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ON STRUCTURE OF SOLUTIONS OF A SYSTEM OF FOUR DIFFERENTIAL INEQUALITIES

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ABSTRACT. The aim of the paper is to study a global structure of solutions of four differential inequalities

$$\alpha_i y_i'(t) y_{i+1} \geq 0, \quad y_{i+1}(t) = 0 \Rightarrow y_i'(t) = 0, \quad i = 1,2,3,4,$$
$$\alpha_i \in \{-1,1\}, \alpha_1\alpha_2\alpha_3\alpha_4 = -1$$

with respect to their zeros. The structure of an oscillatory solution is described, and the number of points with trivial Cauchy conditions is investigated.

1. Introduction

The aim of this paper is to investigate the global structure with respect to zeros of solutions of the system of differential inequalities

$$\begin{gathered} \alpha_i y_i'(t) y_{i+1} \geq 0, \\ y_{i+1}(t) = 0 \Rightarrow y_i'(t) = 0, \quad i \in N_4,\ t \in J, \end{gathered} \tag{1}$$

where $J = (a,b),\ -\infty \leq a < b \leq \infty,\ y_5 = y_1,\ N_4 = \{1,2,3,4\}$

$$\alpha_i \in \{-1,1\}, \quad \alpha_1\alpha_2\alpha_3\alpha_4 = -1\,. \tag{2}$$

$y = (y_1, y_2, y_3, y_4)$ is called a solution of (1) if $y_i : J \to R$, $R = (-\infty, \infty)$ is locally absolutely continuous and (1) holds for all $t \in J$ such that y_i' exists.

Let us mention two special cases of (1) which are often studied; see, for example, [1-4] (and the references therein).

(a) A system of four differential equations

$$\begin{gathered} y_i' = f_i(t, y_1, y_2, y_3, y_4), \quad i \in N_4\,, \\ \alpha_i f_i(t, x_1, x_2, x_3, x_4) x_{i+1} \geq 0\,, \\ x_{i+1} = 0 \Rightarrow f_i(t, x_1, x_2, x_3, x_4) = 0 \quad \text{on} \quad D, \quad i \in N_4, \end{gathered} \tag{3}$$

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where (2) holds, $x_5 = x_1$, $f_i : D = R^5 \to R$ fulfills the local Carathéodory conditions, $i \in N_4$.

(b) A fourth-order differential equation with quasiderivatives

$$\begin{gathered} L_4 x(t) = f(t, L_0 x, L_1 x, L_2 x, L_3 x)\,, \\ f(t, x_1, x_2, x_3, x_4) x_1 \le 0, \quad f(t, 0, x_2, x_3, x_4) = 0, \end{gathered} \tag{4}$$

where $f : R^5 \to R$ fulfills the local Carathéodory conditions, $a_j : R \to R$ is continuous and positive, $j = 0, 1, 2, 3, 4$, and $L_j x$ is the jth quasiderivative of x : $L_0 x = a_0(t) x$, $L_i x = a_i(t)(L_{i-1} x)'$, $i \in N_4$.

By the use of the standard transformation we can see that (4) is a special case of (3): $y_j = L_{j-1} x, \quad j \in N_4$,

$$y_i' = \frac{y_{i+1}}{a_i(t)}, \quad i = 1, 2, 3, \quad y_4' = \frac{1}{a_4(t)} f(t, y_1, y_2, y_3)\,. \tag{5}$$

In [1] the structure of oscillatory solutions (defined in the usual sense) was studied for the fourth-order differential equation (4), $a_j \equiv 1$. It was shown that two different types of them can exist and their structure was described. For example, for every type the zeros of a solution y and its derivatives $y^{(i)}$, $i = 1, 2, 3$ are uniquely ordered. This information allows a more profound study of the asymptotic behavior. In [2] it was shown that the zeros of components of a solution of (1) (under further assumptions) are simple in some neighborhood of their cluster point (the zero τ of y_i, $i \in N_4$, is simple if $y_{i+1}(\tau) \neq 0$ holds).

In the present paper the above-mentioned results are generalized for (1).

