GEORGIAN MATHEMATICAL JOURNAL: Vol. 2, No. 2, 1995, 211-224

THE CAUCHY–NICOLETTI PROBLEM WITH POLES

T. WERNER

Abstract. The Cauchy–Nicoletti boundary value problem for a system of ordinary differential equations with pole-type singularities is investigated. The conditions of the existence, uniqueness, and non-uniqueness of a solution in the class of continuously differentiable functions are given. The classical Banach contraction principle is combined with a special transformation of the original problem.

Introduction

This paper deals with the Cauchy–Nicoletti problem for a system of differential equations with poles, i.e., with the problem

$$(t-a_i)^{r_i} x_i' = \left(\sum_{j=0}^{r_i-1} A_{i,j}\cdot(t-a_i)^j\right) x_i + \\ + f_i(t,x_1,\dots,x_n) + g_i(t), \quad t\in I_i, \tag{1}$$

$$x_i(a_i)=0, \qquad i=1,\dots,n, \tag{2}$$

where x_i are unknown vector variables, $A_{i,j}$ are constant matrices, f_i, g_i are given vector functions, a_i are given real numbers, and I_i are intervals of real numbers, all specified below.

The systematic research of singular problems for ordinary defferential equations (ODE) with nonsummable right-hand side was started by Czeczik [1]. The first monograph on some classes of singular boundary value problems was written by Kiguradze [2]. There are very general results on the Cauchy and especially the Cauchy–Nicoletti problems in this monograph (see [2], Part II). Recently many different properties and applications of singular problems have been investigated. In [3] three boundary value problems arising in gas dynamics are investigated. In [4] the question of when solutions of singular equations are bounded in some general sense is discussed.

1991 *Mathematics Subject Classification.* 34B10, 34B15.

Key words and phrases. Differential inequalities, global structure of solutions, oscillatory solution.

1072-947X/95/0300-0211$07.50/0 © 1995 Plenum Publishing Corporation

In [5] the existence theorems for nonlinear problems with a singularity "less" than pole of degree 1 but with a "great" nonlinearity are given. In [6] the existence, the uniqueness and also the absence of solutions of the two-point boundary value problem of second order with one pole of degree 2 is investigated. The investigation of the general local Cauchy initial value problem with poles at infinity was carried out by Konyuchova [7]. The boundary value problems with poles are investigated in different ways. For example, in [8] the topological method is used while in [9] formal fundamental solutions are calculated at poles using formal power series. This paper uses the Laurent expansion method similarly to [7,9]. The main result shows how the existence and uniqueness of the solution of the problem (1), (2) depends on the real parts of the eigenvalues of matrices $A_{i,j}$ in (1) if suitable restrictions are imposed on functions f_i and g_i.

Notation. The letters $\mathbb{R}$, $\mathbb{C}$, $\mathbb{N}$ denote the sets of real, complex, and natural numbers, respectively. Let $\mathbb{K} \in \{\mathbb{R}, \ \mathbb{C}\}$ and $N := \{1, \ldots, n\}$. $C[X, Y]$ (resp. $C^1[X, Y]$) stand for spaces of continuous and continuously differentiable mappings from the normed space X into the normed space Y. The scalar product of two vectors u, $v \in \mathbb{K}^m$ will be denoted by

$$(u|v) = \sum_{i=1}^{m} u_i \bar{v}_i.$$

For $i \in N$, $m_i \in \mathbb{N}$, and $x_i \in \mathbb{K}^{m_i}$ put $m = \sum_{i \in N} m_i$ and $x = (x_1, \ldots, x_n) \in \mathbb{K}^m$. For real $a_1 < a_2 < \cdots < a_n$ we denote $I_i = (a_1, a_i) \cup (a_i, a_n)$, $\quad I = [a_1, a_n]$. The symbols $A_{i,j}$ will stand for constant $(m_i \times m_i)$ matrices, $f_i \in C[I_i \times \mathbb{K}^m, \mathbb{K}^{m_i}]$, and $g_i \in C[I_i, \mathbb{K}^{m_i}]$. We will use the following norms:

$$\|x_i\| := \sqrt{(x_i|x_i)},$$
$$\|A\| := \sqrt{\sum_{i,j=1}^{m_i} |a_{i,j}|}$$

for any matrix $A = (a_{i,j})_{i,j=1}^{m_i}$,

$$\|\varphi_i\| := \max_{t \in I} \|\varphi_i(t)\|$$

for $\varphi_i \in C[I, \mathbb{K}^{m_i}]$ and

$$\|\varphi\| := \sum_{i \in N} \|\varphi_i\|$$

for $\varphi = (\varphi_1, \ldots, \varphi_n) \in C[I, \mathbb{K}^m]$.

