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THE DIRICHLET PROBLEM FOR HARMONIC FUNCTIONS WITH BOUNDARY VALUES FROM ZYGMUND CLASSES

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Abstract. We investigate the Dirichlet problem in domains with nonsmooth boundaries when a boundary function belongs to functional classes of the Zygmund class type. A complete picture of solvability is established also in the case of existence of solutions for which explicit expressions are found.

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Since harmonic functions of two variables are closely related to analytic functions, in solving boundary value problems one can apply the theory of analytic functions, in particular the methods used in boundary value problems for analytic functions. N. Muskhelishvili indicated quite a useful method of reducing boundary value problems for harmonic functions to a problem of linear conjugation with a certain additional condition and, having solved the latter problem, he constructed solutions of the initial problem (see [1], §§41–43). He considered the case where the unknown function is continuous in the closed domain bounded by the Lyapunov curve and with a boundary function belonging to the Hölder class. A subsequent development of this method made it possible to study the Dirichlet and Riemann–Hilbert (Neumann) boundary value problems in the classes $e_p(D)$ ($e_p'(D)$) of those harmonic functions in the simply connected domain D which are the real parts of analytic functions from the Smirnov class $E^p(D)$ (the derivative of which belongs to $E^p(D)$, $p > 1$) (see [2]–[5]). Moreover, using the two-weight properties of the Cauchy integral, it also became possible to construct solutions when the domain D is bounded by an arbitrary piecewise-smooth curve. In that case the situation as regards solvability turned out to be essentially dependent on the boundary geometry. For some other curves and weighted Smirnov classes analogous results were established in [6]–[8].

When the above-mentioned problems are considered in $e_p(D)$, $p > 1$, the given boundary functions belong as a rule to the Lebesgue space $L^p(\Gamma)$, $p > 1$. For $p = 1$ the assumption that these functions are summable is not sufficient for constructing a more or less clear vision of the solvability of boundary value problems in $e^1(D)$. Hence it is natural to pose the problem of narrowing the set of given function so that we could succeed in solving a boundary value problem in the important class $e^1(D)$.

By such a set we mean wide subsets of summable functions, namely Zygmund classes. In our work [9] it is shown that for the problem of linear conjugation

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with a piecewise-Hölder coefficient, which plays an important role in investigating the Dirichlet problem by the method mentioned above, the well-known results on its solvability remain in force and solutions are constructed in the classes of functions representable by Cauchy integrals inside and outside domains bounded by a Jordan curve Γ and with boundary values belonging to a given Zygmund class. A Jordan curve Γ is taken from a wide class containing in particular piecewise-smooth curves.

In §3 of [10], the results on the Dirichlet problem are established in the class $e^1(D)$, when the boundary function belongs to the Zygmund class. This paper contains a detailed exposition of the results announced in [10]. It should be noted here that in work [11], too, Zygmund classes are used in considering boundary value problems for elliptic equations.

1^0. Let Γ be a rectifiable Jordan closed oriented curve bounding the domain D, ν an arc length measure given on Γ, w a measurable function which is positive almost everywhere on Γ in the sense of the measure ν, and α be a nonnegative real number.

Denote by $Z_{\alpha,w}(\Gamma)$ the class of measurable on Γ functions f for which $f\log^\alpha\big(2+\frac{|f|}{w}\big)$ is summable in the Lebesgue sense, i.e.,

$$\int_\Gamma \Big|f(t)\,\log^\alpha\Big(2+\frac{|f(t)|}{w(t)}\Big)\Big|d\nu<\infty. \tag{1}$$

Assume that $Z_\alpha(\Gamma)\equiv Z_{\alpha,1}(\Gamma)$ for $w(t)\equiv 1$. If $\gamma=\{t:\ |t|=1\}$, then $Z_\alpha(\gamma)$ are Zygmund classes (see, for example, [12]). The class $Z_0(\Gamma)$ coincides with the class $L(\Gamma)$ of functions summable in the Lebesgue sense. Denote the class $Z_1(\Gamma)$ by $Z(\Gamma)$.

A Smirnov class $E^p(D)$, $p>0$ is said to be a set of analytic functions Φ in D for which

$$\sup_{0<r<1}\int_{\Gamma_r}\big|\Phi(z)\big|^p\,|dz|<\infty,$$

where Γ_r are the images of circumferences $\gamma_r=\{z:\ |z|=r\}$ for the conformal mapping of the unit circle $U=\{z:\ |z|<1\}$ onto the domain D (see, for example [13], p. 422]. The class $E^p(U)$ coincides with the Hardy class H^p. Functions Φ from $E^p(D)$ have angular boundary values $\Phi^+(t)$ almost everywhere on Γ and, moreover, $\Phi^+\in L^p(\Gamma)$ ([13], p. 422).