Notation. Let y be a solution of (1). Put $Y_1 = y_1$, $Y_2 = \alpha_1 y_2$, $Y_3 = \alpha_1 \alpha_2 y_3$, $Y_4 = \alpha_1 \alpha_2 \alpha_3 y_4$, $Y_{i+4k} = Y_i$, $y_{i+4k} = y_i$, $i \in N_4$, $k \in Z = \{\dots, -1, 0, 1, \dots\}$.

Definition 1. Let $y : (a, b) \to R^4$ be a solution of (1). Then y is called trivial if $y_i(t) = 0$ in (a, b), $i \in N_4$. Let c be a point such that $c \in (a, b)$, $y_i(c) = 0$, $i \in N_4$, holds. Then c is called a Z-point of y.

Definition 2. Let $y : (a, b) \to R^4$ be a solution of (1). Then y has Property W if for every $i \in N_4$

(a) there exists at most one maximal bounded interval $J \subset (a, b)$ such that either

$$y_i(t) = y_{i+1}(t) = y_{i+2}(t) = 0, y_{i+3}(t) \neq 0, \quad t \in J,$$

or

$$y_i(t) = y_{i+2}(t) = 0, y_{i+1}(t) y_{i+3}(t) \neq 0, \quad t \in J\,;$$

(b) there exist at most two maximal bounded intervals $J, J_1 \subset (a, b)$, $J \cap J_1 = \emptyset$ such that $y_i(t) = y_{i+1}(t) = 0$, $y_{i+2}(t) y_{i+3}(t) \neq 0$ in $J \cup J_1$.

For the study of the structure of solutions of (1) we define the following types. Let $y : J = (a,b) \to R^4$.

Type I(s, $\bar{\mathbf{s}}$): For given $s \in Z \cup \{-\infty\}$, $\bar{s} \in Z \cup \{\infty\}$, $\bar{s} \geq s-1$ there exist $s_i, \bar{s}_i$ and the sequences $\{t_k^i\}$, $\{\bar{t}_k^i\}$, $k \in \{s_i, s_i+1, \dots, \bar{s}_i\}$, $i \in N_4$ such that $s_1 = s$, $\bar{s}_1 = \bar{s}$, $s_i \in \{s_1 - 1, s_1\}$, $s_j \geq s_{j-1}$, $\bar{s}_i \in \{\bar{s}_1 - 1, \bar{s}_1\}$, $\bar{s}_j \leq \bar{s}_{j+1}$, $j = 2,3,4$, $\bar{s}_5 = \bar{s}_1$ holds and for all admissible k we have

$$t_k^1 \leq \bar{t}_k^1 < t_k^4 \leq \bar{t}_k^4 < t_k^3 \leq \bar{t}_k^3 < t_k^2 \leq \bar{t}_k^2 < t_{k+1}^1 \leq \bar{t}_{k+1}^1 \,,$$

$$Y_i(t) = 0 \quad \text{for} \quad t \in [t_k^i, \bar{t}_k^i], Y_i(t) \neq 0 \quad \text{for} \quad t \in J - \bigcup_{k=s_i}^{\bar{s}_i} [t_k^i, \bar{t}_k^i] \,,$$

$$\begin{aligned} Y_j(t)Y_1(t) &> 0 \quad \text{for} \quad t \in (\bar{t}_k^1, t_k^j) \,, \\ &< 0 \quad \text{for} \quad t \in (\bar{t}_k^j, t_{k+1}^1), \ j = 2,3,4, \quad i \in N_4 \,. \end{aligned}$$

Moreover, for $i \in N_4$, $cd = -1$, where $c(d)$ is the sign of Y_i in some left (right) neighborhood of t_k^i $(\bar{t}_k^i)$. If $s = -\infty$ $(\bar{s} = \infty)$, then $\lim_{k \to -\infty} t_k^i = a$ $(\lim_{k \to \infty} t_k^i = b)$ holds.