Definitions and lemmas

Definition 1. We say that

i) the function $\varphi = (\varphi_1, \dots, \varphi_n) \in C[I, \mathbb{K}^m]$ is a solution of system (1) if $\varphi_i \in C^1[I_i, \mathbb{K}^{m_i}]$ and if (1) is satisfied for $x_i = \varphi_i(t)$, $i = 1, \dots, n$;

ii) the solution φ of system (1) is a solution of problem (1), (2) if $\varphi_i(a_i) = 0$ $(i = 1, \dots, n)$.

Consider the homogeneous part of system (1):

$$(t-a_i)^{r_i} y_i' = \left(\sum_{j=0}^{r_i-1} A_{i,j} \cdot (t-a_i)^j \right) y_i, \quad t \in I_i,\ i = 1, \dots, n, \tag{3}$$

which consists of n independent subsystems.

The following lemma on local transformation is an easy reformulation of the basic lemma in [10].

Lemma 1. *There exist T_i, $S_i \in \mathbb{R}$, $S_1 = a_1$, $T_1 > a_1$, $S_n < a_n$, $T_n = a_n$, $S_i < a_i < T_i$, $i = 2, \dots, n-1$, and transformations*

$$z_i = P_i(t) y_i, \quad i = 1, \dots, n,\ t \in (S_i, T_i) \setminus \{a_i\}, \tag{4}$$

with continuously differentiable matrices $P_i(t)$, which transform the subsystems of (3) *into systems of the following special form:*

$$(t-a_i)^{r_i} z_i' = \left(\sum_{j=0}^{r_i-1} B_{i,j} \cdot (t-a_i)^j + C_i(t) \right) z_i,\ t \in (S_i, T_i) \setminus \{a_i\}, \tag{5}$$
$$i = 1, \dots, n,$$

where $B_{i,j}$ are quasidiagonal constant matrices with non-zero blocks $B^1_{i,j}, \dots, B^{k_i}_{i,j}$,

$$B^k_{i,0} = \begin{pmatrix} \lambda^k_{i0} & \gamma_i & 0 & \cdots & 0 & 0 \\ 0 & \lambda^k_{i0} & \gamma_i & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \lambda^k_{i0} & \gamma_i \\ 0 & 0 & 0 & \cdots & 0 & \lambda^k_{i0} \end{pmatrix}, \quad k = 1, \dots, k_i,$$

γ_i, $i = 1, \dots, n$, are arbitrary small pozitive numbers,

$$\operatorname{rank} B^k_{i,j} = \operatorname{rank} B^k_{i,0} \quad \textit{for}\ \ j \neq 0$$

and C_i $(i = 1, \dots, n)$ are continuous and bounded remainder terms such that there exist limits

$$\lim_{t \to a_i} \left\| \frac{C_i(t)}{(t-a_i)^{r_i}} \right\| < +\infty, \quad i = 1, \dots, n.$$

Remark 1. As the proof of the basic lemma in [10] shows, the transformation matrix-functions $P_i(t)$, $i = 1, \dots, n$, can be constructed in the polynomial form

$$P_i(t) = \sum_{j=0}^{r_i - 1} P_{i,j} \cdot (t - a_i)^j$$

with the constant matrices $P_{i,j}$. It is the classical result that the matrices P_{i0} transforming A_{i0} to the modified Jordan matrices are regular (see, for example, [11]).

Definition 2. We say that system (1) satisfies the condition

CO1: if $f_i(t, 0, \dots, 0) \equiv 0$ on the interval I and if there exist functions $\mu_i \in C[I_i, \mathbb{R}^+]$ such that

$$\|f_i(t,x) - f_i(t,\tilde{x})\| \leqq \mu_i(t)\|x - \tilde{x}\|, \quad \text{for each } (t,x),(t,\tilde{x}) \in I_i \times \mathbb{K}^m$$

and

$$\int_{I_i} \mu_i(t)|t - a_i|^{-r_i} dt = M_i < \infty, \quad i = 1, \dots, n;$$

CO2: if

$$\int_{I_i} \|g_i(t)\| \; |t - a_i|^{-r_i} dt = G_i < \infty, \quad i = 1, \dots, n.$$

Definition 3. We say that the ith subsystem of the transformed system (5)

$$(t - a_i)^{r_i} z_i' = \left(\sum_{j=0}^{r_i - 1} B_{i,j} \cdot (t - a_i)^j + C_i(t) \right) z_i, \; t \in (S_i, T_i) \setminus \{a_i\}$$

satisfies the condition

CO3: if all eigenvalues of the matrix $B_{i,0}$ have nonpositive real parts and those of them which lie on the imaginary axis of $\mathbb{C}$ are simple, i.e.,

$$\operatorname{Re} \lambda_{i0}^k \leqq 0,$$

and if $\operatorname{Re} \lambda_{i0}^k = 0$ then $\operatorname{mult} \lambda_{i0}^k = 1, \quad l = 1, \dots, k_i$;