Denote by $E^p(\Gamma)$ the set of analytic functions Φ in the plane cut along Γ, the restrictions of which on the domains D^+ and D^- bounded by Γ ($\infty\in D^-$) belong to the classes $E^p(D^+)$ and $E^p(D^-)$, respectively.

Assume that

$$\widetilde{E}^{\,p}(\Gamma)=\big\{\Phi:\ \Phi=\Phi_1+\mathrm{const},\ \Phi_1\in E^p(\Gamma)\big\}, \tag{2}$$

$$\widetilde{E}^{\,p,\alpha}_{w}(\Gamma)=\big\{\Phi:\ \Phi\in\widetilde{E}^{\,p}(\Gamma),\ \Phi^\pm\in Z_{\alpha,w}(\Gamma)\big\}, \tag{3}$$

$$e^p(D)=\big\{u:\ u=\operatorname{Re}\Phi,\ \Phi\in E^p(D)\big\}. \tag{4}$$

Let us consider the Dirichlet problem formulated as follows: Define a function u satisfying the conditions

$$\begin{cases} \Delta u = 0, & u \in e^1(D), \\ u^+(t) = f(t), & t \in \Gamma, \quad f \in Z(\Gamma). \end{cases} \tag{5}$$

2^0. Following [1] (§43) (see also [2], [5], Ch. IV), we reduce problem (5) to the problem of linear conjugation.

Since $u \in e^1(D)$, there exists a function $\Phi \in E^1(D)$ such that

$$u(z) = \operatorname{Re} \Phi(z). \tag{6}$$

Let $z = z(w)$ be a function that maps conformally the circle U onto D, and

$$\Psi(w) = \Phi(z(w))\, z'(w). \tag{7}$$

Assume that

$$\Omega(z) = \begin{cases} \Psi(w), & |w| < 1, \\ \overline{\Psi\Big(\dfrac{1}{\overline{w}}\Big)}, & |w| > 1, \end{cases} \tag{8}$$

$$\Omega_*(w) \;=\; \overline{\Omega\Big(\frac{1}{\overline{w}}\Big)}, \qquad |w| \neq 1. \tag{9}$$

Proposition 1. *If $\Phi \in E^1(D)$, then the function $\Omega(w)$ defined by (7)–(8) belongs to the class $\widetilde{E}(\Gamma)$.*

Proof. Since $\Phi \in E^1(D)$, we have $\Psi \in H^1$ ([13], p. 422). Therefore it can be represented by the Cauchy integral in the circle U, i.e.,

$$\Psi(w) = \frac{1}{2\pi i} \int_\gamma \frac{\Psi^+(t)}{t - w}$$

(see 13, p. 423). The function $\Omega(w) - \Omega(\infty)$ can be represented in the domain $|w| > 1$ by the Cauchy integral ([14], Lemma 4; see also [5], p. 131). This means that $\Omega \in \widetilde{E}^{\,1}(\Gamma)$. □

Let us write the boundary condition from (5) in the form

$$\operatorname{Re} \Phi^+(z(\tau)) = \operatorname{Re} \frac{\Psi^+(\tau)}{z'(\tau)} = f(z(\tau)), \quad \tau \in \gamma.$$

Then we have

$$\operatorname{Re} \Phi^+(z(\tau)) = \frac{1}{2}\left[\frac{\Psi^+(\tau)}{z'(\tau)} + \overline{\left(\frac{\Psi^+(\tau)}{z'(\tau)}\right)}\right] = f(z(\tau)),$$

i.e.,

$$\frac{\Psi^+(\tau)}{z'(\tau)} + \frac{\overline{\Psi^+(\tau)}}{\overline{z'(\tau)}} = 2f(z(\tau)).$$

Taking into account that

$$\frac{\Psi^+(\tau)}{z'(\tau)} = \Omega^+(\tau), \quad \frac{\overline{\Psi^+(\tau)}}{\overline{z'(\tau)}} = \Omega^-(\tau))$$

([1], §41) we eventually conclude that if the harmonic function $u = \operatorname{Re}\Phi$ is a solution of problem (5), then the analytic function Ω defined by (7), (8) is a solution of the problem of linear conjugation

$$\Omega^+(\tau) = -\frac{z'(\tau)}{\overline{z'(\tau)}}\,\Omega^-(\tau) + g(\tau), \quad \tau \in \gamma, \quad \Omega \in E^1(\gamma),$$

where $g(\tau) = 2f(z(\tau))\,z'(\tau)$.