Type II(s, $\bar{\mathbf{s}}$): For given $s \in Z \cup \{-\infty\}$, $\bar{s} \in Z \cup \{\infty\}$, $\bar{s} \geq s-1$ there exist $s_i, \bar{s}_i$ and the sequences $\{t_k^i\}$, $\{\bar{t}_k^i\}$, $k \in \{s_i, s_i+1, \dots, \bar{s}_i\}$, $i \in N_4$ such that $s_1 = s$, $\bar{s}_1 = \bar{s}$, $s_i \in \{s_1 - 1, s_1\}$, $s_j \leq s_{j-1}$, $\bar{s}_j \in \{\bar{s}_1 - 1, \bar{s}_1\}$, $\bar{s}_j \leq s_{j-1}$, $j = 2,3,4$, and for all admissible k

$$t_{k-1}^1 \leq \bar{t}_{k-1}^1 < t_k^2 \leq \bar{t}_k^2 < t_k^3 \leq \bar{t}_k^3 < t_k^4 \leq \bar{t}_k^4 < t_k^1 \leq \bar{t}_k^1 \,,$$

$$Y_i(t) = 0 \quad \text{for} \quad t \in [t_k^i, \bar{t}_k^i], \ Y_i(t) \neq 0 \quad \text{for} \quad t \in J - \bigcup_{k=s_i}^{\bar{s}_i} [t_k^i, \bar{t}_k^i] \,,$$

$$\begin{aligned} (-1)^{j+1} Y_j(t)Y_1(t) &> 0 \quad \text{for} \quad t \in (\bar{t}_{k-1}^1, t_k^j) \\ &< 0 \quad \text{for} \quad t \in (\bar{t}_k^j, t_k^1), \quad j = 2,3,4, \quad i \in N_4, \end{aligned}$$

hold. Moreover, if $i \in N_4$, $cd = -1$, where c (d) is the sign of Y_i in some left (right) neighborhood of t_k^i $(\bar{t}_k^i)$. If $s = -\infty (\bar{s} = \infty)$, then $\lim_{k \to -\infty} t_k^i = a$ $(\lim_{k \to \infty} t_k^i = b)$ holds.

Type III. There exist $j \in N_4$, $\tau \in \{-1, 1\}$ such that

$$\tau Y_j(t) \geq 0, \quad \tau c_i Y_{j+1}(t) > 0, \quad t \in J, \quad i = 1,2,3 \,,$$

$|Y_{j+k}|$ is nondecreasing, $|Y_{j+3}|$ is nonincreasing on J, $k = 0,1,2$, where $c_1 = 1$ for $j = 1,2,3$, $c_1 = -1$ for $j = 4$, $c_2 = 1$ for $j = 1,2$, $c_2 = -1$ for $j = 3,4$, $c_3 = 1$ for $j = 1$, $c_3 = -1$ for $j = 2,3,4$.

Type IV. There exist $j \in N_4$, $\tau \in \{-1, 1\}$ such that

$$\tau Y_j(t) \geq 0, \quad \tau c_i Y_{j+i}(t) > 0, \quad t \in J, \quad i = 1, 2, 3,$$

$|Y_j|$ is nondecreasing, $|Y_{j+k}|$ is nonincreasing in J, $k = 1, 2, 3$, where $c_1 = 1$ for $j = 1, 2, 3$, $c_1 = -1$ for $j = 4$, $c_2 = 1$ for $j = 3, 4$, $c_2 = -1$ for $j = 1, 2$, $c_3 = 1$ for $j = 1$, $c_3 = -1$ for $j = 2, 3, 4$.

Type V. There exist $j \in N_4$, $\tau \in \{-1, 1\}$ such that

$$Y_j = 0, \quad \tau Y_{j+1}(t) \geq 0, \quad \tau c Y_{j+3}(t) > 0, \quad \operatorname{sign} Y_{j+2}(t) \quad \text{is}$$

constant in J where $c = 1$ for $j = 3, 4$, $c = -1$ for $j = 1, 2$.

Type VI. $y \equiv 0$ in J.

Remark 1. The solutions of either Type I(s, ∞) or II(s, ∞) are called oscillatory. The solutions of Type III, IV, V are usually called nonoscillatory.

Definition 3. Let $y : (a, b) \to R^4$ and let A_i, $i = 0, 1, \dots, s$ be one of Types I - VI. y is successively of Types $A_1, A_2, \dots, A_s$ if numbers $\tau_0, \dots, \tau_s$ exist such that $a = \tau_0 \leq \tau_1 \leq \dots \leq \tau_s = b$ and y is of Type A_j on (τ_{j-1}, τ_j), $j = 1, 2, \dots, s$. At the same time, if y is of Type A in (τ, τ), then Type A is missing.

