20), (1.4).

Lemma 3.5. *Let assumptions* (H_4) *and* (H_5) *be satisfied, let* $\mathcal{B}(r,\varepsilon)$ *be given by* (2.27) *and* r^* *be from Lemma* 2.6. *Let* f_m, $m \in \mathbb{N}$, *be defined by* (3.19). *Then, for each* $m \in \mathbb{N}$, *BVP* (3.20), (1.4) *has a solution* $u_m \in \mathcal{B}(r,\varepsilon)$ *such that*

$$\|u_m^{n-1}\| < r^*. \tag{3.21}$$

Proof. Fix an arbitrary $m \in \mathbb{N}$. (H_4) and (3.19) yield $f_m \in Car(J \times \mathbb{R}^n)$. Now, put

$$g_m(t) = \sup\Big\{\big|f(t, x_0, \dots, x_{n-1})\big| : \frac{1}{m} \leq |x_i| \leq \rho_i,\ 0 \leq i \leq n-1\Big\},$$

where $\rho_i, 0 \leq i \leq n-1$, are given by (3.18). We see that $g_m \in L_1(J)$ and $|f_m(t, x_0, \dots, x_{n-1})| \leq g_m(t)$ for a.e. $t \in J$ and all $(x_0, \dots, x_{n-1}) \in \mathbb{R}^n$. Since the problem $(-1)^{n-p}x^{(n)}(t) = 0$, (1.4) has only a trivial solution, the Nonlinear Fredholm Alternative implies that (3.20), (1.4) has a solution u_m.

Further, by virtue of (H_4) and (H_5), we see that for a.e. $t \in J$ and all $(x_0, \dots, x_{n-1}) \in \mathbb{R}^n$ the inequalities

$$\varepsilon(T-t)^r \le f_m(t, x_0, \dots, x_{n-1}), \tag{3.22}$$

$$f_m(t, x_0, \dots, x_{n-1}) \le \phi^*(t) + \sum_{i=0}^{n-1} q_i(t)\omega_i(|x_i|) + \sum_{i=0}^{n-1} h_i(t)|x_i|^{\alpha_i} \tag{3.23}$$

are true. Note that inequality (3.23) follows from the relations

$$\left|\sigma_i\Big(\frac{1}{m}, x_i\Big)\right|^{\alpha_i} \le \Big(\frac{1}{m}\Big)^{\alpha_i} + |x_i|^{\alpha_i} \le 1 + |x_i|^{\alpha_i},$$

and

$$\omega_i\Big(\Big|\sigma_i\Big(\frac{1}{m}, x_i\Big)\Big|\Big) \le \omega_i(\rho_i) + \omega_i(|x_i|) \le \omega_i(1) + \omega_i(|x_i|)$$

which are valid for $0 \le i \le n-1$. In view of (3.22) we have $u_m \in \mathcal{B}(r, \varepsilon)$ and therefore using (3.23) and Lemma 2.6, we get (3.21). □

4. Main Results, Examples

4.1. Sturm–Liouville boundary value problems.

Theorem 4.1. *Let assumptions* (H_1) *and* (H_2) *be satisfied and let* $\min\{\beta, \delta\} = 0$ *in* (1.2). *Then BVP* (1.1), (1.2) *has a solution.*

Proof. By Lemma 3.1 there is a solution x_m of BVP (3.4), (1.2) for each $m \in \mathbb{N}$. Consider the sequence $\{x_m\}$. Then inequalities (3.7)–(3.10) are satisfied by Lemmas 2.1 and 3.1, where $\xi_m \in (0, T)$ denotes the unique zero of $x_m^{(n-1)}$, A is given by (2.5) and $\widehat{M}$ is a positive constant. Moreover,

