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INTEGRAL REPRESENTATIONS FOR THE SOLUTION OF DYNAMIC BENDING OF A PLATE WITH DISPLACEMENT-TRACTION BOUNDARY DATA

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Abstract. The existence of distributional solutions is investigated for the time-dependent bending of a plate with transverse shear deformation under mixed boundary conditions. The problem is then reduced to nonstationary boundary integral equations and the existence and uniqueness of solutions to the latter are studied in appropriate Sobolev spaces.

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1. Introduction

Mathematical models play an essential role in the study of mechanical phenomena and processes. Owing to their complexity, solutions of initial-boundary value problems for such models are usually approximated by means of various numerical procedures. But before methods of this type can be developed and applied, the practitioner must have a guarantee that the problem is well posed; in other words, that it has a unique solution depending continuously on the data.

In this paper we address the questions of existence, uniqueness, and stability of the solution to the time-dependent bending of a plate with transverse shear deformation where displacements are prescribed on one part of the boundary and the bending and twisting moments and shear force are given on the remaining part. The analysis is performed in Sobolev spaces, whose norms are particularly well suited to the construction of global error estimates.

The corresponding equilibrium problem has been fully investigated in [1]. A discussion of other mathematical models in elasticity can be found in [2]–[4].

2. Formulation of the Problem

Consider a homogeneous and isotropic elastic plate of thickness $h_0 = \text{const} > 0$, which occupies a domain $\bar{S} \times [-h_0/2, h_0/2]$ in $\mathbb{R}^3$, where S is a domain in $\mathbb{R}^2$ with a simple closed boundary ∂S. In the transverse shear deformation model proposed in [5] it is assumed that the displacement vector at (x, x_3), $x = (x_1, x_2) \in \mathbb{R}^2$, at time $t \geq 0$, is of the form

$$\big(x_3 u_1(x,t), x_3 u_2(x,t), u_3(x,t)\big)^{\mathrm{T}},$$

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where the superscript T denotes matrix transposition. Then $u = (u_1, u_2, u_3)^{\mathrm{T}}$ satisfies the equation of motion

$$B(\partial_t^2 u)(x,t) + (Au)(x,t) = q(x,t), \quad (x,t) \in G = S \times (0,\infty);$$

here $B = \operatorname{diag}\{\rho h^2, \rho h^2, \rho\}$, $h^2 = h_0^2/12$, ρ is the density of the material, $\partial_t = \partial/\partial t$,

$$A = \begin{pmatrix} -h^2\mu\Delta - h^2(\lambda+\mu)\partial_1^2 + \mu & -h^2(\lambda+\mu)\partial_1\partial_2 & \mu\partial_1 \\ -h^2(\lambda+\mu)\partial_1\partial_2 & -h^2\mu\Delta - h^2(\lambda+\mu)\partial_2^2 + \mu & \mu\partial_2 \\ -\mu\partial_1 & -\mu\partial_2 & -\mu\Delta \end{pmatrix},$$

$\partial_\alpha = \partial/\partial x_\alpha$, $\alpha = 1, 2$, λ and μ are the Lamé constants satisfying $\lambda + \mu > 0$ and $\mu > 0$, and q is a combination of forces and moments acting on the plate and its faces $x_3 = \pm h_0/2$.

In what follows we work with three-component distributions; however, for simplicity, we use the same symbols for their spaces and norms as in the scalar case.

We denote by $H_{m,p}(\mathbb{R}^2)$, $m \in \mathbb{R}$, $p \in \mathbb{C}$, the space that coincides with $H_m(\mathbb{R}^2)$ as a set but is endowed with the norm

$$\|u\|_{m,p} = \left\{ \int\limits_{\mathbb{R}^2} (1 + |p|^2 + |\xi|^2)^m |\tilde{u}(\xi)|^2 \, d\xi \right\}^{1/2},$$

where $\tilde{u}$ is the distributional Fourier transform of $u \in \mathcal{S}'(\mathbb{R}^2)$. Next, $\mathring{H}_{m,p}(S)$ is the subspace of $H_{m,p}(\mathbb{R}^2)$ consisting of all $u \in H_{m,p}(\mathbb{R}^2)$ with $\operatorname{supp} u \subset \bar{S}$, and $H_{m,p}(S)$ is the space of the restrictions to S of all $v \in H_{m,p}(\mathbb{R}^2)$. The norm of $u \in H_{m,p}(S)$ is defined by

$$\|u\|_{m,p;S} = \inf_{v \in H_{m,p}(\mathbb{R}^2):\, v|_S = u} \|v\|_{m,p}.$$

Also, $H_{-m,p}(\mathbb{R}^2)$ is the dual of $H_{m,p}(\mathbb{R}^2)$ with respect to the duality generated by the inner product $(\cdot\,,\cdot)_0$ in $L^2(\mathbb{R}^2)$; the dual of $\mathring{H}_{m,p}(S)$ is $H_{-m,p}(S)$. Let γ be the trace operator that maps $H_{1,p}(S)$ continuously to the space $H_{1/2,p}(\partial S)$, which coincides as a set with $H_{1/2}(\partial S)$ but is endowed with the norm

$$\|f\|_{1/2,p;\partial S} = \inf_{u \in H_{1,p}(S):\, \gamma u = f} \|u\|_{1,p;S}.$$

The continuity of γ from $H_{1,p}(S)$ to $H_{1/2,p}(\partial S)$ is uniform with respect to $p \in \mathbb{C}$. Finally, $H_{-1/2,p}(\partial S)$ is the dual of $H_{1/2,p}(\partial S)$ with respect to the duality generated by the inner product $(\cdot\,,\cdot)_{0;\partial S}$ in $L^2(\partial S)$.

We fix $\kappa > 0$ and introduce the complex half-plane $\mathbb{C}_\kappa = \{p = \sigma + i\tau \in \mathbb{C} : \sigma > \kappa\}$. Consider the space $H^{\mathcal{L}}_{m,k,\kappa}(S)$, m, $k \in \mathbb{R}$, of all $\widehat{u}(x,p)$, $x \in S$, $p \in \mathbb{C}_\kappa$, such that $U(p) = \widehat{u}(\cdot\,,p)$ is a holomorphic mapping from $\mathbb{C}_\kappa$ to $H_m(S)$ (which implies that $U(p)$ also belongs to $H_{m,p}(S)$ for every $p \in \mathbb{C}_\kappa$) and for which

$$\|\widehat{u}\|^2_{m,k,\kappa;S} = \sup_{\sigma > \kappa} \int\limits_{-\infty}^{\infty} (1 + |p|^2)^k \|U(p)\|^2_{m,p;S} \, d\tau < \infty.$$

The norm on $H^{\mathcal{L}}_{m,k,\kappa}(S)$ is defined by this equality. In what follows, we use the symbol $\widehat{u}(x,p)$ when we want to emphasize that this is a distribution in $H_{m,p}(S)$, and the symbol $U(p)$ when we need to regard it as a mapping from $\mathbb{C}_\kappa$ to $H_m(S)$. The spaces $H^{\mathcal{L}}_{\pm 1/2,k,\kappa}(\partial S)$ and its norm $\|\cdot\|_{\pm 1/2,k,\kappa;\partial S}$ are introduced similarly.