CO4: if for each k such that λ_{i0}^k is simple and $\operatorname{Re} \lambda_{i0}^k = 0$ there exists $j_k \in \{1, \dots, r_i - 1\}$ such that

$$\operatorname{Re} \lambda_{i0}^k = \operatorname{Re} B_{i,1}^k = \dots = \operatorname{Re} B_{i,j_k - 1}^k = 0, \; \operatorname{Re} B_{i,j_k}^k > 0; \qquad (6)$$

CO5: if for each k such that λ_{i0}^k is simple and $\operatorname{Re} \lambda_{i0}^k = 0$ there is no $j_k \in \{1, \dots, r_i - 1\}$ with property (6);

CO3': if all eigenvalues of the matrix $B_{i,0}$ have nonnegative real parts and those which lie on the imaginary axis of $\mathbb{C}$ are simple, i.e.,

$$\operatorname{Re} \lambda_{i0}^k \geqq 0,$$

and if

$$\operatorname{Re} \lambda_{i0}^k = 0 \text{ then } \operatorname{mult} \lambda_{i0}^k = 1, \quad l = 1, \dots, k_i;$$

CO4': if for each k such that λ_{i0}^k is simple and $\operatorname{Re} \lambda_{i0}^k = 0$ there exists $j_k \in \{1, \dots, r_i - 1\}$ such that

$$\operatorname{Re} \lambda_{i0}^k = \operatorname{Re} B_{i,1}^k = \dots = \operatorname{Re} B_{i,j_k-1}^k = 0, \ \operatorname{Re} B_{i,j_k}^k < 0; \tag{7}$$

CO5': if for each k such that λ_{i0}^k is simple and $\operatorname{Re} \lambda_{i0}^k = 0$ there is no $j_k \in \{1, \dots, r_i - 1\}$ with property (7).

Definition 4. Let system (1) satisfy conditions **CO1** and **CO2**. We say that the ith equation of (1)

$$(t - a_i)^{r_i} x_i' = \left(\sum_{j=0}^{r_i - 1} A_{i,j} \cdot (t - a_i)^j \right) x_i + f_i(t, x) + g_i(t), \quad t \in I_i,$$

has:

a) property U_R,
b) property $\overline{U}_R$,
c) property U_L,
d) property $\overline{U}_L$,

if its transformed homogeneous part, i.e., the ith subsystem of (5) satisfies the conditions

a) **CO3** $\wedge$ **CO5**,
b) **CO3'** $\wedge$ **CO4**,
c) (**CO3** for r_i odd $\vee$ **CO3'** for r_i even) $\wedge$ (**CO5** with j_k odd $\vee$ **CO5'** with j_k even),
d) (**CO3'** for r_i odd $\vee$ **CO3** for r_i even) $\wedge$ (**CO4** with j_k odd $\vee$ **CO4'** with j_k even).

The following lemma summarizes some local topics of [7].

Lemma 2. *Let the ith equation of* (1) *have property U_R for some fixed $i \in \{1, \dots, n-1\}$ [resp. have property U_L for some fixed $i \in \{2, \dots, n\}$]. Then for T_i (resp. S_i) sufficiently close to a_i ($T_i > a_i$, resp. $S_i < a_i$) there exist constants L_i^+ (resp. L_i^-) such that for any fundamental matrix Ψ_i of the ith transformed subsystem of* (5) *we have*

$$\|\Psi_i(t) \Psi_i^{-1}(s)\| \leqq L_i^+, \quad \textit{for } \ a_i < s \leqq t \leqq T_i,$$

resp.

$$\|\Psi_i(t)\Psi_i^{-1}(s)\| \leqq L_i^-, \quad \textit{for } S_i \leqq t \leqq s < a_i.$$

Moreover, there are only trivial solutions of the ith subsystem of (5) *on the intervals* $(a_i, T_i]$ (*resp.* $[S_i, a_i)$) *which vanish at the singular point* a_i.

Corollary 1. *Let the assumptions of Lemma 2 hold. Then there exists a constant K_i^+ (resp. K_i^-) such that for any fundamental matrix Φ_i of the ith subsystem of* (3) *we have*

$$\|\Phi_i(t)\Phi_i^{-1}(s)\| \leqq K_i^+, \quad \textit{for } a_i < s \leqq t \leqq a_n,$$

resp.

$$\|\Phi_i(t)\Phi_i^{-1}(s)\| \leqq K_i^-, \quad \textit{for } a_1 \leqq t \leqq s < a_i.$$

Moreover, there are only trivial solutions of the ith subsystem of (3) *on the intervals* (a_i, a_n) [*resp.* (a_1, a_i)] *which vanish at the singular point* a_i.