Let now Ω be some solution of this problem. As seen from the definition of Ω (see (8)), the function $u(w(z)) = \operatorname{Re}\frac{\Omega(w(z))}{w'(z)}$, where $w = w(z)$ is the inverse function to $z = z(w)$, give a solution of problem (5) if the function Ω satisfies the condition $\Omega_*(w) = \Omega(w)$ (see (9)). Thus a solution of problem (5) is equivalent to a solution of the problem

$$\begin{cases} \Omega^+(\tau) = -\dfrac{z'(\tau)}{\overline{z'(\tau)}}\,\Omega^-(\tau) + g(\tau), \quad \Omega \in \widetilde{E}^{\,1}(\gamma), \\ \Omega_*(w) = \Omega(w), \quad |w| \neq 1, \end{cases} \tag{10}$$

where $g(\tau) = 2f(z(\tau))\,z'(\tau)$. This means that any solution $u(z)$ of problem (5) generates some solution of problem (10) defined by equalities (6)–(8) and, conversely, any solution $\Omega(w)$ of problem (10) generates some solution $u(z)$ of problem (5), which a restriction of the function $\operatorname{Re}\Omega(w(z))\,w'(z)$ on D.

In [9] the problem of linear conjugation

$$\Omega^+ = G\,\Omega^- + g$$

is studied in the class $\widetilde{E}^{\,1,\alpha}_{w}(\Gamma)$ under the assumption that G is piecewise-continuous in the Hölder sense, the (power) function w is defined by means of discontinuity points of G, and Γ is a regular curve which is smooth in a neighborhood of discontinuity points of G, (A curve is called regular if $\sup\limits_{\zeta\in\Gamma, r>0} \frac{\nu B(\zeta,r)\cap\Gamma}{r} < \infty$, where $B(\zeta, r)$ is a circle of radius r with center at the point ζ).

In our case, i.e., in the boundary value problem from (10) the boundary condition is given on the unit circumference (and therefore in problem (10) we have no difficulty connected with the boundary curve). As for the function $G(\tau) = -\frac{z'(\tau)}{\overline{z'(\tau)}}$, in order that it be piecewise-continuous in the Hölder sense we will assume in what follows that the boundary of the domain D, where a solution of problem (5) is to be found, is a piecewise-Lyapunov curve with angular points $A_1, A_2, \dots, A_n$; moreover, the angle values at these points are equal to $\pi\nu_k$, $k = 1, 2, \dots, n$, $0 < \nu_k \leq 2$. We denote by $C^{1,\nu_1,\nu_2,\dots,\nu_n}_{A_1,A_2,\dots,A_n}$ the set of curves of this type.

3^0. In this subsection it is assumed for the sake of simplicity that $\Gamma \in C^{1,\nu}_{A}$, i.e., that the curve has one angular point A at which the value of the angle, internal with respect to D, is $\nu\pi$, $0 < \nu \leq 2$.

In this case we know that

$$z'(w) = (w-a)^{\nu-1}\, z_0(w), \quad a \in \gamma, \;\; z(a) = A, \tag{11}$$

where $z_0(w)$ belongs to the Hölder class $H(\gamma)$ and $z_0(w) \neq 0$ ([15], [16]; see also [5], p. 146).

Let us consider three cases i) $\nu > 1$; ii) $0 < \nu < 1$; iii) $\nu = 1$, separately.

i) $\nu > 1$. Assume that

$$X(w) = \begin{cases} z'(w)\,(w-a)^{-1}, & |w| < 1, \\ \overline{z'\Big(\dfrac{1}{\overline{w}}\Big)}\,(w-a)^{-1}, & |w| > 1. \end{cases} \tag{12}$$

Then the boundary condition from (10) can be written as

$$\frac{\Omega^+(\tau)}{X^+(\tau)} - \frac{\Omega^-(\tau)}{X^-(\tau)} = \frac{g(\tau)}{X^+(\tau)}, \quad \tau \in \gamma. \tag{13}$$

Let us show that under our assumptions, i.e., under $f \in Z(\Gamma)$ and $\nu > 1$ the function g belongs to the class $Z_{1,w}(\gamma)$, $w = |z'(\tau)|$.