$$\big\{f_{m,\widehat{M}}\big(t, x_m(t), \dots, x_m^{(n-1)}(t)\big)\big\}$$

is uniformly absolutely continuous on J by Lemma 3.2, which implies that $\{x_m^{(n-1)}(t)\}$ is equicontinuous on J. Without loss of generality we can assume that $\{x_m\}$ is convergent in $C^{n-1}(J)$ and $\{\xi_m\}$ is convergent in $\mathbb{R}$. Let $\lim\limits_{m\to\infty} x_m = x$, $\lim\limits_{m\to\infty} \xi_m = \xi$. Then x satisfies the boundary conditions (1.2) and from (3.7)–(3.9) we deduce that

$$\begin{aligned} x^{(n-1)}(t) &\ge \frac{\varepsilon}{r+1}(\xi - t)^{r+1} \ \text{ for } \ t \in [0, \xi], \\ x^{(n-1)}(t) &\le -\frac{\varepsilon}{r+1}(\xi - t)^{r+1} \ \text{ for } \ t \in (\xi, T], \\ x^{(n-2)}(t) &\ge \begin{cases} At & \text{for } \ t \in \Big[0, \dfrac{T}{2}\Big], \\ A(T-t) & \text{for } \ t \in \Big(\dfrac{T}{2}, T\Big] \end{cases} \end{aligned} \tag{4.1}$$

and

$$x^{(j)}(t) \ge \frac{A}{4(n-j-1)!} t^{n-j-1}, \ \ t \in J, \ \ 0 \le j \le n-3.$$

From the construction of the auxiliary functions $f_{m,\widehat{M}} \in Car(J \times \mathbb{R}^n)$ it follows that there exists of a set $\mathcal{U} \subset J$, $\mu(\mathcal{U}) = 0$, such that $f_{m,\widehat{M}}(t, \cdot, \ldots, \cdot)$ is continuous on $\mathbb{R}^n$ for $t \in J \setminus \mathcal{U}$ and $m \in \mathbb{N}$. Hence

$$\lim_{m\to\infty} f_{m,\widehat{M}}\big(t, x_m(t), \ldots, x_m^{(n-1)}(t)\big) = f\big(t, x(t), \ldots, x^{(n-1)}(t)\big)$$

for $t \in J \setminus (\mathcal{U} \cup \{0, \xi, T\})$. Now the Vitali's convergence theorem gives

$$f\big(t, x(t), \ldots, x^{(n-1)}(t)\big) \in L_1(J)$$

and

$$\lim_{m\to\infty} \int_0^t f_{m,\widehat{M}}\big(s, x_m(s), \ldots, x_m^{(n-1)}(s)\big)\, ds = \int_0^t f\big(s, x(s), \ldots, x^{(n-1)}(s)\big)\, ds, \ \ t \in J.$$

Taking the limit as $m \to \infty$ in the equalities

$$x_m^{(n-1)}(t) = x_m^{(n-1)}(0) + \int_0^t f_{m,\widehat{M}}(s, x_m(s), \ldots, x_m^{(n-1)}(s))\, ds, \ \ t \in J, \ \ m \in \mathbb{N},$$

we obtain

$$x^{(n-1)}(t) = x^{(n-1)}(0) + \int_0^t f\big(s, x(s), \ldots, x^{(n-1)}(s)\big)\, ds, \ \ t \in J.$$

Consequently, $x \in AC^{n-1}(J)$ and x satisfies (1.1) a.e. on J. We have proved that x is a solution of BVP (1.1), (1.2). $\square$

Theorem 4.2. *Let assumptions* (H_1) *and* (H_3) *be satisfied and let* $\min\{\beta, \delta\} > 0$ *in* (1.2). *Then BVP* (1.1), (1.2) *has a solution.*