Proof. Lemma 2 implies that

$$\|\Phi_i(t)\Phi_i^{-1}(s)\| \leqq \|P_i^{-1}(t)\| \cdot \|P_i(s)\| L_i^+, \quad \text{for } a_i < s \leqq t \leqq T_i,$$

resp.

$$\|\Phi_i(t)\Phi_i^{-1}(s)\| \leqq \|P_i^{-1}(t)\| \cdot \|P_i(s)\| L_i^-, \quad \text{for } S_i \leqq t \leqq s < a_i,$$

where the terms on the right-hand sides are bounded for T_i (resp. S_i) sufficiently close to a_i, because P_{i0} in Remark 1 is regular. For each fixed s the columns of $\Phi_i(t)\Phi_i^{-1}(s)$ are the solutions of the ith subsystem of (3) with their norms bounded by $\sup_{t\in(a_i,T_i]} \|P_i^{-1}(t)\| \cdot \|P_i(s)\| L_i^+ =: Q_i^+$ (resp. $\sup_{t\in[S_i,a_i)} \|P_i^{-1}(t)\| \cdot \|P_i(s)\| L_i^- =: Q_i^-$) at some point of the interval $[T_i, a_n)$ (resp. $(a_1, S_i]$). Since $\| \sum_{j=0}^{r_i-1} A_{i,j} \cdot (t-a_i)^{j-r_i} \|$ is bounded on the interval $[T_i, a_n]$ (resp. $[a_1, S_i]$), we have

$$\|\varphi_i(t)\| \leqq Q_i^+ e^{A_i^+(a_n-T_i)} \text{ resp. } Q_i^- e^{A_i^-(S_i-a_1)}$$

for any column φ_i of $\Phi_i(t)\Phi_i^{-1}(s)$. Here

$$A_i^+ \text{ denote } \sup \| \sum_{j=0}^{r_i-1} A_{i,j} \cdot (t-a_i)^{j-r_i} \|, \quad t \in [T_i, a_n],$$

resp.

$$A_i^- \text{ denote } \sup \| \sum_{j=0}^{r_i-1} A_{i,j} \cdot (t-a_i)^{j-r_i} \|, \quad t \in [a_1, S_i].$$

Consequently,

$$\|\Phi_i(t)\Phi_i^{-1}(s)\| \leqq \sqrt{m_i} Q_i^+ e^{A_i^+(a_n-T_i)} =: K_i^+ \ (\text{resp.} \sqrt{m_i} Q_i^- e^{A_i^-(S_i-a_1)} =: K_i^-)$$

for $a_i < s \leqq t \leqq a_n$ (resp. $a_1 \leqq t \leqq s < a_i$). The nonexistence of any nontrivial solution vanishing at the point a_i directly follows, as in Lemma 2, from the regularity of P_{i0} and from the uniqueness of solutions of the ith subsystem of (3) on the intervals $[T_i, a_n]$ (resp. $[a_1, S_i]$). □

Lemma 3. *Let the ith equation of* (1) *has property U_R for some fixed $i \in \{1, \dots, n-1\}$ (resp. has property U_L for some fixed $i \in \{2, \dots, n\}$). Then for any $(n-1)$-tuple of functions $\varphi_j \in C[I, \mathbb{K}^{m_i}]$, $j = 1, \dots, n$, $j \neq i$, all solutions of the equation*

$$(t-a_i)^{r_i} x_i' = \left(\sum_{j=0}^{r_i - 1} A_{i,j} \cdot (t-a_i)^j \right) x_i +$$
$$+ f_i(t, \varphi_1(t), \dots, \varphi_{i-1}(t), x_i, \varphi_{i+1}(t), \dots, \varphi_n(t)) + g_i(t),\ t \in I_i, \tag{8}$$

satisfy the ith subcondition of (2), *i.e., $x_i(a_i) = 0$.*

Proof. Without loss of generality, we can consider only the case of property U_R. Let us consider the mentioned ith equation of (1) transformed by the the ith transformation from (4):

$$(t-a_i)^{r_i} z_i' = \left(\sum_{j=0}^{r_i - 1} B_{i,j} \cdot (t-a_i)^j \right) z_i +$$
$$+ \tilde{f}_i(t, \varphi_1(t), \dots, \varphi_{i-1}(t), z_i, \varphi_{i+1}(t), \dots, \varphi_n(t)) + \tilde{g}_i(t), \quad t \in (a_i, T_i), \tag{9}$$

where $\tilde{f}_i$ contain also the remainder term $C_i(t) z_i$ from (5):