Let $H^{\mathcal{L}^{-1}}_{m,k,\kappa}(G)$ and $H^{\mathcal{L}^{-1}}_{\pm 1/2,k,\kappa}(\Gamma)$, $\Gamma = \partial S \times (0,\infty)$, be the spaces of the inverse Laplace transforms u and f of all $\widehat{u} \in H^{\mathcal{L}}_{m,k,\kappa}(S)$ and $\widehat{f} \in H^{\mathcal{L}}_{\pm 1/2,k,\kappa}(\partial S)$, with norms

$$\|u\|_{m,k,\kappa;G} = \|\widehat{u}\|_{m,k,\kappa;S}, \quad \|f\|_{\pm 1/2,k,\kappa;\partial G} = \|\widehat{f}\|_{\pm 1/2,k,\kappa;\partial S}.$$

We assume that ∂S is a C^2-curve consisting of two arcs ∂S_ν, $\nu = 1,2$, such that $\partial S = \overline{\partial S_1} \cup \overline{\partial S_2}$, $\partial S_1 \cap \partial S_2 = \varnothing$, and $\operatorname{mes} \partial S_\nu > 0$, $\nu = 1,2$. Let S^+ and S^- be the interior and exterior domains into which ∂S divides $\mathbb{R}^2$, and let $G^\pm = S^\pm \times (0,\infty)$ and $\Gamma_\nu = \partial S_\nu \times (0,\infty)$, $\nu = 1,2$. We denote by $\gamma^\pm$ the trace operators corresponding to $S^\pm$. For simplicity, we use the same symbols for the trace operators in the spaces of originals and in those of their Laplace transforms. Thus, $\gamma^\pm$ also denote the trace operators mapping $H^{\mathcal{L}^{-1}}_{1,k,\kappa}(G^\pm)$ continuously to $H^{\mathcal{L}^{-1}}_{1/2,k,\kappa}(\Gamma)$ for any $k \in \mathbb{R}$. Also, we denote by π_ν, $\nu = 1,2$, the operators of restriction from Γ to Γ_ν (and from ∂S to ∂S_ν), and write $\gamma^\pm_\nu = \pi_\nu \gamma^\pm$, $\nu = 1,2$. Finally, let $\pi^\pm$ be the operators of restriction from $\mathbb{R}^2 \times (0,\infty)$ to $G^\pm$ (or from $\mathbb{R}^2$ to $S^\pm$).

We introduce the subspace $\mathring{H}_{1/2,p}(\partial S_\nu)$ of $H_{1/2,p}(\partial S)$ consisting of all $\varphi \in H_{1/2,p}(\partial S)$ such that $\operatorname{supp} \varphi \in \overline{\partial S_\nu}$, $\nu = 1,2$, and denote by $H_{1/2,p}(\partial S_\nu)$ the space of restrictions from ∂S to ∂S_ν of the elements of $H_{1/2,p}(\partial S)$. The norm of $\varphi \in H_{1/2,p}(\partial S_\nu)$ is defined by

$$\|\varphi\|_{1/2,p;\partial S_\nu} = \inf_{f \in H_{1/2,p}(\partial S):\pi_\nu f = \varphi} \|f\|_{1/2,p;\partial S}, \quad \nu = 1,2.$$

Let l_ν, $\nu = 1,2$, be extension operators from ∂S_ν to ∂S which map $H_{1/2,p}(\partial S_\nu)$ to $H_{1/2,p}(\partial S)$ continuously and uniformly with respect to p and satisfy

$$\|l_\nu f\|_{1/2,p;\partial S} \le c\|f\|_{1/2,p;\partial S_\nu} \quad \forall f \in H_{1/2,p}(\partial S_\nu).$$

Also, let $l^\pm$ be operators of extension from ∂S to $S^\pm$ which map $H_{1/2,p}(\partial S)$ to $H_{1,p}(S^\pm)$ continuously and uniformly with respect to p.

By $\mathring{H}_{-1/2,p}(\partial S_\nu)$ and $H_{-1/2,p}(\partial S_\nu)$, $\nu = 1,2$, we denote the duals of $H_{1/2,p}(\partial S_\nu)$ and $\mathring{H}_{1/2,p}(\partial S_\nu)$, respectively, with respect to the duality generated by the inner product in $[L^2(\partial S_\nu)]^3$; their norms are $\|\cdot\|_{-1/2,p;\partial S}$ and $\|\cdot\|_{-1/2,p;\partial S_\nu}$. The corresponding spaces $H^{\mathcal{L}^{-1}}_{\pm 1/2,k,\kappa}(\Gamma_\nu)$ and $\mathring{H}^{\mathcal{L}^{-1}}_{\pm 1/2,k,\kappa}(\Gamma_\nu)$, $\nu = 1,2$, and their norms $\|\cdot\|_{\pm 1/2,k,\kappa,\Gamma_\nu}$ and $\|\cdot\|_{\pm 1/2,k,\kappa,\Gamma}$, $k \in \mathbb{R}$, $\kappa > 0$, are introduced in the usual way.

In what follows we denote by

$$a_\pm(u,v) = 2\int\limits_{S^\pm} E(u,v)\,dx$$

the sesquilinear form of the internal energy density, where

$$\begin{aligned}2E(u,v) &= h^2E_0(u,v) + h^2\mu(\partial_2 u_1 + \partial_1 u_2)(\partial_2\bar{v}_1 + \partial_1\bar{v}_2)\\ &\qquad + \mu[(u_1 + \partial_1 u_3)(\bar{v}_1 + \partial_1\bar{v}_3) + (u_2 + \partial_2 u_3)(\bar{v}_2 + \partial_2\bar{v}_3)],\\ E_0(u,v) &= (\lambda + 2\mu)\big[(\partial_1 u_1)(\partial_1\bar{v}_1) + (\partial_2 u_2)(\partial_2\bar{v}_2)\big]\\ &\qquad + \lambda\big[(\partial_1 u_1)(\partial_2\bar{v}_2) + (\partial_2 u_2)(\partial_1\bar{v}_1)\big].\end{aligned}$$

The classical formulation of the dynamic mixed problems (DM$^\pm$) consists in finding $u \in C^2(G^\pm) \cap C^1(\bar{G}^\pm)$ that satisfies

$$\begin{gathered}B(\partial_t^2 u)(x,t) + (Au)(x,t) = 0, \quad (x,t) \in G^+ \text{ or } G^-,\\ u(x,0+) = (\partial_t u)(x,0+) = 0, \quad x \in S^+ \text{ or } S^-,\\ u^\pm(x,t) = f_1(x,t), \quad (x,t) \in \Gamma_1, \quad (Tu)^\pm(x,t) = g_2(x,t), \quad (x,t) \in \Gamma_2,\end{gathered}$$

where T is the moment-force boundary operator defined by

$$\begin{pmatrix} h^2(\lambda + 2\mu)n_1\partial_1 + h^2\mu n_2\partial_2 & h^2\mu n_2\partial_1 + h^2\lambda n_1\partial_2 & 0\\ h^2\lambda n_2\partial_1 + h^2\mu n_1\partial_2 & h^2\mu n_1\partial_1 + h^2(\lambda + 2\mu)n_2\partial_2 & 0\\ \mu n_1 & \mu n_2 & \mu(n_1\partial_1 + n_2\partial_2)\end{pmatrix},$$

$n = (n_1, n_2)$ is the outward unit normal to ∂S, the superscripts $\pm$ denote the limiting values of the corresponding functions as $(x,t) \to \Gamma$ from inside $G^\pm$ (or $x \to \partial S$ from inside $S^\pm$), and f_1 and g_2 are given functions.

We call $u \in H_{1,0,\kappa}^{\mathcal{L}^{-1}}(G^\pm)$ a weak solution of the corresponding problem (DM$^\pm$) if it satisfies

$$\begin{gathered}\int_0^\infty \{a_\pm(u,v) - (B^{1/2}\partial_t u, B^{1/2}\partial_t v)_{0;S^\pm}\}\,dt = \pm\int_0^\infty (g_2, v)_{0;\partial S_2}\,dt,\\ \gamma_1^\pm u = f_1\end{gathered} \tag{1}$$

for all $v \in C_0^\infty(\bar{G}^\pm)$ such that $\gamma_1^\pm v = 0$.

3. Solvability of the Problems

In what follows we use the same symbol c for all positive constants that occur in various estimates and are independent of the functions in those estimates and of $p \in \mathbb{C}_\kappa$ (but may depend on κ).