Proof. By Lemma 3.3 there exists a solution x_m of BVP (3.11), (1.2) for each $m \in \mathbb{N}$. Consider the sequence $\{x_m\}$. Then inequalities (3.7) and (3.13)–(3.15) are satisfied for each $m \in \mathbb{N}$, which follows from Lemmas 2.3 and 2.4, where $\xi_m \in (0, T)$ is the unique zero of $x_m^{(n-1)}$ and $B, \widehat{V}$ are positive constants independent of x_m. Moreover, by Lemma 3.4, the sequence $\{f_{m,\widehat{V}}(t, x_m, \ldots, x_m^{(n-1)}(t))\}$ is uniformly absolutely continuous on J, which yields that $\{x_m^{(n-1)}(t)\}$ is equicontinuous on J. Without loss of generality we can assume that $\{x_m\}$ and $\{\xi_m\}$ are convergent in $C^{n-1}(J)$ and $\mathbb{R}$, respectively. Let $\lim\limits_{m\to\infty} x_m = x$, $\lim\limits_{m\to\infty} \xi_m = \xi$. Then x satisfies the boundary conditions (1.2) and from (3.7), (3.13) and (3.14) we conclude that (4.1) holds and

$$x^{(n-2)}(t) \geq B, \ \ t \in J,$$
$$x^{(j)}(t) \geq \frac{B}{(n-j-2)!}\, t^{n-j-2}, \ \ t \in J, \ \ 0 \leq j \leq n-3.$$

The next part of the proof is the same as that of the proof of Theorem 4.1 and therefore it is omitted. $\square$

4.2. **Focal boundary value problems.** First, we consider the singular $(p, n-p)$ right focal BVP (1.3), (1.4) with $1 \le p \le n-1$.

Theorem 4.3. *Let assumptions* (H_4) *and* (H_5) *be satisfied. Then there exists a solution of BVP* (1.3), (1.4).

Proof. Define for $m \in \mathbb{N}$ functions f_m by (3.19). According to Lemma 3.5 there is a positive number r^* such that for each $m \in \mathbb{N}$ problem (3.20), (1.4) has a solution $u_m \in \mathcal{B}(r, \varepsilon)$ satisfying (3.21). By Lemma 2.5 there exists $c > 0$ such that for $m \in \mathbb{N}$ and $t \in J$ we have

$$\begin{aligned} u_m^{(i)}(t) &\ge ct^{r+n-i} \ \text{ for } \ 0 \le i \le p-1, \\ (-1)^{i-p} u_m^{(i)}(t) &\ge c(T-t)^{r+n-i} \ \text{ for } \ p \le i \le n-1. \end{aligned} \tag{4.2}$$

Conditions (1.4) and (3.21) yield

$$\|u_m^{(i)}\| < r^* T^{n-i-1} < \rho_i, \quad 0 \le i \le n-1. \tag{4.3}$$

Moreover, by virtue of (3.23) we have for $0 \le t_1 \le t_2 \le T$

$$\left|u_m^{(n-1)}(t_2) - u_m^{(n-1)}(t_1)\right| \le \int_{t_1}^{t_2} h(t)\,dt + \sum_{i=0}^{n-1} \|q_i\|_\infty \int_{t_1}^{t_2} \omega_i(|u_m^{(i)}(t)|)\,dt, \tag{4.4}$$

where

$$h(t) = \phi^*(t) + \sum_{i=0}^{n-1} \rho_i^{\alpha_i} h_i(t), \quad h \in L_1(J).$$

Since $u_m \in \mathcal{B}(r, \varepsilon)$, we can use Lemma 2.7 and conclude that the sequence $\{u_m^{(n-1)}\}$ is equicontinuous on J. Estimates (4.3) mean that $\{u_m\}$ is bounded in $C^{n-1}(J)$. Thus, by the Arzelà–Ascoli theorem, we can choose a subsequence, which is denoted by $\{u_k\}$ and which converges in $C^{n-1}(J)$ to a function $u \in C^{n-1}(J)$. Clearly, u satisfies (1.4). Letting $k \to \infty$ and using (4.2), we get

$$\begin{aligned} u^{(i)}(t) &\ge ct^{r+n-i} \ \text{ for } \ 0 \le i \le p-1, \\ (-1)^{i-p} u^{(i)}(t) &\ge c(T-t)^{r+n-i} \ \text{ for } \ p \le i \le n-1. \end{aligned}$$