$$\tilde{f}_i(t, \varphi_1(t), \dots, \varphi_{i-1}(t), z_i, \varphi_{i+1}(t), \dots, \varphi_n(t)) \equiv$$
$$\equiv P_i(t) f_i(t, \varphi_1(t), \dots, \varphi_{i-1}(t), P_i^{-1} z_i, \varphi_{i+1}(t), \dots, \varphi_n(t)) + C_i(t) z_i$$

and

$$\tilde{g}_i(t) \equiv P_i(t) g_i(t).$$

Due to the behavior of the ith transformation from (4), the continuity of f_i, g_i and conditions **CO1** and **CO2** are invariant with respect to this transformation. Let ψ_i be any solution of (9). If T_i is sufficiently close to

a_i, then for the derivative of the norm of ψ_i we have

$$\frac{d}{dt}\|\psi_i(t)\|^2 = 2\operatorname{Re}(\psi_i(t)| \left(\sum_{j=0}^{r_i-1} B_{i,j}(t-a_i)^{j-r_i}\right)\psi_i(t) +$$
$$+\tilde{f}_i(t,\varphi_1(t),\dots,\varphi_{i-1}(t),\psi_i(t),\varphi_{i+1}(t),\dots,\varphi_n(t)) + \tilde{g}_i(t)) \geqq$$
$$\geqq \left((2\operatorname{Re}\lambda - \epsilon)(t-a_i)^{L-r_i} - \tilde{\mu}_i(t)\left(\sum_{j\in N\setminus\{i\}}\|\varphi_j(t)\|\right)\right)\cdot\|\psi_i(t)\|^2 -$$
$$-\|\tilde{g}_i(t)\|\cdot\|\psi_i(t)\|,$$

where $\epsilon > 0$ is arbitrarily small, the term $\tilde{\mu}_i(t)\cdot\left(\sum_{j\in N\setminus\{i\}}\|\varphi_j(t)\|\right)$ is bounded on the considered interval, $L \leqq r_i - 1$ is a greater value of j_k-s in the condition **CO4** or $L = 0$ if no j_k exists, and $\operatorname{Re}\lambda$ is the smallest value of $\operatorname{Re}\lambda^k_{j_k}$ for $j_k = L$, i.e.,

$$\operatorname{Re}\lambda = \min\{\operatorname{Re}\lambda^k_{j_k},\ j_k = L\} > 0.$$

Consequently,

$$\|\psi_i(t)\|' \geqq \frac{\operatorname{Re}\lambda}{2}(t-a_i)^{L-r_i}\|\psi_i(t)\| - \|\tilde{g}_i\| \geqq \frac{\operatorname{Re}\lambda}{2(t-a_i)}\|\psi_i(t)\| - \|\tilde{g}_i\|$$

for T_i sufficiently close to a_i; hence

$$\|\psi_i(t)\| \leqq \left(\Big(\frac{\|\psi_i(T_i)\|}{(T_i-a_i)^{\frac{\operatorname{Re}\lambda}{2}}} + \frac{(T_i-a_i)^{1-\frac{\operatorname{Re}\lambda}{2}}}{1-\frac{\operatorname{Re}\lambda}{2}}\|\tilde{g}_i\|\Big)(t-a_i)^{\frac{\operatorname{Re}\lambda}{2}} -\right.$$
$$\left. -\frac{\|\tilde{g}_i\|}{1-\frac{\operatorname{Re}\lambda}{2}}(t-a_i)\right),$$

where the last term tends to zero as $t \to a_i+$. This implies that each solution $\varphi_i = P_i^{-1}\psi_i$ of (8) on (a_i, T_i) vanishes at the point a_i, too. □

Corollary 2. *Let the assumptions of Lemma* 3 *hold. Then there exists a constant* $K_i^+ > 0$ (*resp.* $K_i^- > 0$) *such that any fundamental matrix* Φ_i *of the* i*th subsystem of* (3) *satisfies*

$$\|\Phi_i(t)\Phi_i^{-1}(s)\| \leqq K_i^+ \ \ \textit{resp.}\ \ K_i^-$$

for

$$a_i < t \leqq s < a_n \quad \textit{resp.}\ \ a_1 < s \leqq t < a_i.$$

Proof. Each column of $\Phi_i(t)\Phi_i^{-1}(s)$ is a solution of the ith equation of (1) for $f_i \equiv g_i \equiv 0$ on some interval (a_i, T_i) (resp. (S_i, a_i)). Such a solution vanishes at the point a_i and its extension on the interval (s, a_n) (resp. (a_1, s)) is bounded because $\sum\limits_{j=0}^{r_i-1} A_{i,j} \cdot (t-a_i)^{j-r_i}$ is bounded there, too. Thus $\|\Phi_i(t)\Phi_i^{-1}(s)\|$ is bounded on the interval $[a_i, a_n]$ (resp. $[a_1, a_i]$) (and vanishes at a_i), too. $\square$