2. Main results

Theorem 1. *Let $y : J = (a, b) \to R^4$, $-\infty \leq a < b \leq \infty$ be a solution of* (1).

(i) *Let Z-points of y not exist in J. Then numbers s, $\bar{s}$, r, $\bar{r}$ exist such that s, $r \in Z \cup \{-\infty\}$, $\bar{s}$, $\bar{r} \in Z \cup \{\infty\}$ and y is successively of Types* IV, II $(s, \bar{s})$, V, I $(r, \bar{r})$, III *on J; if $r = -\infty$ $(\bar{s} = +\infty)$, then Types* IV, II, V (*Types* V, I, III) *are missing, if $\bar{r} = \infty$ $(s = -\infty)$, then Type* III (*Type* IV) *is missing. Moreover, y has Property W.*

(ii) *Let y have the only Z-point τ in J. Then*

(a) *y is either of Type* I(s, ∞) *or of Type* II(s, ∞) *in some left neighborhood of τ, where $s \in Z$ is a suitable number;*

(b) *y is either of Type* I$(-\infty, s)$ *or of Type* II$(-\infty, s)$ *in some right neighborhood of τ, where $s \in Z$ is a suitable number.*

(iii) *Let τ, τ_1, $a < \tau < \tau_1 < b$ be Z-points of y and let no Z-point of y exist in (τ, τ_1). Then y is either of Type* I$(-\infty, \infty)$ *or of Type* II$(-\infty, \infty)$ *or numbers s, r exist such that s, $r \in Z$ and y is successively of Types* II$(-\infty, s)$, V, I(r, ∞) *in (τ, τ_1). In the last case Types* I, II *are always present.*

Remark 2. Let $i \in N_4$. Consider (1) with an extra condition

$$\alpha_i y_i'(t) y_{i+1}(t) > 0 \quad \text{for} \quad y_{i+1}(t) \neq 0.$$

Let y be a solution of this problem, $y : J = (a,b) \to R^4$, $a < b$. If y is either of Type I or II, then $t_k^i = \bar{t}_k^i$ for all admissible k. If y is either of Type III or IV or V, then the relation

$$y_{i+1} \neq 0 \quad \text{on I} \quad \Rightarrow y_i \neq 0 \quad \text{on I}$$

holds. Note that in the case of the system (3) an extra condition

$$\alpha_i f_i(t, x_1, x_2, x_3, x_4) x_{i+1} > 0 \quad \text{for} \quad x_{i+1} \neq 0$$

is added. The following theorem gives some conditions under which a nontrivial solution of (3) has no Z-points.

Theorem 2. *Let $\varepsilon > 0$, $\bar{\varepsilon} > 0$, $K > 0$, and $y : J = (a,b) \to R^4$ be a nontrivial solution of* (3). *Let nonnegative functions $a_i \in L_{loc}(R)$, $g_i \in C^\circ([0,\varepsilon])$, $i \in N_4$ exist such that g_i is nondecreasing, $g_i(0) = 0$,*

$$|f_i(t, x_1, \dots, x_4)| \leqq a_i(t)\ g_i(|x_{i+1}|) \quad on \quad R \times [-\varepsilon, \varepsilon]^4, \quad i \in N_4, \tag{6}$$

and

$$g_1(\bar{\varepsilon} g_2(\bar{\varepsilon} g_3(\bar{\varepsilon} g_4(z)))) \leq Kz, \quad z \in [0,\varepsilon], \tag{7}$$

hold, where $x_{4i+j} = x_i$, $i \in Z$, $j \in N_4$. Then y has no Z-point in J and the statement (i) *of Theorem* 1 *holds.*

Remark 3.

1. Theorem 2 generalizes the well-known condition for the nonexistence of Z-points of a nontrivial solution:

$$\varepsilon > 0, \quad |f_i(t, x_1, \dots, x_4)| \leq d(t) \sum_{i=1}^{4} |x_i|,$$
$$t \in R,\ |x_i| \leq \varepsilon, \quad i \in N_4, \quad d \in L_{\text{loc}}(R)$$

(i.e., the Lipschitz condition for $y \equiv 0$; for (4), $a_j \equiv 1$ see [3]).