In the first place it shoud be noted that in this case the function z' is bounded by virtue of (11), which the condition $f \in Z(\Gamma)$ implies that

$$\int_\gamma \Big| f(z(\tau)) \, \log\big(2 + |f(z(\tau))|\big)\Big| \, |z'(\tau)|\, d\nu < \infty.$$

Therefore

$$\int_\gamma |g(\tau)| \, \log\Big(2 + \Big|\frac{g(\tau)}{z'(\tau)}\Big|\Big) d\nu < \infty, \tag{14}$$

i.e., $g \in Z_{1,w}(\gamma)$, $w = |z'(\tau)|$. Hence in particular it follows that $g \in L(\Gamma)$. Further, we have

$$\begin{aligned} \int_\gamma |g(\tau)| \, \log\big(2 + |g(\tau)|\big) d\nu &= \int_\gamma |g(\tau)| \, \log\Big(2 + \Big|\frac{g(\tau)}{z'(\tau)}\Big|\, |z'(\tau)|\Big) d\nu \\ &\leq \int_\gamma |g(\tau)| \Big[\log\Big(2 + \Big|\frac{g(\tau)}{|z'(\tau)|}\Big|\Big) + \log\big(2 + |z'(\tau)|\big)\Big] d\nu. \end{aligned} \tag{15}$$

Since $g \in L(\Gamma)$ and z' is a bounded function, by (14), (15) we conclude that $g \in Z(\Gamma)$.

From (11) and (12) it follows that

$$X(w) = (w-a)^{v-2} z_0(w), \quad z_0 \neq 0, \quad z_0 \in H(\gamma), \tag{16}$$

for points w near to γ; moreover, under the assumption i) we have

$$1 < \nu - 2 \leq 0.$$

As usual (see, for example, [17], Ch. IV), by means of (11), (12) we conclude that all solutions of the boundary value problem from (10) are contained in the set of functions

$$\Omega(w) = \frac{X(w)}{2\pi i} \int_\gamma \frac{g(\tau)}{X^+(\tau)} \, \frac{d\tau}{\tau - w} + X(w)(Bw + C) = \Omega_g(w) + \Omega_0(w). \tag{17}$$

Here $\Omega_0(w)=X(w)(Bw+C)$, where B and C are arbitrary complex constants.

Like in [2] (see also [5], p. 159), the condition $u_0(z(w))=\operatorname{Re}\left[\frac{\Omega_0(w)}{z'(w)}\right]$ from (10) and equality $f=0$ (u_0 is a solution of problem (5)) give

$$u_0(z)=M\operatorname{Re}\left[\frac{a+w(z)}{a-w(z)}\right],\quad a=w(A), \tag{18}$$

where M is an arbitrary real constant.

Therefore $g\in Z(\Gamma)$, and near γ we have $|X^+(\tau)|\sim|\tau-a|^{\nu-2}$ (see (16)), where $(\nu-2)\in(-1,0]$. But by Theorem $1'$ from [9] the function

$$(Tg)(t)=\frac{X^+(t)}{2\pi i}\int_\gamma\frac{g(\tau)}{X^+(\tau)}\,\frac{d\tau}{\tau-t},\quad t\in\gamma,$$

belongs to the class $L(\gamma)$ when $g\in Z_{1,w}(\gamma)$, $\beta\in(-1,0]$. In our case $\beta=\nu-2$, $\beta\in(-1,0)$. Therefore $T_g\in L(\gamma)$. Hence it follows that the function

$$\Omega_g^+(t)=\frac12\,g(t)+\frac12\,(Tg)(t)$$

belongs to $L(\gamma)$. But $\Omega_g(w)=\frac{X(w)}{2\pi i}\int_\gamma\frac{g(\tau}{X^+(\tau)}\,\frac{d\tau}{\tau-w}$, $|w|<1$, belongs to some Hardy class H^η, $\eta>0$, because so do the functions $X(w)$ (which is obvious) and $\int_\gamma\frac{g(\tau)}{X^+(\tau)}\,\frac{d\tau}{\tau-w}$ (see, for example [5], p. 29). Since, as has been proved, $\Omega_g^+\in L(\gamma)$, we are able to apply Smirnov's theorem according to which Ω_g in this case is a function of the class H^1 (see, for example, [13], p. 393). But then the function Ω defined by (17) belongs to $E^1(\gamma)$ by virtue of Proposition 1. It is thus a particular solution of the problem of linear conjugation from (10). The function $\frac12\left[\Omega_g(w)+(\Omega_g(w))_*\right]$ will satisfy all conditions from (10). We eventually conclude that for $\nu>1$ all solutions of problem (5) are given by the equality