Theorem 1. *For every $\kappa > 0$, $f_1 \in H_{1/2,1,\kappa}^{\mathcal{L}^{-1}}(\Gamma_1)$, and $g_2 \in H_{-1/2,1,\kappa}^{\mathcal{L}^{-1}}(\Gamma_2)$, problems* (1) *have a unique solution $u \in H_{1,0,\kappa}^{\mathcal{L}^{-1}}(G^\pm)$. Furthermore, if $f_1 \in H_{1/2,k,\kappa}^{\mathcal{L}}(\Gamma_1)$ and $g_2 \in H_{-1/2,k,\kappa}^{\mathcal{L}}(\Gamma_2)$, then $u \in H_{1,k-1,\kappa}^{\mathcal{L}}(G^\pm)$ and for every $k \in \mathbb{R}$*

$$\|u\|_{1,k-1,\kappa;G^\pm} \le c\big(\|f_1\|_{1/2,k,\kappa;\Gamma_1} + \|g_2\|_{-1/2,k,\kappa;\Gamma_2}\big).$$

Proof. We prove the assertion for (DM$^+$); the case of (DM$^-$) is treated similarly. We begin by rewriting (1) in terms of Laplace transforms.

Let $\mathring{H}_{1,p}(S^+, \partial S_\nu)$ be the subspace of $H_{1,p}(S^+)$ of all elements u such that $\pi_{3-\nu}\gamma^+ u = 0$, $\nu = 1,2$; that is, $\gamma^+ u \in \mathring{H}_{1/2,p}(\partial S_\nu)$. Going over to Laplace transforms, (1) becomes the problem (DM_p^+) of seeking $u \in H_{1,p}(S^+)$ such that for every $p \in \mathbb{C}_\kappa$

$$\begin{gathered} p^2(B^{1/2}u,\, B^{1/2}v)_{0;S^+} + a_+(u,v) = (g_2, v)_{0;\partial S_2} \quad \forall v \in \mathring{H}_{1,p}(S^+, \partial S_2), \\ \pi_1\gamma^+ u = f_1. \end{gathered} \tag{2}$$

Since for any $v \in \mathring{H}_{1,p}(S^+, \partial S_2)$

$$\big|(g_2, v)_{0;\partial S_2}\big| \le \|g_2\|_{-1/2,p;\partial S_2}\|v\|_{1/2,p;\partial S} \le c\|g_2\|_{-1/2,p;\partial S_2}\|v\|_{1,p;S^+},$$

$(g_2, v)_{0;\partial S_2}$ defines an antilinear functional on $\mathring{H}_{1,p}(S^+, \partial S_2)$; hence, it can be written as

$$(g_2, v)_{0;\partial S_2} = (q_2, v)_{0;S^+} \quad \forall v \in H_{1,p}(S^+, \partial S_2), \tag{3}$$

where $q_2 \in [\mathring{H}_{1,p}(S^+, \partial S_2)]^*$ and

$$\|q_2\|_{[\mathring{H}_{1,p}(S^+,\partial S_2)]^*} \le c\|g_2\|_{-1/2,p;\partial S_2}. \tag{4}$$

First, we assume that $f_1 = 0$. Taking (3) into account, we write (2) in the form

$$p^2(B^{1/2}u_0, B^{1/2}v)_{0;S^+} + a_+(u_0, v) = (q_2, v)_{0;S^+} \quad \forall v \in \mathring{H}_{1,p}(S^+, \partial S_2), \tag{5}$$

where $u_0 \in \mathring{H}_{1,p}(S^+, \partial S_2)$ is an unknown vector function. The unique solvability of (5) and the estimate

$$\|u_0\|_{1,p;S^+} \le c|p|\ \|q_2\|_{[\mathring{H}_{1,p}(S^+,\partial S_2)]^*} \tag{6}$$

are proved in the usual way (see, for example, [6]).

If $f_1 \ne 0$, then we set $w = l^+ l_1 f_1 \in H_{1,p}(S^+)$ and remark that

$$\|w\|_{1,p;S^+} \le c\|f_1\|_{1/2,p;\partial S_1}. \tag{7}$$

We seek the solution u of (2) as $u = u_0 + w$. Clearly, $u_0 \in \mathring{H}_{1,p}(S^+, \partial S_2)$ satisfies

$$\begin{aligned} &p^2(B^{1/2}u_0, B^{1/2}v)_{0;S^+} + a_+(u_0, v) \\ &\qquad = (q_2, v)_{0;S^+} - p^2(B^{1/2}w,\, B^{1/2}v)_{0;S^+} - a_+(w, v) \\ &\qquad\qquad \forall v \in \mathring{H}_{1,p}(S^+, \partial S_2). \end{aligned} \tag{8}$$

By (7),

$$\begin{aligned} |p^2(B^{1/2}w,\, B^{1/2}v)_{0,S^+} + a_+(w,v)| &\le c\|w\|_{1,p;S^+}\|v\|_{1,p;S^+} \\ &\le c\|f_1\|_{1/2,p;\partial S_1}\|v\|_{1,p;S^+}; \end{aligned}$$

hence,

$$p^2(B^{1/2}w,\, B^{1/2}v)_{0;S^+} + a_+(w,v) = (q_1, v)_{0;S^+} \quad \forall v \in \mathring{H}_{1,p}(S^+, \partial S_2),$$

where $q_1 \in [\mathring{H}_{1,p}(S^+, \partial S_2)]^*$ and

$$\|q_1\|_{[\mathring{H}_{1,p}(S^+,\partial S_2)]^*} \le c\|f_1\|_{1/2,p;\partial S_1}. \tag{9}$$

Equation (8) now takes the form

$$p^2(B^{1/2}u_0,\, B^{1/2}v)_{0;S^+} + a_+(u_0, v) = (q_2 - q_1, v)_{0;S^+} \quad \forall v \in \mathring{H}_{1,p}(S^+, \partial S_2). \tag{10}$$

We have already stated that (10) has a unique solution $u_0 \in \mathring{H}_{1,p}(S^+, \partial S_2)$. By (6), (4), and (9),

$$\begin{aligned}\|u_0\|_{1,p;S^+} &\le c|p|\|q_1 - q_2\|_{[\mathring{H}_{1,p}(S^+,\partial S_2)]^*} \\ &\le c|p|\big(\|f_1\|_{1/2,p;\partial S_1} + \|g_2\|_{-1/2,p;\partial S_2}\big).\end{aligned}$$

Therefore, for $u = u_0 + w$ we have

$$\|u\|_{1,p;S^+} \le c|p|\big(\|f_1\|_{1/2,p;\partial S_1} + \|g_2\|_{-1/2,p;\partial S_2}\big). \tag{11}$$

Returning to the spaces of originals and repeating the scheme used in [6] and [7], we complete the proof of the assertion. □

4. Dynamic Plate Potentials and their Properties

Consider a matrix $D(x,t)$ of fundamental solutions for the equation of motion, that is, a (3×3)-matrix such that

$$\begin{gathered}B(\partial_t^2 D)(x,t) + AD(x,t) = \delta(x,t)I, \quad (x,t)\in\mathbb{R}^2\times\mathbb{R},\\ D(x,t) = 0, \quad (x,t)\in\mathbb{R}^2\times(-\infty,0),\end{gathered}$$

where δ is the Dirac delta distribution and I is the identity (3×3)-matrix. Clearly, its Laplace transform $\widehat{D}(x,p)$ satisfies

$$Bp^2\widehat{D}(x,p) + A\widehat{D}(x,p) = \delta(x)I, \quad x\in\mathbb{R}^2.$$

The explicit form of $\widehat{D}(x,p)$ can be found in [7].