This yields that

$$\begin{aligned} u^{(i)}(t) &> 0 \ \text{ on } \ (0, T] \ \text{ for } \ 0 \le i \le p-1, \\ (-1)^{i-p} u^{(i)}(t) &> 0 \ \ mboxon \ \ [0, T) \ \text{ for } \ p \le i \le n-1. \end{aligned} \tag{4.5}$$

Finally, let us show that $u \in AC^{n-1}(J)$ and that u fulfils (1.3) a.e. on J. Consider the sequence of equalities

$$u_k^{(n-1)}(t) = u_k^{(n-1)}(0) + \int_0^t f_k\big(s, u_k(s), \dots, u_k^{(n-1)}(s)\big)\,ds \ \text{ for } \ t \in J. \tag{4.6}$$

Denote the set of all $t \in J$ such that $f(t, \cdot, \dots, \cdot) : X \to \mathbb{R}$ is not continuous by $\mathcal{U}$. Then $\mu(\mathcal{U}) = 0$ and, by virtue of (4.5),

$$\lim_{k\to\infty} f_k\big(t, u_k(t), \dots, u_k^{(n-1)}(t)\big) = f\big(t, u(t), \dots, u^{(n-1)}(t)\big)$$

for all $t \in J \setminus (\mathcal{U} \cup \{0, T\})$. Further, using (H_4), (H_5) and (4.2), we see that for all $k \in \mathbb{N}$ and a.e. $t \in J$, the inequalities

$$\left| f_k(t, u_k(t), \ldots, u_k^{(n-1)}(t)) \right| \le g(t)$$

are satisfied, where

$$g(t) = \phi^*(t) + \sum_{i=0}^{n-1} \rho_i^{\alpha_i} h_i(t) + \sum_{i=0}^{n-1} \|q_i\|_\infty \omega_i^*(t)$$

and

$$\omega_i^*(t) = \begin{cases} \omega_i(ct^{r+n-i}) & \text{for } 0 \le i \le p-1, \\ \omega_i(c(T-t)^{r+n-i}) & \text{for } p \le i \le n-1. \end{cases}$$

Since $g \in L_1(J)$, we can use the Lebesgue dominated convergence theorem by which $f(t, u(t), \ldots, u^{(n-1)}(t)) \in L_1(J)$ and letting $k \to \infty$ in (4.6) we have that

$$u^{(n-1)}(t) = u^{(n-1)}(0) + \int_0^t f\big(s, u(s), \ldots, u^{(n-1)}(s)\big)\, ds \quad \text{for } t \in J$$

is valid, i.e., $u \in AC^{n-1}(J)$ and u satisfies (1.3) a.e. on J. $\square$

Theorem 4.3 together with Remark 1.5 yields the existence result for the singular $(n-p, p)$ left focal BVP (1.12), (1.13) with $1 \le p \le n-1$.

Theorem 4.4. *Let assumptions* (H_6) *and* (H_7) *be satisfied. Then there exists a solution of BVP* (1.12), (1.13).

For the continuous function f in equations (1.1) and (1.3) we get immediately from Theorems 4.1 and 4.3 and our previous considerations the following corollaries. Similar results could be obtained from Theorems 4.2 and 4.4.

Corollary 4.5. *Let* $f \in C^0(J \times D)$ *satisfy assumptions* (H_1) *and* (H_2) *and let* $\min\{\beta, \delta\} = 0$ *in* (1.2). *Then there exists a solution* x *of BVP* (1.1), (1.2) *such that* $x \in AC^{n-1}(J) \cap C^n(J \setminus \{0, T, \xi\})$ *and* (1.1) *holds for each* $t \in J \setminus \{0, T, \xi\}$ *where* $\xi \in (0, T)$ *is the unique zero of* $x^{(n-1)}$ *in* J.