$$u(z)=u_f(z)+u_0(z), \tag{19}$$

where $u_0(z)$ is the function defined by (18), and

$$u_f(z)=\operatorname{Re}\left[\frac{1}{a-w(z)}\,\frac{1}{2\pi i}\int_\gamma\frac{f(z(\tau))}{\tau-w(z)}\Big(\tau-a-\frac{aw^2(z)(\tau-a)}{\tau}\Big)d\tau\right], \tag{20}$$

(cf. [5], p. 160).

ii) $-1<\nu-1<0$.

Let

$$Y(w)=\begin{cases}z'(w), & |w|<1,\\[4pt] \dfrac{}{z'\Big(\dfrac{1}{|w|}\Big)}, & |w|>1.\end{cases} \tag{21}$$

Then all possible solutions of problem (10) are contained in the set of functions

$$\Omega(w)=\frac{Y(w)}{2\pi i}\int_\gamma\frac{g(\tau)}{Y^+(\tau)}\,\frac{d\tau}{\tau-w}+CY(w),\quad c\in\mathbb{C}. \tag{22}$$

Like in [5] (pp. 157, 158), we establish that $C=0$ and therefore $u_0=0$. As for the nonhomogeneous problem, we cannot claim this time that $g\in Z(\gamma)$

because in this case z' is an unbounded function. Therefore we have to require of the function f to satisfy a more rigorous condition ensuring the inclusion $g \in Z(\gamma)$ than the requirement that $f \in Z(\gamma)$.

Let us assume the condition

$$f(t)\,\log\Big(2+\Big|\frac{f(t)}{w(t)-a}\Big|\Big) \in L(\Gamma), \quad a = w(A), \tag{23}$$

is fulfilled and show that under the assumptions ii) $-1 < \nu < 0$ and (23) the function g belongs to $Z(\gamma)$. For this, we have first to establish the validity of the inclusion

$$f \in L(\Gamma), \quad |f|\,\log(2+|f|) \in L(\Gamma).$$

Indeed, the first of these inclusions is obvious. Further, assuming that $M_f = \int_\Gamma |f(t)|\,\log\big(2+\frac{|f(t)|}{|w(t)-a|}\big)d\nu$, we have

$$\begin{aligned}\int_\Gamma |f|\,\log(2+|f|)d\nu &\le \int_{|w(t)-a|<1} |f|\,\log\Big(2+\Big|\frac{f(t)}{w(t)-a}\Big|\Big)d\nu \\ &+ \int_{|w(t)-a|>1} |f|\,\log\Big(2+\Big|\frac{f(t)(w(t)-a)}{w(t)-a}\Big|\Big)d\nu \\ &\le M_f + \int_{|w(t)-a|>1} |f|\,\log\Big(2+\Big|\frac{f(t)}{w(t)-a}\Big|\Big)d\nu \\ &+ \int_{|w(t)-a|>1} |f|\,\log\big(2+|w(t)-a|\big)d\nu \le 2M_f + \|f\|\,\mathrm{diam}\,\Gamma < \infty.\end{aligned}$$

Now we obtain

$$\begin{aligned}I &= \int_\Gamma |g(\tau)|\,\log\big(2+|g(\tau)|\big)d\nu = \int_\gamma |f(z(\tau))z'(\tau)\,\log\big(2+|f(z(\tau))z'(\tau)|\big)d\nu \\ &= \int_\gamma |f(z(\tau))z'(\tau)|\,\log\big(2+\big|f(z(\tau))|\tau-a|^{\nu-1}z_0(\tau)\big|\big)d\nu \\ &\le \int_\gamma \big|f(z(\tau))z'(\tau)\big|\,\Big[\log\Big(2+\frac{|f(z(\tau))|}{|\tau-a|^{1-\nu}}\Big)+\log\big(2+|z_0(\tau)|\big)\Big]d\nu.\end{aligned} \tag{24}$$

Here $0 < 1-\nu < 1$ and therefore

$$\frac{|f(z(\tau))|}{|\tau-a|^{1-\nu}} \le \begin{cases} \dfrac{|f(z(\tau))|}{|\tau-a|} & \text{for } \ |\tau-a|<1, \\ |f(z(\tau))| & \text{for } \ |\tau-a|\ge 1. \end{cases}$$