Let α, $\beta \in C^2(\partial S\times\mathbb{R})$ be functions with compact support in $\bar{\Gamma}$, and let $\widehat{\alpha}$ and $\widehat{\beta}$ be their Laplace transforms. We define the single-layer and double-layer potentials

$$\begin{aligned}(V_p\widehat{\alpha})(x,p) &= \int\limits_{\partial S}\widehat{D}(x-y,p)\widehat{\alpha}(y,p)\,ds_y, \quad x\in\mathbb{R}^2,\ p\in\mathbb{C}_0,\\ (W_p\widehat{\beta})(x,p) &= \int\limits_{\partial S}\big(T_y\widehat{D}(y-x,p)\big)^{\mathrm{T}}\widehat{\beta}(y,p)\,ds_y, \quad x\in S^+\cup S^-,\ p\in\mathbb{C}_0,\end{aligned}$$

where T_y is the boundary operator T acting with respect to y.

Since $\widehat{D}(x,p)$ has a polynomial growth with respect to $p\in\mathbb{C}_\kappa$, $\kappa>0$, we may now define the dynamic (retarded) single-layer and double-layer potentials

$$\begin{aligned}(V\alpha)(x,t) &= (\mathcal{L}^{-1}V_p\widehat{\alpha})(x,t)\\ &= \int\limits_0^\infty\int\limits_{\partial S} D(x-y,t-\tau)\alpha(y,\tau)\,ds_y\,d\tau, \quad (x,t)\in\mathbb{R}^2\times(0,\infty),\\ (W\beta)(x,t) &= (\mathcal{L}^{-1}W_p\widehat{\beta})(x,t)\\ &= \int\limits_0^\infty\int\limits_{\partial S}\big(T_yD(y-x,t-\tau)\big)^{\mathrm{T}}\beta(y,\tau)\,ds_y\,d\tau, \quad (x,t)\in G^+\cup G^-.\end{aligned}$$

We consider the boundary operators V_0, $W^{\pm}$, and N defined by

$$V_0\alpha = \gamma^{\pm}\pi^{\pm}V\alpha, \quad W^{\pm}\beta = \gamma^{\pm}\pi^{\pm}W\beta, \quad N\beta = T^{\pm}\pi^{\pm}W\beta.$$

Their corresponding transformed versions are defined by

$$V_{p,0}\widehat{\alpha} = \gamma^{\pm}\pi^{\pm}V_p\widehat{\alpha}, \quad W_p^{\pm}\widehat{\beta} = \gamma^{\pm}\pi^{\pm}W_p\widehat{\beta}, \quad N_p\widehat{\beta} = T^{\pm}\pi^{\pm}W_p\widehat{\beta}.$$

These operators can be extended by continuity to much wider classes of densities [7]. For convenience, we gather here some of the main results.

First we introduce the Poincaré–Steklov operators $\mathcal{T}_p^{\pm}$ acting in Sobolev spaces with a parameter. Let $f \in H_{1/2,p}(\partial S)$, $p \in \mathbb{C}_0$, and let $u \in H_{1,p}(S^{\pm})$ be the weak solution [8] of the problem

$$\begin{gathered} p^2(B^{1/2}u, B^{1/2}v)_{0,;S^{\pm}} + a_{\pm}(u,v) = 0 \quad \forall v \in \mathring{H}_{1,p}(S^{\pm}), \\ \gamma^{\pm}u = f. \end{gathered} \tag{12}$$

Also, let $\varphi \in H_{1/2,p}(\partial S)$ be arbitrary and let $w \in H_{1,p}(S^{\pm})$ be such that $\gamma^{\pm}w = \varphi$. We define the Poincaré–Steklov operators $\mathcal{T}_p^{\pm}$ depending on the parameter $p \in \mathbb{C}_0$ by

$$(\mathcal{T}_p^{\pm}f, \varphi)_{0;\partial S} = \pm\big\{p^2(B^{1/2}u, B^{1/2}w)_{0;S^{\pm}} + a_{\pm}(u,w)\big\}. \tag{13}$$

It is obvious that (13) defines $\mathcal{T}_p^{\pm}$ correctly. For if w_1, $w_2 \in H_{1,p}(S^{\pm})$ are such that $\gamma^{\pm}w_1 = \gamma^{\pm}w_2 = \varphi$, then $v = w_1 - w_2 \in \mathring{H}_{1,p}(S^{\pm})$ and, by (12),

$$\begin{aligned} p^2(B^{1/2}u, B^{1/2}w_1)_{0,S^{\pm}} + a_{\pm}(u,w_1) &= p^2(B^{1/2}u, B^{1/2}(v+w_2))_{0;S^{\pm}} + a_{\pm}(u, v+w_2) \\ &= p^2(B^{1/2}u, B^{1/2}w_2)_{0;S^{\pm}} + a_{\pm}(u,w_2). \end{aligned}$$

Lemma 1. *For every $p \in \mathbb{C}_0$, the operators $\mathcal{T}_p^{\pm}$ are homeomorphisms from $H_{1/2,p}(\partial S)$ to $H_{-1/2,p}(\partial S)$, and for every $\kappa > 0$ and $p \in \bar{\mathbb{C}}_\kappa$*

$$\|\mathcal{T}_p^{\pm}f\|_{-1/2,p;\partial S} \le c|p|\|f\|_{1/2,p;\partial S}, \tag{14}$$

$$\|f\|_{1/2,p;\partial S} \le c|p|\|\mathcal{T}_p^{\pm}f\|_{-1/2,p;\partial S}. \tag{15}$$

Returning to the spaces of originals, we introduce the Poincaré–Steklov operators $\mathcal{T}^{\pm}$. From (14) and (15) we easily deduce the following assertion.

Theorem 2. *For any $\kappa > 0$ and $k \in \mathbb{R}$, the operators $\mathcal{T}^{\pm}$ define injective maps*

$$\mathcal{T}^{\pm} : H^{\mathcal{L}^{-1}}_{1/2,k,\kappa}(\Gamma) \to H^{\mathcal{L}^{-1}}_{-1/2,k-1,\kappa}(\Gamma).$$

Their inverses, extended by continuity to the corresponding spaces, define injections

$$(\mathcal{T}^{\pm})^{-1} : H^{\mathcal{L}^{-1}}_{-1/2,k,\kappa}(\Gamma) \to H^{\mathcal{L}^{-1}}_{1/2,k-1,\kappa}(\Gamma).$$

We now use the above results to investigate the properties of the single-layer and double-layer potentials. Obviously, both these potentials satisfy the equation

$$Bp^2\widehat{u}(x,p) + A\widehat{u}(x,p) = 0, \quad x \in S^+ \cup S^- .$$

Lemma 2. *For any $p \in \mathbb{C}_0$, the operator $V_{p,0}$, extended by continuity from $C^2(\partial S)$ to the space $H_{-1/2,p}(\partial S)$, is a homeomorphism from $H_{-1/2,p}(\partial S)$ to $H_{1/2,p}(\partial S)$, and for every $p \in \bar{\mathbb{C}}_\kappa$*

$$\|V_{p,0}\widehat{\alpha}\|_{1/2,p;\partial S} \leq c|p|\|\widehat{\alpha}\|_{-1/2,p;\partial S},$$
$$\|\widehat{\alpha}\|_{-1/2,p;\partial S} \leq c|p|\|V_{p,0}\widehat{\alpha}\|_{1/2,p;\partial S}.$$

For any density $\widehat{\alpha} \in H_{-1/2,p}(\partial S)$, the single-layer potential is continuous in the sense that $\gamma^+ V_p\widehat{\alpha} = \gamma^- V_p\widehat{\alpha} = V_{p,0}\widehat{\alpha}$. Also, the jump formulas

$$\mathcal{T}_p^+ V_{p,0}\widehat{\alpha} = \tfrac{1}{2}\,\widehat{\alpha} + (\mathcal{T}_p^+ V_{p,0}\widehat{\alpha})_0,$$
$$\mathcal{T}_p^- V_{p,0}\widehat{\alpha} = -\tfrac{1}{2}\,\widehat{\alpha} + (\mathcal{T}_p^+ V_{p,0}\widehat{\alpha})_0$$

hold, where $(\mathcal{T}_p^+ V_{p,0}\widehat{\alpha})_0$ is the integral operator generated by the direct value on ∂S of the corresponding singular integral.