Main results

Let us consider system (1). Denote by N_L and N_R the sets of all indices i for which the ith equation of (1) has property $\overline{U}_L$ or $\overline{U}_R$, respectively, and by N_L^0, N_R^0 the sets of all indices i for which the ith equation of (1) has property U_L or U_R, respectively. Then we have

Theorem 1. *Let*

$$\operatorname{card} N_L + \operatorname{card} N_R + \operatorname{card} N_L^0 + \operatorname{card} N_R^0 = 2(n-1)$$

and let the inequality

$$\sum_{i=1}^{n} K_i M_i < 1 \tag{10}$$

hold.

Then there exists just a $\left(\sum_{i\in N_L} m_i + \sum_{i \in N_R} m_i\right)$ *-parametric family of solutions of problem* (1), (2).

Proof. Let $\varphi = (\varphi_1, \ldots, \varphi_n)$ be a solution of problem (1), (2). Then its ith component φ_i can be written in one of the following forms. If $i \in N \setminus (N_L \cup N_R)$, then

$$\varphi_i(t) = \int\limits_{a_i}^{t} \Phi_i(t)\Phi_i^{-1}(s)[f_i(s, \varphi(s)) + g_i(s)](s-a_i)^{-r_i} ds, \quad t \in I; \tag{11}$$

if $i \in N \setminus (N_L^0 \cup N_R^0)$, then

$$\varphi_i(t) = \begin{cases} \Phi_i(t)\Phi_i^{-1}(a_1)\varphi_i(a_1) + \int_{a_1}^{t} \Phi_i(t)\Phi_i^{-1}(s)[f_i(s,\varphi(s))+ \\ \qquad +g_i(s)](s-a_i)^{-r_i} ds, \ \ t \in [a_1, a_i], \\ \Phi_i(t)\Phi_i^{-1}(a_n)\varphi_i(a_n) + \int_{a_n}^{t} \Phi_i(t)\Phi_i^{-1}(s)[f_i(s,\varphi(s))+ \\ \qquad +g_i(s)](s-a_i)^{-r_i} ds, \ \ t \in [a_i, a_n]; \end{cases} \tag{12}$$

if $i \in N_R^0 \cap (N \setminus N_L^0)$, then

$$\varphi_i(t) = \Phi_i(t)\Phi_i^{-1}(a_1)\varphi_i(a_1) + \\ + \int\limits_{a_1}^{t} \Phi_i(t)\Phi_i^{-1}(s)[f_i(s,\varphi(s)) + g_i(s)](s-a_i)^{-r_i}ds, \quad t \in I, \tag{13}$$

and, finally, if $i \in N_L^0 \cap (N \setminus N_R^0)$, then

$$\varphi_i(t) = \Phi_i(t)\Phi_i^{-1}(a_n)\varphi_i(a_n) + \\ + \int\limits_{a_n}^{t} \Phi_i(t)\Phi_i^{-1}(s)[f_i(s,\varphi(s)) + g_i(s)](s-a_i)^{-r_i}ds, \quad t \in I. \tag{14}$$

Here $\varphi_i(a_1) \in \mathbb{K}^{m_i}$, $i \in N_L$, $\varphi_i(a_n) \in \mathbb{K}^{m_i}$, $i \in N_R$, are arbitrary constants. Lemma 3 and Corollaries 1 and 2 ensure that the above integrations are correct. On the other hand, the solutions of the system of integral equations (11)–(14) are the solutions of problem (1), (2) which satisfy the boundary conditions

$$\begin{aligned} x_i(a_1) &= \varphi_i(a_1), \quad i \in N_L, \\ x_i(a_n) &= \varphi_i(a_n), \quad i \in N_R. \end{aligned} \tag{15}$$

Thus for any fixed values of $\varphi_i(a_1) \in \mathbb{K}^{m_i}$, $i \in N_L$, $\varphi_i(a_n) \in \mathbb{K}^{m_i}$, $i \in N_R$, problem (1), (2), (15) is equivalent to the system of integral equations (11)–(14). Define the integral operator F by means of the right sides of (11)–(14), which maps $C[I, \mathbb{K}^m]$ into itself and denote

$$\mathcal{I}_i(\zeta,\xi) := \int\limits_{\zeta}^{\xi} \Phi_i(t)\Phi_i^{-1}(s)[f_i(s,\varphi(s)) + g_i(s)](s-a_i)^{-r_i}ds.$$