2. The condition (7) cannot be replaced by

$$\delta > 0,\ g_1(\bar{\varepsilon} g_2(\bar{\varepsilon} g_3(\bar{\varepsilon} g_4(z)))) \leq K z^{1-\delta}, \quad z \in [0, \varepsilon]\,. \tag{8}$$

In [4] where sufficient conditions are given for the existence of a solution of (3) with a Z-point (when studying singular solutions of the 1st kind), the inequality (8) is fulfilled but (7) is not.

3. Let $k \in N_4$. Then (7) can be replaced by

$$g_k(\bar{\varepsilon} g_{k+1}(\bar{\varepsilon} g_{k+2}(\bar{\varepsilon} g_{k+3}(z)))) \leq Kz, \quad z \in [0,\varepsilon],$$

where $g_{4k+j} = g_j$, $j \in N_4$, $k \in N$. The proof is similar.

The last part is devoted to equation (4). It is proved that for every solution x at most one maximal interval exists on which x is trivial.

Theorem 3. *Let $x : J = (a, b) \to R$ be a nontrivial solution of* (4) *and let $y_i = L_{i-1}x$, $i \in N_4$, $y = (y_i)_{i=1}^4$.*

(i) *Let the number $\varepsilon > 0$ and a nonnegative function $d \in L_{loc}(R)$ exist such that*

$$|f(t, x_1, x_2, x_3, x_4)| \le d(t)|x_1| \quad \text{for} \quad t \in R,\ |x_i| \le \varepsilon,\ i \in N_4.$$

Then the statement (i) *of Theorem* 1 *holds.*

(ii) *Let $a_j \in C^1(R)$, $j = 1, 2$, $\frac{a_3}{a_1} \in C^2(R)$. Then either the statement* (i) *of Theorem* 1 *holds or numbers $s \in Z \cup \{-\infty\}$, $r \in Z \cup \{\infty\}$ exist such that y is successively of Types* IV, II(s, ∞), VI, I$(-\infty, r)$, III *in J; if $s = -\infty$ ($r = \infty$), then Type* IV (*Type* III) *is missing; if Type* I (II) *is missing, then Type* III (IV) *is missing, too.*

(iii) *y has Property W.*

Remark 4. Let y have a Z-point. Then the intervals from the definition of Property W do not exist.

3. Proof of the main results

We start with some lemmas.

Lemma 1. *Let y be a solution of* (1) *defined on the interval I.*

(a) *Let $j \in \{2, 3, 4\}$ and $Y_j(t) \ge 0$ (≤ 0) on I. Then Y_{j-1} is nondecreasing (nonincreasing) in I.*

(b) *If $Y_1 \ge 0$ ($Y_1 \le 0$) on I, then Y_4 is nonincreasing (nondecreasing) in I.*

(c) *Let $j \in N_4$, $Y_j(t) = 0$ on I. Then Y_{j-1} is constant in I.*

Proof. (a) Let $j = 2$, $Y_2(t) = \alpha_1 y_2(t) \ge 0$ on I. As by (1) $\alpha_1 y_1'(t) y_2(t) \ge 0$ we have $y_1' = Y_1' \ge 0$ for almost all $t \in I$. In the other cases the proof is similar.

(b) Let $Y_1 \ge 0$ in I. By (1) $\alpha_4 y_4' y_1 \ge 0$ holds. Then, according to (2)

$$Y_4'(t) = \alpha_1\alpha_2\alpha_3 y_4'(t) = -\alpha_4 y_4'(t) y_1(t) \le 0\,.$$

The case (c) is a consequence of (a), (b). □

Remark 5. The conclusions about the monotonicity in the definitions of Types III–V follow directly from Lemma 1.

Lemma 2. *Let y $I = [t_1, t_2] \to R^4$ be a solution of* (1), *$i \in N_4$, $Y_i(t_1) = Y_{i+1}(t) = 0$, $Y_{i+2}(t) \ne 0$ on I. Then either*

$$Y_i \equiv Y_{i+1} \equiv 0 \quad on \quad I \tag{9}$$

or there exists a number $\tau \in [t_1, t_2)$ such that $Y_i(t) \equiv Y_{i+1} \equiv 0$ in $[t_1, \tau]$ and $c\ Y_{i+1}(t)Y_{i+2}(t) > 0$ in $(\tau, t_2]$ hold, where $c = 1$ ($c = -1$) for $i = 1, 2, 4$ ($i = 3$).