Lemma 3. *For any $p \in \mathbb{C}_0$, the operators $W_p^\pm$, extended by continuity from $C^2(\partial S)$ to $H_{1/2,p}(\partial S)$, are homeomorphisms from $H_{1/2,p}(\partial S)$ to $H_{1/2,p}(\partial S)$, and for every $p \in \bar{\mathbb{C}}_\kappa$*

$$\|W_p^\pm\widehat{\beta}\|_{1/2,p;\partial S} \leq c|p|^2\|\widehat{\beta}\|_{1/2,p;\partial S},$$
$$\|\widehat{\beta}\|_{1/2,p;\partial S} \leq c|p|^2\|W_p^\pm\widehat{\beta}\|_{1/2,p;\partial S}.$$

The jump formulas for the double-layer potential with $\widehat{\beta} \in H_{1/2,p}(\partial S)$ have the form

$$W_p^+\widehat{\beta} = -\tfrac{1}{2}\,\widehat{\beta} + (W_p\widehat{\beta})_0,$$
$$W_p^-\widehat{\beta} = \tfrac{1}{2}\,\widehat{\beta} + (W_p\widehat{\beta})_0,$$

where $(W_p\widehat{\beta})_0$ is the integral operator generated by the direct value on ∂S of the corresponding singular integral.

Finally, we consider the operator $N_p = \mathcal{T}_p^+ W_p^+ = \mathcal{T}_p^- W_p^-$, which is an integral (pseudodifferential) operator with a hypersingular kernel.

Lemma 4. *For any $p \in \mathbb{C}_0$, the operator N_p, extended by continuity from $C^2(\partial S)$ to $H_{1/2,p}(\partial S)$, is a homeomorphism from $H_{1/2,p}(\partial S)$ to $H_{-1/2,p}(\partial S)$, and for every $p \in \bar{\mathbb{C}}_\kappa$*

$$\|N_p\widehat{\beta}\|_{-1/2,p;\partial S} \leq c|p|^3\|\widehat{\beta}\|_{1/2,p;\partial S},$$
$$\|\widehat{\beta}\|_{1/2,p;\partial S} \leq c|p|\|N_p\widehat{\beta}\|_{-1/2,p;\partial S}.$$

We remark that if $\widehat{\alpha} \in H_{-1/2,p}(\partial S)$ and $\widehat{\beta} \in H_{1/2,p}(\partial S)$, then the single-layer and double-layer potentials belong to $H_{1,p}(S^\pm)$.

Returning to the spaces of originals, we establish the properties of the boundary operators, which are summarized in the next assertion.

Theorem 3. (i) *The operator V_0, extended by continuity from $C_0^2(\Gamma)$ to $H^{\mathcal{L}^{-1}}_{-1/2,k,\kappa}(\Gamma)$, is continuous and injective from $H^{\mathcal{L}^{-1}}_{-1/2,k,\kappa}(\Gamma)$ to $H^{\mathcal{L}^{-1}}_{1/2,k-1,\kappa}(\Gamma)$ for every $\kappa > 0$ and $k \in \mathbb{R}$, and its range is dense in $H^{\mathcal{L}^{-1}}_{1/2,k-1,\kappa}(\Gamma)$. The inverse*

V_0^{-1}, *extended by continuity from the range of* V_0 *to* $H^{\mathcal{L}^{-1}}_{1/2,k,\kappa}(\Gamma)$, *is continuous and injective from* $H^{\mathcal{L}^{-1}}_{1/2,k,\kappa}(\Gamma)$ *to* $H^{\mathcal{L}^{-1}}_{-1/2,k-1,\kappa}(\Gamma)$ *for any* $k \in \mathbb{R}$, *and its range is dense in* $H^{\mathcal{L}^{-1}}_{-1/2,k-1,\kappa}(\Gamma)$. *Furthermore,*

$$\|\pi^+ V\alpha\|_{1,k-1,\kappa;G^+} + \|\pi^- V\alpha\|_{1,k-1,\kappa;G^-} \le c\|\alpha\|_{-1/2,k,\kappa;\Gamma}.$$

(ii) *The operators* $W^{\pm}$, *extended by continuity from* $C_0^2(\Gamma)$ *to* $H^{\mathcal{L}^{-1}}_{1/2,k,\kappa}(\Gamma)$, *are continuous and injective from* $H^{\mathcal{L}^{-1}}_{1/2,k,\kappa}(\Gamma)$ *to* $H^{\mathcal{L}^{-1}}_{1/2,k-2,\kappa}(\Gamma)$ *for any* $\kappa > 0$ *and* $k \in \mathbb{R}$, *and their ranges are dense in* $H^{\mathcal{L}^{-1}}_{1/2,k-2,\kappa}(\Gamma)$. *The inverses* $(W^{\pm})^{-1}$, *extended by continuity from the ranges of* $W^{\pm}$ *to* $H^{\mathcal{L}^{-1}}_{1/2,k,\kappa}(\Gamma)$, *are continuous and injective from* $H^{\mathcal{L}^{-1}}_{1/2,k,\kappa}(\Gamma)$ *to* $H^{\mathcal{L}^{-1}}_{1/2,k-2,\kappa}(\Gamma)$ *for all* $k \in \mathbb{R}$, *and their ranges are dense in* $H^{\mathcal{L}^{-1}}_{1/2,k-2,\kappa}(\Gamma)$. *Furthermore,*

$$\|\pi^+ W\beta\|_{1,k-2,\kappa;G^+} + \|\pi^- W\beta\|_{1,k-2,\kappa;G^-} \le c\|\beta\|_{1/2,k,\kappa;\Gamma}.$$

(iii) *The operator* N, *extended by continuity from* $C_0^2(\Gamma)$ *to* $H^{\mathcal{L}^{-1}}_{1/2,k,\kappa}(\Gamma)$, *is continuous and injective from* $H^{\mathcal{L}^{-1}}_{1/2,k,\kappa}(\Gamma)$ *to* $H^{\mathcal{L}^{-1}}_{-1/2,k-3,\kappa}(\Gamma)$ *for any* $\kappa > 0$ *and* $k \in \mathbb{R}$, *and its range is dense in* $H^{\mathcal{L}^{-1}}_{-1/2,k-3,\kappa}(\Gamma)$. *The inverse* N^{-1}, *extended by continuity from the range of* N *to* $H^{\mathcal{L}^{-1}}_{-1/2,k,\kappa}(\Gamma)$, *is continuous and injective from* $H^{\mathcal{L}^{-1}}_{-1/2,k,\kappa}(\Gamma)$ *to* $H^{\mathcal{L}^{-1}}_{1/2,k-1,\kappa}(\Gamma)$ *for any* $k \in \mathbb{R}$ *and its range is dense in* $H^{\mathcal{L}^{-1}}_{1/2,k-1,\kappa}(\Gamma)$.

(iv) *For any* $\alpha \in H^{\mathcal{L}^{-1}}_{-1/2,k,\kappa}(\Gamma)$, $\kappa > 0$, *and* $k \ge 1$, *the potential* $V\alpha$ *is a weak solution of the homogeneous equation* (1) *in* $G^+ \cup G^-$ *with homogeneous initial conditions. For any* $\beta \in H^{\mathcal{L}^{-1}}_{1/2,k,\kappa}(\Gamma)$, $\kappa > 0$, *and* $k \ge 2$, *the potential* $W\beta$ *is a weak solution of the homogeneous equation* (1) *in* $G^+ \cup G^-$ *with homogeneous initial conditions.*

The jump formulas for the single-layer and double-layer dynamic potentials are obvious and we omit them.