Thus

$$\log\Big(2+\frac{|f(z(\tau))|}{|\tau-a|^{1-\nu}}\Big) \le \log\Big(2+\frac{|f(z(\tau))|}{|\tau-a|}\Big) + \log\big(2+|f(z(\tau))|\big), \quad \tau\in\gamma.$$

Moreover, $z_0(\tau)$ is an bounded function (and it is assumed that $M = \max_{\tau\in\gamma}|z_0(\tau)|$). Therefore (24) implies

$$I = \int_\Gamma |g(\tau)|\,\log\big(2+|g(\tau)|\big)d\nu \le \int_\gamma |f(z(\tau))|\,\log\Big(2+\frac{|f(z(\tau))|}{|\tau-a|}\Big)|z'(\tau)|d\nu$$

$$+\int_\gamma |f(z(\tau))z'(\tau)| \log\big(2+|f(z(\tau))|d\nu + M_1\int_\gamma \big|f(z(\tau))z'(\tau)\big|d\nu,$$

$$M_1=\log(2+M).$$

Hence

$$I\le M_f+\big\|f\,\log(2+|f|)\big\|_{L(\gamma)}+M_1\|f\|_{L(\Gamma)}<\infty.$$

Thus $g\in Z(\gamma)$ and, like above, we conclude that the function

$$\Omega_g(w)=\frac{Y(w)}{2\pi i}\int_\gamma \frac{g(\tau)}{Y^+(\tau)}\,\frac{d\tau}{\tau-w}$$

belongs to $\widetilde{E}^{\,1}(\gamma)$. Since in the considered case the problem of linear conjugation from(10) has a unique solution, we obtain $(\Omega_g(w))_*=\Omega_g(w)$ and thus this time a solution of problem (5) is given by the equality

$$u(z)=\mathrm{Re}\left[\frac{1}{2\pi i}\int_\gamma \frac{f(z(\tau))}{\tau}\,\frac{\tau+w(z)}{\tau-w(t)}d\tau\right] \tag{25}$$

iii) $\nu=1$. In this case Γ is a Lyapunov curve, i.e., $z'(w)$ is a nonzero Hölder function and from the condition $f\in Z(\Gamma)$ it immediately follows that $g\in Z(\gamma)$. Due to this fact the function $u(z)$ defined by (25) gives an unique solution of problem (5). But in the considered case we could avoid the requirement that f belong to $Z(\gamma)$. The matter is that $u(z)$ is the wanted solution even if it is only assumed that the function

$$\Big(S_\Gamma\frac{f}{\tau}\Big)(t)=\frac{1}{\pi i}\int_\gamma \frac{f(z(\tau))}{\tau}\,\frac{d\tau}{\tau-t}$$

belongs to $L(\gamma)$. For this, in turn, it is sufficient to have $S_\Gamma f\in L(\Gamma)$.

The arguments given above result in

Theorem 1. *Let Γ be a simple piecewise-Lyapunov closed curve with one angular point A, with angle value $\nu\pi$, $0<\nu\le 2$, and bounding the finite domain D, $z=z(w)$ be a conformal mapping of the unit circle onto D, and $w=w(z)$ be an inverse function to it, and $w(A)=a$.*

Then:

i) *if $1<\nu\le 2$ and $f\in Z(\Gamma)$, then the Dirichlet problem* (5) *is solvable in the class $e^1(D)$ and has a general solution which depends on one real parameter and is given by equalities* (19), (18), (20), (12)*;*

ii) *if $0<\nu<1$ and*

$$f(t)\,\log\Big(2+\Big|\frac{f(t)}{w(t)-a}\Big|\Big)\in L(\Gamma) \tag{23}$$

then problem (5) *is uniquely solvable and the solution is given by* (25)*;*

iii) *if $\nu=1$, $f\in L(\Gamma)$ and $S_\Gamma(f)\in L(\Gamma)$, the the problem is uniquely solvable and the solution is again given by* (25).

Remark 1. Statement iii) of Theorem 1 can also be formulated as follows: if Γ is a Lyapunov curve, f and $S_\Gamma f$ belong to $L(\Gamma)$, then the Dirichlet problem (5) has, in the class $e^1(D)$, a unique solution given by (25).