Denote by $\mathcal{B}(c, R)$ a ball in the space $C[I, \mathbb{K}^m]$ with radius R and center at the fundamental solution of the homogeneous part of (1) $c = (c_1, \ldots, c_n)$, where

$$c_i(t) = \begin{cases} 0, & t \in I,\ i \in N \setminus (N_L \cup N_R), \\ \Phi_i(t)\Phi_i^{-1}(a_1)\varphi_i(a_1), & t \in I, \quad i \in N_L \setminus N_R, \\ \Phi_i(t)\Phi_i^{-1}(a_1)\varphi_i(a_1), & t \in [a_1, a_i],\ i \in N_L \cap N_R, \\ \Phi_i(t)\Phi_i^{-1}(a_n)\varphi_i(a_n), & t \in [a_i, a_n],\ i \in N_L \cap N_R, \\ \Phi_i(t)\Phi_i^{-1}(a_n)\varphi_i(a_n), & t \in I, \quad i \in N_R \setminus N_L. \end{cases}$$

For any $\varphi \in \mathcal{B}(c, R)$ we get

$$\begin{aligned}
\|F\varphi - c\| \leqq & \sum_{i \in N \setminus (N_L \cup N_R)} \sup_{t \in I_i} \|\mathcal{I}_i(a_i, t)\| + \\
+ \sum_{i \in (N_L \cap N_R)} & \max\{ \sup_{t \in (a_1, a_i)} \|\mathcal{I}_i(a_1, t)\|, \sup_{t \in (a_i, a_n)} \|\mathcal{I}_i(a_n, t)\|\} + \\
+ \sum_{i \in (N_L \setminus N_R)} & \sup_{t \in I_i} \|\mathcal{I}_i(a_1, t)\| + \sum_{i \in (N_R \setminus N_L)} \sup_{t \in I_i} \|\mathcal{I}_i(a_n, t)\| \leqq \\
& \leqq \sum_{i \in N} K_i(M_i \|\varphi\| + G_i) \leqq \\
& \leqq \left(\sum_{i \in N} K_i M_i \right) \|\varphi - c\| + \sum_{i \in N} K_i(M_i \|c\| + G_i). \qquad (16)
\end{aligned}$$

Since $\sum_{i \in N} K_i M_i < 1$, we can select a radius R such that

$$\left(1 - \sum_{i \in N} K_i M_i \right) R > \left(\sum_{i \in N} K_i(M_i \|c\| + G_i) \right).$$

The last inequality implies that the operator F maps the ball $\mathcal{B}(c, R)$ into itself.

Similarly, we obtain the estimate

$$\|F\varphi - F\tilde{\varphi}\| \leqq \left(\sum_{i \in N} K_i M_i \right) \|\varphi - \tilde{\varphi}\| \text{ for any pair } \varphi,\ \tilde{\varphi} \in \mathcal{B}(c, R); \qquad (17)$$

hence F is a contraction. The Banach theorem gives the existence of a unique solution of the system of integral equations (11)–(14) satisfying condition (15). This solution is simultaneously the solution of problem (1), (2) which satisfies the condition (15). The values $\varphi_i(a_1)$, $i \in N_L$, $\varphi_i(a_n)$, $i \in N_R$ occurring in (15) can be selected arbitrarily and the total dimension of (15) is $\sum_{i \in N_L} m_i + \sum_{i \in N_R} m_i$. $\square$

Remark 2. Condition (10) is substantial in view of the following example.

Example. Consider a linear problem

$$x_1' = -\frac{x_1}{t} + 2(x_1 - x_2),\ t \in (0, 1), \qquad (18)$$

$$x_2' = -\frac{x_2}{t - 1} + 2(x_1 - x_2),\ t \in (0, 1), \qquad (19)$$

$$x_1(0) = x_2(1) = 0. \qquad (20)$$

Equation (18) has property U_R at its singular point $a_1 = 0$ and equation (19) has property U_L at the singular point $a_2 = 1$. However, there exists a one-parametric system of solutions of problem (18), (19), (20)

$$x_1 = ct,\ x_2 = c(t-1),\ c \in \mathbb{K}$$

where c is arbitrary. This happens because condition (10) does not hold. In fact, we have

$$K_1 M_1 + K_2 M_2 \geqq 4 > 1$$

where

$$\begin{aligned}
&K_1 \geqq \sup_{0<s\leq t\leq 1} \frac{s}{t} = 1,\\
&K_2 \geqq \sup_{0\leq t\leq s<1} \frac{s-1}{t-1} = 1,\\
&\mu_1 \geqq 2, \quad \mu_2 \geqq 2\\
&\text{and so} \quad M_1 \geqq \int_0^1 2dt = 2, \quad M_2 \geqq 2.
\end{aligned}$$

The next Theorem 2 indicates the special case where condition (10) can be omitted.