5. Integral Representations of the Solutions

For every $p \in \mathbb{C}_0$ we introduce three pairs of special boundary operators. Let α, $\beta = 1, 2$, $\alpha \ne \beta$. For $f \in H_{1/2,p}(\partial S)$ we define the operators $\pi^{\pm}_{\alpha\beta}$ by

$$\pi^{\pm}_{\alpha\beta} f = \{\pi_\alpha f,\, \pi_\beta \mathcal{T}^{\pm}_p f\}.$$

For $g \in H_{-1/2,p}(\partial S)$, we define the operators $\theta^{\pm}_{\alpha\beta}$ by

$$\theta^{\pm}_{\alpha\beta} g = \pi^{\pm}_{\alpha\beta} (\mathcal{T}^{\pm}_p)^{-1} g.$$

Finally, for $\{f_\alpha, g_\beta\} \in H_{1/2,p}(\partial S_\alpha) \times H_{-1/2,p}(\partial S_\beta)$ we define the operators $\rho^{\pm}_{\beta\alpha}$ by

$$\rho^{\pm}_{\beta\alpha}\{f_\alpha, g_\beta\} = \pi^{\pm}_{\beta\alpha}(\pi^{\pm}_{\alpha\beta})^{-1}\{f_\alpha, g_\beta\}$$

We denote by $\|\{f_\alpha, g_\beta\}\|_{1/2,p;\partial S_\alpha;-1/2,p;\partial S_\beta}$ the norm of $\{f_\alpha, g_\beta\}$ in the space $H_{1/2,p}(\partial S_\alpha) \times H_{-1/2,p}(\partial S_\beta)$; that is,

$$\|\{f_\alpha, g_\beta\}\|_{1/2,p;\partial S_\alpha;-1/2,p;\partial S_\beta} = \|f_\alpha\|_{1/2,p;\partial S_\alpha} + \|g_\beta\|_{-1/2,p;\partial S_\beta}.$$

Lemma 5. (i) *The operators* $\pi^\pm_{\alpha\beta}$ *are homeomorphisms from* $H_{1/2,p}(\partial S)$ *to* $H_{1/2,p}(\partial S_\alpha) \times H_{-1/2,p}(\partial S_\beta)$, *and for every* $\kappa > 0$ *and* $p \in \bar{\mathbb{C}}_\kappa$

$$\|\pi^\pm_{\alpha\beta} f\|_{1/2,p;\partial S_\alpha;-1/2,p;\partial S_\beta} \le c|p|\|f\|_{1/2,p;\partial S}, \tag{16}$$

$$\|f\|_{1/2,p;\partial S} \le c|p|\|\pi^\pm_{\alpha\beta} f\|_{1/2,p,\partial S_\alpha;-1/2,p;\partial S_\beta}. \tag{17}$$

(ii) *The operators* $\theta^\pm_{\alpha\beta}$ *are homeomorphisms from* $H_{-1/2,p}(\partial S)$ *to the space* $H_{1/2,p}(\partial S_\alpha) \times H_{-1/2,p}(\partial S_\beta)$, *and for every* $\kappa > 0$ *and* $p \in \bar{\mathbb{C}}_\kappa$

$$\|\theta^\pm_{\alpha\beta} g\|_{1/2,p;\partial S_\alpha;-1/2,p;\partial S_\beta} \le c|p|\|g\|_{-1/2,p;\partial S}, \tag{18}$$

$$\|g\|_{-1/2,p;\partial S} \le c|p|\|\theta^\pm_{\alpha\beta} g\|_{1/2,p;\partial S_\alpha;-1/2,p;\partial S_\beta}. \tag{19}$$

(iii) *The operators* $\rho^\pm_{\alpha\beta}$ *are homeomorphisms from the space* $H_{1/2,p}(\partial S_\alpha) \times H_{-1/2,p}(\partial S_\beta)$ *to* $H_{1/2,p}(\partial S_\beta) \times H_{-1/2,p}(\partial S_\alpha)$, *and for every* $\kappa > 0$, $p \in \bar{\mathbb{C}}_\kappa$

$$\|\rho^\pm_{\alpha\beta}\{f_\alpha, g_\beta\}\|_{1/2,p;\partial S_\beta;-1/2,p,\partial S_\alpha} \le c|p|\|\{f_\alpha, g_\beta\}\|_{1/2,p;\partial S_\alpha;-1/2,p;\partial S_\beta}, \tag{20}$$

$$\|\{f_\alpha, g_\beta\}\|_{1/2,p;\partial S_\alpha;-1/2,p;\partial S_\beta} \le c|p|\|\rho^\pm_{\alpha\beta}\{f_\alpha, g_\beta\}\|_{1/2,p;\partial S_\beta;-1/2,p;\partial S_\alpha}. \tag{21}$$

Proof. (i) The continuity of $\pi^\pm_{\alpha\beta}$ and (16) follow from Lemma 1. Let $u \in H_{1,p}(S^\pm)$ be a unique solution of the problem

$$\begin{gathered} p^2(B^{1/2}u,\, B^{1/2}v)_{0;S^\pm} + a_\pm(u,v) = \pm(g_\beta, v)_{0;\partial S_\beta} \quad \forall v \in \mathring{H}_{1,p}(S^\pm, \partial S_\beta), \\ \pi_\alpha \gamma^\pm u = f_\alpha, \end{gathered} \tag{22}$$

where $\{f_\alpha, g_\beta\} \in H_{1/2,p}(\partial S_\alpha) \times H_{-1/2,p}(\partial S_\beta)$. By (11),

$$\|u\|_{1,p;S^\pm} \le c|p|\|\{f_\alpha, g_\beta\}\|_{1/2,p,\partial S_\alpha;-1/2,p;\partial S_\beta}. \tag{23}$$

If $f = \gamma^\pm u \in H_{1/2,p}(\partial S)$, then $\pi^\pm_{\alpha\beta} f = \{f_\alpha, g_\beta\}$; hence, $\pi^\pm_{\alpha\beta}$ are surjective. The trace theorem and (23) imply that (17) holds.

(ii) From Lemma 1 and (i) it follows that $\theta^\pm_{\alpha\beta}$ are homeomorphisms from $H_{-1/2,p}(\partial S)$ to $H_{1/2,p}(\partial S_\alpha) \times H_{-1/2,p}(\partial S_\beta)$. Let $u \in H_{1,p}(S^\pm)$ be the unique solution of the problem

$$p^2(B^{1/2}u,\, B^{1/2}v)_{0;S^\pm} + a_\pm(u,v) = \pm(g, v)_{0;\partial S} \quad \forall v \in H_{1,p}(S^\pm),$$

where $g \in H_{-1/2,p}(\partial S)$. By Theorem 3 in [8],

$$\|u\|_{1,p;S^\pm} \le c|p|\|g\|_{-1/2,p;\partial S}.$$

If $f = (\mathcal{T}^\pm_p)^{-1} g$, then $\|f\|_{1/2,p;\partial S} \le c|p|\ \|g\|_{-1/2,p;\partial S}$. We now have

$$\|\pi_\alpha f\|_{1/2,p;\partial S_\alpha} + \|\pi_\beta g\|_{-1/2,p;\partial S_\beta} \le \|f\|_{1/2,p;\partial S} + \|g\|_{-1/2,p;\partial S} \le c|p|\|g\|_{-1/2,p;\partial S},$$

which proves (18).

Next, let u be the solution of (2), $\gamma^\pm u = f$ and $\mathcal{T}^\pm_p f = g$. From the estimate [7]

$$\|g\|_{-1/2,p;\partial S} \le c\|u\|_{1,p;S^\pm}$$

and (11) it follows that (19) is holds.