Remark 2. From the arguments given above it easily follows that for $0 < \nu < 1$, in order that problem (5) be solvable it is necessary and sufficient that the function $\big(Y^+S_\gamma\frac{f}{Y^+}\big)(t)$ belong to the class $L(\gamma)$. As has been proved above, this condition is fulfilled under assumption (23).

4^0. We will now give some conditions which provide the fulfilment of inclusion (23).

Proposition 2. *If* $\Gamma \in C^{1,\nu}_A$, $\nu \in (0,1)$ *and* $a = w(A)$, *then condition* (23) *is equivalent to the collection of conditions:*

$$a)\ f \in Z(\Gamma), \quad b)\ |f(t)| \log\Big(2 + \frac{1}{|w(t)-a|}\Big) \in L(\Gamma). \tag{23_1}$$

Proof. As was shown in considering the case ii), when condition (23) is fulfilled, we have $f \in Z(\Gamma)$. Let us show that the condition b) is fulfilled, too.

We have

$$\int_{|f(t)|>1} |f(t)| \log\Big(2 + \frac{1}{|w(t)-a|}\Big) d\nu \le \int_{|f(t)|>1} |f(t)| \Big(\log\Big(2 + \frac{|f(t)|}{|w(t)-a|}\Big) + \log\Big(2 + \frac{1}{|f(t)|}\Big)\Big) d\nu, \tag{26}$$

But if $|f| > 1$, then $\log\big(2 + \frac{1}{|f(t)|}\big) \le \log\frac{1}{|f|} + \log 3 \le \frac{M}{|f|^\varepsilon} + \log 3$ for some $\varepsilon \in (0,1)$. Therefore (26) implies

$$\int_{|f(t)|>1} |f(t)| \log\Big(2 + \frac{1}{|w(t)-a|}\Big| d\nu < \infty. \tag{27}$$

Further,

$$\int_{|f(t)|\le 1} |f| \log\Big(2 + \frac{1}{|w(t)-a|}\Big) d\nu \le \int_\Gamma \log\Big(2 + \frac{1}{|w(t)-a|}\Big) d\nu$$
$$= \int_\gamma |z'(\tau)| \log\Big(2 + \frac{1}{|\tau-a|}\Big) |d\tau| \le \int_\gamma \frac{M}{|\tau-a|^{1-\nu-\varepsilon}} |d\tau| < \infty. \tag{28}$$

From (27) and (28) it follows that b) is valid.

Let us show that the assumptions a)–b) lead to (23). This immediately follows from the inequality

$$|f(t)| \log\Big(2 + \frac{|f(t)|}{|w(t)-a|}\Big) \le |f(t)| \log\big(2 + |f|\big) + |f| \log\Big(2 + \frac{1}{|w(t)-a|}\Big). \quad \square$$

Proposition 3. *For* $0 < \nu < 1$, *in Theorem* 1 *condition* (23) *together with conditions* (23_1) *can be replaced by the condition*

$$|f(t)| \log\Big(2 + \frac{|f(t)|}{|t-A|}\Big) \in L(\Gamma) \tag{23_2}$$

or by the collection of conditions

$$f \in Z(\Gamma), \quad |f(t)| \log\Big(2 + \frac{1}{|t-A|}\Big) \in L(\Gamma). \tag{23_3}$$

Proof. We have

$$|w(t)-a| = |w(t)-w(A)| = \big|(t-a)^{\frac{1}{\nu}-1}\, w_0(t)\big|, \quad w_0 \in H(\Gamma), \quad w_0 \neq 0$$

(see [15] and also [5], pp. 153–5). Hence it follows that

$$\log\Big(2+\frac{1}{|w(t)-a|}\Big) \leq M_1 + \log\frac{1}{|t-A|^{\frac{1}{\nu}-1}} = M_1 + M_2 \log\frac{1}{|t-A|}.$$

Moreover,

$$\log\Big(2+\frac{|f(t)|}{|w(t)-a|}\Big) \leq M_1 + \log\Big(2+\frac{|f(t)|}{|t-A|}\Big)$$
$$+\log\Big(2+\frac{|t-A|}{|w(t)-a|}\Big) \leq \log\Big(2+\frac{|f(t)|}{|t-A|}\Big) + M_2\log\Big(\frac{1}{|t-A|}\Big).$$

Taking into account that the function $\varphi(t) = \log\frac{1}{|t-A|}$ is summable on the considered curve, the above inequalities imply that Proposition 3 is valid. □