Corollary 4.6. *Let* $f \in C^0(J \times X)$ *satisfy assumptions* (H_4) *and* (H_5). *Then BVP* (1.3), (1.4) *has a solution* x *such that* $x \in AC^{n-1}(J) \cap C^n(J \setminus \{0, T\})$ *and* (1.3) *holds for each* $t \in J \setminus \{0, T\}$.

Example 4.7. Let $n \in \mathbb{N}$, $n \ge 2$, γ, $\alpha_i \in (0, 1)$, $h_i \in L_1(J)$ be nonnegative and β_i, $c_i \in \mathbb{R}_+$, $0 \le i \le n-1$. Consider the differential equation

$$-x^{(n)}(t) = \Big(\frac{t}{T-t}\Big)^\gamma + \sum_{i=0}^{n-1} \frac{c_i}{|x^{(i)}(t)|^{\beta_i}} + \sum_{i=0}^{n-1} h_i(t)|x^{(i)}(t)|^{\alpha_i}. \tag{4.7}$$

If we set $r = \gamma$, $\varepsilon = T^{-\gamma}$ and $\omega_i(z) = z^{-\beta_i}$, $0 \le i \le n-1$, then assumptions (H_1) and (H_2) are satisfied for

$$\beta_{n-1} \in \Big(0, \frac{1}{1+\gamma}\Big), \quad \beta_i \in \Big(0, \frac{1}{n-i-1}\Big), \quad 0 \le i \le n-2, \tag{4.8}$$

and assumptions (H_1) and (H_3) are satisfied for

$$\beta_{n-1} \in \Big(0, \frac{1}{1+\gamma}\Big), \quad \beta_{n-2} \in \mathbb{R}_+, \quad \beta_j \in \Big(0, \frac{1}{n-j-2}\Big), \quad 0 \le j \le n-3. \tag{4.9}$$

Hence problem $(4.7), (1.2)$ with $\min\{\beta, \delta\} = 0$ has a solution for β_i satisfying (4.8) by Theorem 4.1 and the solvability of problem $(4.7), (1.2)$ with $\min\{\beta, \delta\} > 0$ for β_i satisfying (4.9) is guaranteed by Theorem 4.2.

Example 4.8. Let $n \in \mathbb{N}$, $n \ge 2$ and $1 \le p \le n-1$. Consider the differential equation

$$(-1)^{n-p} x^{(n)}(t) = \Big(\frac{T-t}{t}\Big)^{\gamma} + \sum_{i=0}^{n-1} \frac{c_i}{|x^{(i)}(t)|^{\beta_i}} + \sum_{i=0}^{n-1} h_i(t)|x^{(i)}(t)|^{\alpha_i}, \tag{4.10}$$

where (for $0 \le i \le n-1$)

$$\begin{gathered} \gamma,\ \alpha_i \in (0,1), \quad c_i \in \mathbb{R}_+, \quad \beta_i \in \Big(0, \frac{1}{n+\gamma-i}\Big), \\ h_i \in L_1(J) \ \text{ is nonnegative.} \end{gathered} \tag{4.11}$$

Then we can see that if we put $r = \gamma$, $\varepsilon = T^{-\gamma}$ and $\omega_i(z) = z^{-\beta_i}$, $0 \le i \le n-1$, assumptions (H_4) and (H_5) are satisfied. Hence, by Theorem 4.3, problem $(4.10), (1.4)$ has a solution.

Similarly, having the differential equation

$$(-1)^{p} x^{(n)}(s) = \Big(\frac{s}{T-s}\Big)^{\gamma} + \sum_{i=0}^{n-1} \frac{c_i}{|x^{(i)}(s)|^{\beta_i}} + \sum_{i=0}^{n-1} h_i(s)|x^{(i)}(s)|^{\alpha_i} \tag{4.12}$$

and assuming (4.11), we can easily check that (H_6) and (H_7) are fulfilled, which yields, by Theorem 4.4, that problem $(4.12), (1.4)$ is solvable.

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