(iii) The definition of $\rho^{\pm}_{\alpha\beta}$ and assertion (i) indicate that these operators are homeomorphisms from $H_{1/2,p}(\partial S_\alpha)\times H_{-1/2,p}(\partial S_\beta)$ to $H_{1/2,p}(\partial S_\beta)\times H_{-1/2,p}(\partial S_\alpha)$. Let u be the solution of (2), $f=\gamma^{\pm}u$, and $g=\mathcal{T}_p^{\pm}f$. We have

$$\begin{aligned}\|\pi_\alpha g\|_{-1/2,p;\partial S_\alpha}+\|\pi_\beta f\|_{1/2,p;\partial S_\beta}\le \|g\|_{-1/2,p;\partial S}+\|f\|_{1/2,p;\partial S}\le c\|u\|_{1,p;S^{\pm}}\\ \le c|p|\ \|\{f_\alpha,g_\beta\}\|_{1/2,p;\partial S_\alpha;-1/2,p;\partial S_\beta},\end{aligned}$$

which proves (20). Estimate (21) is proved similarly. □

We now consider four representations for the solutions of problems (DM$^{\pm}$) in terms of dynamic potentials and prove the unique solvability of the corresponding systems of boundary integral equations. We begin with the representation

$$u(x,t)=(V\alpha)(x,t),\quad (x,t)\in G^{+}\text{ or }G^{-}, \tag{24}$$

which yields the system of boundary equations

$$\begin{aligned}(\pi_1 V_0\alpha)(x,t)&=f_1(x,t),\quad (x,t)\in\Gamma_1,\\ (\pi_2\mathcal{T}^{\pm}V_0\alpha)(x,t)&=g_2(x,t),\quad (x,t)\in\Gamma_2.\end{aligned} \tag{25}$$

Theorem 4. *For every $\kappa>0$, $k\in\mathbb{R}$, $f_1\in H^{\mathcal{L}^{-1}}_{1/2,k,\kappa}(\Gamma_1)$, and $g_2\in H^{\mathcal{L}^{-1}}_{-1/2,k,\kappa}(\Gamma_2)$ system* (25) *has a unique solution $\alpha\in H^{\mathcal{L}^{-1}}_{-1/2,k-2,\kappa}(\Gamma)$, in which case u defined by* (24) *belongs to $H^{\mathcal{L}^{-1}}_{1,k-1,\kappa}(G^{\pm})$. If $k\ge 1$, then u is the solution of* (DM$^{\pm}$).

Proof. We give the proof of this assertion and of the next three ones only for (DM^{+}). The exterior problem (DM^{-}) is treated similarly.

In terms of Laplace transforms, (25) takes the form

$$\pi_1 V_{p,0}\widehat{\alpha}=f_1,\quad \pi_2\mathcal{T}_p^{+}V_{p,0}\widehat{\alpha}=g_2,$$

or

$$\pi^{+}_{12}V_{p,0}\widehat{\alpha}=\{f_1,g_2\}. \tag{26}$$

By Lemmas 5 and 2, (26) has a unique solution $\widehat{\alpha}\in H_{-1/2,p}(\partial S)$ and

$$\|\widehat{\alpha}\|_{-1/2,p;\partial S}\le c|p|^2\|\{f_1,g_2\}\|_{1/2,p;\partial S_1;-1/2,p,\partial S_2} \tag{27}$$

for all $p\in\mathbb{C}_\kappa$. Taking (27) and Theorem 1 into account, we complete the proof by the standard scheme used in [8]. □

The second representation is

$$u(x,t)=(W\beta)(x,t),\quad (x,t)\in G^{+}\text{ or }G^{-}, \tag{28}$$

and it leads to the system

$$\begin{aligned}(\pi_1 W^{\pm}\beta)(x,t)&=f_1(x,t),\quad (x,t)\in\Gamma_1,\\ (\pi_2 N\beta)(x,t)&=g_2(x,t),\quad (x,t)\in\Gamma_2.\end{aligned} \tag{29}$$

Theorem 5. *For all $\kappa>0$, $k\in\mathbb{R}$, $f_1\in H^{\mathcal{L}^{-1}}_{1/2,k,\kappa}(\Gamma_1)$, and $g_2\in H^{\mathcal{L}^{-1}}_{-1/2,k,\kappa}(\Gamma_2)$ system* (29) *has a unique solution $\beta\in H^{\mathcal{L}^{-1}}_{1/2,k-2,\kappa}(\Gamma)$, in which case u defined by* (28) *belongs to $H^{\mathcal{L}^{-1}}_{1,k-1,\kappa}(G^{\pm})$. If $k\ge 1$, then u is the solution of* (DM$^{\pm}$).

Proof. In the case of (DM$^+$), (29) written in terms of Laplace transforms takes the form

$$\pi_1 W_p^+ \widehat{\beta} = f_1, \quad \pi_2 N_p \widehat{\beta} = g_2,$$

or

$$\pi_{12}^+ W_p^+ \widehat{\beta} = \{f_1, g_2\}. \tag{30}$$

If $g = N_p \widehat{\beta}$, then $W_p^+ \widehat{\beta} = (\mathcal{T}_p^+)^{-1} g$ and (30) can be rewritten as

$$\theta_{12}^+ g = \{f_1, g_2\}. \tag{31}$$

By Lemmas 5 and 1, (31)—hence, also (30)—have a unique solution $\widehat{\beta} \in H_{1/2,p}(\partial S)$ and

$$\|\widehat{\beta}\|_{1/2,p;\partial S} \le c|p|^2 \|\{f_1, g_2\}\|_{1/2,p;\partial S_1;-1/2,p;\partial S_2} \tag{32}$$

for all $p \in \mathbb{C}_\kappa$. Theorem 1 and (32) now enable us to complete the proof in the usual way. $\square$

The third representation is

$$u(x,t) = (V\alpha_1)(x,t) + (W\beta_2)(x,t), \quad (x,t) \in G^+ \text{ or } G^-, \tag{33}$$

where $\alpha_1 \in \mathring{H}^{\mathcal{L}^{-1}}_{-1/2,k,\kappa}(\Gamma_1)$ and $\beta_2 \in \mathring{H}^{\mathcal{L}^{-1}}_{1/2,k,\kappa}(\Gamma_2)$. This representation yields the system of boundary equations

$$\begin{aligned} (\pi_1 V_0 \alpha_1)(x,t) + (\pi_1 W^\pm \beta_2)(x,t) &= f_1(x,t), \quad (x,t) \in \Gamma_1, \\ (\pi_2 \mathcal{T}^\pm V_0 \alpha_1)(x,t) + (\pi_2 N \beta_2)(x,t) &= g_2(x,t), \quad x,t \in \Gamma_2. \end{aligned} \tag{34}$$

Theorem 6. *For all $\kappa > 0$, $k \in \mathbb{R}$, $f_1 \in H^{\mathcal{L}^{-1}}_{1/2,k,\kappa}(\Gamma_1)$, and $g_2 \in H^{\mathcal{L}^{-1}}_{-1/2,k,\kappa}(\Gamma_2)$ system* (34) *has a unique solution $\{\alpha_1, \beta_2\} \in \mathring{H}^{\mathcal{L}^{-1}}_{-1/2,k-1,\kappa}(\Gamma_1) \times \mathring{H}^{\mathcal{L}^{-1}}_{1/2,k-1,\kappa}(\Gamma_2)$, in which case u defined by* (33) *belongs to $H^{\mathcal{L}^{-1}}_{1,k-1,\kappa}(G^\pm)$. If $k \ge 1$, then u is the solution of* (DM$^\pm$).