5^0. By the results of Subsections 3^0–5^0 we obtain

Theorem 2. *Let* $\Gamma \in C^{1,\nu_1,\nu_2,\dots,\nu_n}_{A_1,A_2,\dots,A_n}$, $0<\nu_k\leq 2$, $k=\overline{1,n}$. *Let us renumber the point* A_k *so as to have* $1<\nu_1\leq 2$, $1<\nu_2\leq 2,\dots,1<\nu_j\leq 2$, $0<\nu_{j+1}<1,\dots,0<\nu_n<1$. *Assume that* f *satisfies the conditions*

$$f\in Z(\Gamma), \quad f(t)\log\Big(2+\frac{|f(t)|}{\prod_{k=j+1}^{n}|t-A_k|}\Big)\in L(\Gamma). \tag{29}$$

Then problem (5) *is solvable and its general solution is given by the equality*

$$u(z) = u_f(z) + \sum_{k=1}^{j} M_k \,\mathrm{Re}\Big[\frac{a_k+w(z)}{a_k-w(z)}\Big], \quad a_k = w(A_k),$$

where

$$u_f(z) = \mathrm{Re}\Big[\frac{1}{\rho(w(z))}\Big(\frac{1}{2\pi i}\int_\gamma \frac{f(z(\tau))\rho(\tau)}{\tau-w(z)}\,d\tau$$
$$+\frac{(-1)^j w^{j+1}(z)}{2\pi i}\prod_{k=1}^{j} A_k \int_\gamma \frac{f(z(\tau))\overline{\rho(\tau)}}{\tau(\tau-w(z))}\,d\tau\Big)\Big],$$

and $\rho(w(z)) = \prod_{k=1}^{j}(w(z)-a_k)$, *if* $j\neq 0$ *and* $\rho(w(z))=1$, *if* $j=0$.

6^0. In this subsection we consider the Dirichlet problem in the weight class $e^1_\rho(D)$, $\rho(t)=|w'(t)|$, where $w=w(t)$ is the function mapping conformally the domain D onto the unit circle. If D is bounded by a piecewise-Lyapunov curve with one angular point with the angle value $\nu\pi$, $0<\nu\leq 2$, then $|w'(t)|\sim |t-A|^{\frac{1}{\nu}-1}$. Recall that

$$e^1_\rho = \Big\{u : u=\mathrm{Re}\,\Phi,\ \Phi\in E^1(D),\ \int_\Gamma |\Phi^+(t)|\log\Big(2+\Big|\frac{\Phi^+(t)}{\rho(t)}\Big|\Big)d\nu<\infty.$$

Thus, in the finite domain D bounded by a simple piecewise-Lyapunov closed curve Γ from the class $C_A^{1,\nu}$, $0<\nu\le 2$, it is required to define the function u satisfying the conditions

$$\begin{cases}\Delta u=0, \quad u\in e_\rho^1(D), \quad \rho=|w'(t)|\sim|t-A|^{\frac{1}{\nu}-1},\\ u\big|_\Gamma=f, \quad f\in Z_\rho^1(\Gamma).\end{cases} \tag{30}$$

By virtue of the definition of Smirnov and Zygmund classes it immediately follows that: i) the analytic function $\Psi(w)=\Phi(z(w))z'(w)$ belongs to the Hardy class H^1, while Ψ^+ to the class $Z(\gamma)$; ii) $g(\tau)=2f(z(\tau))z'(\tau)\in Z(\gamma)$. Hence, as above, we conclude that the function Ω defined by (8) belongs to $\widetilde{E}'(\Gamma)$. Therefore, the problem posed again reduces to problem (10) already considered. It is not difficult to verify that if Ω is a solution of the latter problem, then the function $u(z)=\mathrm{Re}[\Omega(w(z)/w'(z)]$ is a solution of problem (30). Since for any $f\in Z_\rho(\Gamma)$ and arbitrary ν, the function g belong to $Z(\gamma)$, we do not have to impose an additional restriction on f and, due to the reasoning given above, we conclude that our next statement is valid.

Proposition 4. *If $\Gamma\in C_A^{1,\nu}$, $0<\nu\le 2$, then:* i) *for $\nu\in(1,2]$ problem* (30) *is solvable and the set of its solution given by equality* (19) *depends on one real parameter;* ii) *for $\nu\in(0,1]$ the problem is uniquely solvable and the solution is given by equality* (25)*.*

A statement analogous to Theorem 2 holds true in the case of a curve with several angular points.

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