Theorem 2. *Let*

$$\operatorname{card} N_L = \operatorname{card} N_R^0 = n-1,$$

or

$$\operatorname{card} N_L^0 = \operatorname{card} N_R = n-1.$$

Then there exists just a $\left(\sum_{i\in N_L} m_i\right)$*-parametric [resp. a* $\left(\sum_{i\in N_R} m_i\right)$*-parametric] family of solutions of problem* (1), (2).

Proof. The system of integral equations in the proof of Theorem 1 reduces to (13) in the first case or to (14) in the second case. Let us consider the first case and define a new norm in the space $C[I,\mathbb{K}^m]$:

$$\|\varphi\|_p := \max_{t\in I}\left(\|\varphi(t)\| e^{-p(t-a_1)}\right).$$

The following estimates hold:

$$\|\mathcal{I}_i(a_1,t)\| e^{-p(t-a_1)} \leqq$$

$$\leqq K_i\left(M_i \int_{a_1}^t \|\varphi(s)\| e^{-p(s-a_1)} e^{p(s-a_1)} ds + G_i\right) e^{-p(t-a_1)} \leqq$$

$$\leqq K_i \left(M_i \|\varphi\|_p \int\limits_{a_1}^{t} e^{p(s-a_1)} ds + G_i \right) e^{-p(t-a_1)} \leqq$$

$$\leqq K_i \left(M_i \frac{\|\varphi\|_p}{p} (e^{p(t-a_1)} - 1) + G_i \right) e^{-p(t-a_1)} \leqq$$

$$\leqq K_i \left(M_i \frac{\|\varphi\|_p}{p} + G_i \right), \quad i \in N.$$

So estimate (16) has the form

$$\|F\varphi - c\|_p \leqq \sum_{i \in N} K_i \left(M_i \frac{\|\varphi\|_p}{p} + G_i \right) \leqq$$

$$\leqq \frac{\sum_{i \in N} K_i M_i}{p} \|\varphi - c\|_p + \sum_{i \in N} K_i \left(M_i \frac{\|c\|_p}{p} + G_i \right).$$

Similarly we obtain the modification of estimate (17)

$$\|F\varphi - F\tilde{\varphi}\|_p \leqq \frac{\sum_{i \in N} K_i M_i}{p} \|\varphi - \tilde{\varphi}\|_p.$$

When we select p such that

$$p > \sum_{i \in N} K_i M_i,$$

the proof can be completed as that of Theorem 1. □

References

1. B. A. Czeczik, Investigation of systems of ordinary differential equations with a singularity. *Trudy Moskov. Matem. obsch.* **8**(1959), 155–197.

2. I. T. Kiguradze, Some singular boundary value problems for ordinary differential equations. (Russian) *Tbilisi University Press, Tbilisi*, 1975.

3. M. M. Ad'yutov, Yu. A. Klokov, and A. P. Mikhailov, Some problems for an ordinary differential equation that arise in gas dynamics. (Russian) *Differentsial'nye Uravneniya* **26**(1990), No. 7, 1107–1110, 1284; *English transl.: Differential Equations* **26**(1990), No. 7, 803–805.

4. J. Diblík, On existence of δ-bounded solutions of a nonhomogeneous linear system of differential equations. *Funkcial. Ekvac.* **34**(1991), No. 1, 1–18.

5. A. G. O'Farrel and D. O'Regan, Existence results for initial and boundary value problems. *Proc. Amer. Math. Soc.* **110**(1990), No. 3, 661–673.

6. G. P. Grizans, A boundary value problem for an equation with nonsummable singularity. (Russian) *Latv. Mat. Ezhegodnik* **29**(1985), 22–35.

7. N. B. Konyukhova, Singular Cauchy problems for systems of ordinary differential equations. (Russian) *Zh. Vychisl. Mat. i Mat. Fiz.* **23**(1983), No. 3, 629–645.

8. J. Diblík, The singular Cauchy-Nicoletti problem for system of two ordinary differential equations. *Math. Bohemica* **117**(1992), No. 1, 55–67.

9. E. Wagenführer, On the computation of formal fundamental solutions of linear differential equations at a singular point. *Analysis* **9**(1989), No. 4, 389–405.

10. E. S. Birger and N. B. Ljalikova, On determining the solutions of certain systems of ordinary differential equations with a given condition at infinity. (Russian) *Zh. Vychisl. Mat. i Mat. Fiz.* **5**(1965), 979–990; **3**(1966), 446–453.

11. W. Wasow, Asymptotic expansions for ordinary differential equations. *Wiley, New York,* 1965.

(Received 20.10.1993)

Author's address:
Department of Mathematics
Czech Academy of Sciences
Žižkova 22, 616 62 Brno
Czech Republic