Proof. Going over to Laplace transforms, we see that in the case of (DM$^+$) system (34) becomes

$$\pi_{12}^+ (V_{p,0} \widehat{\alpha}_1 + W_p^+ \widehat{\beta}_2) = \{f_1, g_2\}. \tag{35}$$

We claim that $\{\widehat{\alpha}_1, \widehat{\beta}_2\}$, where

$$\begin{aligned} \widehat{\alpha}_1 &= [(\theta_{12}^+)^{-1} - (\theta_{12}^-)^{-1}]\{f_1, g_2\}, \\ \widehat{\beta}_2 &= [(\pi_{12}^-)^{-1} - (\pi_{12}^+)^{-1}]\{f_1, g_2\}, \end{aligned}$$

is a solution of (35). Indeed, from the definition of $\pi_{12}^\pm$ and $\theta_{12}^\pm$ it follows that $\pi_1[(\pi_{12}^+)^{-1} - (\pi_{12}^-)^{-1}] = 0$; hence, $\widehat{\beta}_2 \in \mathring{H}_{1/2,p}(\partial S_2)$. Analogously, $\widehat{\alpha}_1 \in \mathring{H}_{-1/2,p}(\partial S_1)$. Then, since $(\theta_{12}^\pm)^{-1} = \mathcal{T}_p^\pm (\pi_{12}^\pm)^{-1}$ and $V_{p,0}(\mathcal{T}_p^+ - \mathcal{T}_p^-) = I$, we have

$$\begin{aligned} V_{p,0}\widehat{\alpha}_1 + W_p^+ \widehat{\beta}_2 &= V_{p,0}\{(\theta_{12}^+)^{-1} - (\theta_{12}^-)^{-1} + \mathcal{T}_p^-[(\pi_{12}^-)^{-1} - (\pi_{12}^+)^{-1}]\}\{f_1, g_2\} \\ &= V_{p,0}\{(\theta_{12}^+)^{-1} - \mathcal{T}_p^-(\pi_{12}^+)^{-1}\}\{f_1, g_2\} = (\pi_{12}^+)^{-1}\{f_1, g_2\}, \end{aligned}$$

which proves the assertion.

The difference $\{\widetilde{\alpha}_1,\widetilde{\beta}_2\}\in\mathring{H}_{-1/2,p}(\partial S_1)\times\mathring{H}_{1/2,p}(\partial S_2)$ of two solutions of (35) will satisfy

$$\pi_{12}^{+}(V_{p,0}\widetilde{\alpha}_1+W_p^{+}\widetilde{\beta}_2)=\{0,0\},$$

so

$$V_{p,0}\widetilde{\alpha}_1+W_p^{+}\widetilde{\beta}_2=0.$$

We take $u(x,p)=(V_p\widetilde{\alpha}_1)(x,p)+(W_p\widetilde{\beta}_2)(x,p)$. This function is a solution of both (DM_p^{+}) and (DM_p^{-}) with $f_1=0$ and $g_2=0$, respectively; therefore, $u(x,p)=0$ in S^{+} and in S^{-}. Then $\widetilde{\beta}_2=\gamma^{-}u-\gamma^{+}u=0$ and $\widetilde{\alpha}_1=\mathcal{T}_p^{+}\gamma^{+}u-\mathcal{T}_p^{-}\gamma^{-}u=0$, which proves the uniqueness of the solution.

The estimates

$$\|\widehat{\alpha}_1\|_{-1/2,p;\partial S}\le c|p|\ \|\{f_1,g_2\}\|_{1/2,p;\partial S_1;-1/2,p;\partial S_2},$$
$$\|\widehat{\beta}_2\|_{1/2,p;\partial S}\le c|p|\ \|\{f_1,g_2\}\|_{1/2,p;\partial S_1;-1/2,p;\partial S_2}$$

follow from (17) and (19). The proof is completed by following the standard procedure. □

The fourth representation is

$$u(x,t)=(W\beta_1)(x,t)+(V\alpha_2)(x,t),\quad (x,t)\in G^{+}\text{ or }G^{-},\tag{36}$$

where $\beta_1\in\mathring{H}^{\mathcal{L}^{-1}}_{1/2,k,\kappa}(\Gamma_1)$ and $\alpha_2\in\mathring{H}^{\mathcal{L}^{-1}}_{-1/2,k,\kappa}(\Gamma_2)$. This yields the system

$$\begin{aligned}(\pi_1W^{\pm}\beta_1)(x,t)+(\pi_1V_0\alpha_2)(x,t)&=f_1(x,t),\quad (x,t)\in\Gamma_1,\\ (\pi_2N\beta_1)(x,t)+(\pi_2\mathcal{T}^{\pm}V_0\alpha_2)(x,t)&=g_2(x,t),\quad (x,t)\in\Gamma_2.\end{aligned}\tag{37}$$

Theorem 7. *For all $\kappa>0$, $k\in\mathbb{R}$, $f_1\in H^{\mathcal{L}^{-1}}_{1/2,k,\kappa}(\Gamma_1)$, and $g_2\in H^{\mathcal{L}^{-1}}_{-1/2,k,\kappa}(\Gamma_2)$ system* (37) *has a unique solution $\{\beta_1,\alpha_2\}\in\mathring{H}^{\mathcal{L}^{-1}}_{1/2,k-2,\kappa}(\Gamma_1)\times\mathring{H}^{\mathcal{L}^{-1}}_{-1/2,k-2,\kappa}(\Gamma_2)$, in which case u defined by* (36) *belongs to $H^{\mathcal{L}^{-1}}_{1,k-1,\kappa}(G^{\pm})$. If $k\ge1$, then u is the solution of* $(\mathrm{DM}^{\pm})$.

Proof. In terms of Laplace transforms, for (DM^{+}) (37) takes the form

$$\pi_{12}^{+}(W_p^{+}\widehat{\beta}_1+V_{p,0}\widehat{\alpha}_2)=\{f_1,g_2\}.\tag{38}$$

We claim that

$$\widehat{\beta}_1=[(\pi_{12}^{-})^{-1}\rho_{21}^{+}-(\pi_{12}^{+})^{-1}]\{f_1,g_2\},$$
$$\widehat{\alpha}_2=[(\theta_{12}^{+})^{-1}-(\theta_{12}^{-})^{-1}\rho_{21}^{+}]\{f_1,g_2\}$$

is the solution of (38). Indeed, we have

$$\begin{aligned}&W_p^{+}\widehat{\beta}_1+V_{p,0}\widehat{\alpha}_2\\ &\quad=V_{p,0}\{\mathcal{T}_p^{-}[(\pi_{12}^{-})^{-1}\rho_{21}^{+}-(\pi_{12}^{+})^{-1}]+(\theta_{12}^{+})^{-1}-(\theta_{12}^{-})^{-1}\rho_{21}^{+}\}\{f_1,g_2\}\\ &\quad=V_{p,0}\{\mathcal{T}_p^{+}(\pi_{12}^{+})^{-1}-\mathcal{T}_p^{-}(\pi_{12}^{+})^{-1}\}\{f_1,g_2\}=(\pi_{12}^{+})^{-1}\{f_1,g_2\};\end{aligned}$$

hence, (38) holds.

Since $\pi_2[(\pi_{12}^{-})^{-1}\rho_{21}^{+}-(\pi_{12}^{+})^{-1}]\{f_1,g_2\}=0$, we have $\widehat{\beta}_1\in\mathring{H}_{1/2,p}(\partial S_1)$. Similarly, $\widehat{\alpha}\in\mathring{H}_{-1/2,p}(\partial S_2)$. To prove the unique solvability of (38), we repeat (with obvious changes) the proof of Theorem 6. From (17), (19), and (20) it follows that for all $p \in \mathbb{C}_\kappa$

$$\|\widehat{\beta}_1\|_{1/2,p;\partial S} \leq c|p|^2 \|\{f_1, g_2\}\|_{1/2,p;\partial S_1;-1/2,p;\partial S_2},$$
$$\|\widehat{\alpha}_2\|_{-1/2,p;\partial S} \leq c|p|^2 \|\{f_1, g_2\}\|_{1/2,p;\partial S_1;-1/2,p;\partial S_2}.$$

The proof is now completed in the usual way. □

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