Proof. Suppose that (9) is not valid and $i = 1$, $Y_3(t) > 0$ on I. Then by Lemma 1 the function Y_2 is nondecreasing, $Y_2 \geq 0$ in I, and Y_1 is nondecreasing, $Y_1 \geq 0$ in I, too. From this and according to Lemma 1(c) there exists $\tau \in [t_1, t_2)$ such that

$$Y_1 \equiv Y_2 \equiv 0 \quad \text{on } [t_1, \tau]\,, \tag{10}$$

$$Y_1^2(t) + Y_2^2(t) > 0 \quad \text{on } (\tau, t_2)\,. \tag{11}$$

Suppose that $Y_2(t) = 0$ in some right neighborhood I_1 of τ. Then by (1) we have $y_1'(t) = 0$ for almost all $t \in I_1$ and according to (10) $y_1(t) = Y_1(t) = 0$ on I. This contradiction to (11) proves the statement for $i = 1$. For the other i the proof is similar. $\square$

Proof of Theorem 1. (i) Let $t_0 \in J$ be an arbitrary number. Divide all possible initial conditions at $t \in J$ into 32 cases:

1°	$Y_1Y_4 \geq 0$,	$Y_iY_4 > 0$,	$i=2,3$,	17°	$Y_1=Y_2=0$,	$Y_3Y_4 > 0$
2°	$Y_1Y_i > 0$,	$Y_4Y_1 \leq 0$,	$i=2,3$,	18°	$Y_1=Y_2=0$,	$Y_3Y_4 < 0$
3°	$Y_iY_4 < 0$,	$Y_3Y_4 \geq 0$,	$i=1,2$,	19°	$Y_1=Y_3=0$,	$Y_2Y_4 > 0$
4°	$Y_1Y_i < 0$,	$Y_1Y_2 \leq 0$,	$i=3,4$,	20°	$Y_1=Y_3=0$,	$Y_2Y_4 < 0$
5°	$Y_1Y_3 \leq 0$,	$Y_iY_3 < 0$,	$i=2,4$,	21°	$Y_1=Y_4=0$,	$Y_2Y_3 > 0$
6°	$Y_iY_3 < 0$,	$Y_2Y_3 \geq 0$,	$i=1,4$,	22°	$Y_1=Y_4=0$,	$Y_2Y_3 < 0$
7°	$Y_iY_2 < 0$,	$Y_2Y_3 \leq 0$,	$i=1,4$,	23°	$Y_2=Y_3=0$,	$Y_1Y_4 > 0$
8°	$Y_iY_2 < 0$,	$Y_2Y_4 \geq 0$,	$i=1,3$,	24°	$Y_2=Y_3=0$,	$Y_1Y_4 > 0$
9°	$Y_1=0$,	$Y_iY_4 < 0$,	$i=2,3$,	25°	$Y_2=Y_4=0$,	$Y_1Y_3 > 0$
10°	$Y_1=0$,	$Y_2Y_i < 0$,	$i=3,4$,	26°	$Y_2=Y_4=0$,	$Y_1Y_3 < 0$
11°	$Y_iY_4 > 0$,	$Y_2=0$,	$i=1,3$,	27°	$Y_3=Y_4=0$,	$Y_1Y_2 > 0$
12°	$Y_iY_4 < 0$,	$Y_2=0$,	$i=1,3$,	28°	$Y_3=Y_4=0$,	$Y_1Y_2 < 0$
13°	$Y_1Y_i > 0$,	$Y_3=0$,	$i=2,4$,	29°	$Y_1=Y_2=Y_3=0$,	$Y_4 \neq 0$
14°	$Y_1Y_i < 0$,	$Y_3=0$,	$i=2,4$,	30°	$Y_1=Y_3=Y_4=0$,	$Y_2 \neq 0$
15°	$Y_iY_3 < 0$,	$Y_4=0$,	$i=1,2$,	31°	$Y_1=Y_2=Y_4=0$,	$Y_3 \neq 0$
16°	$Y_1Y_i < 0$,	$Y_4=0$,	$i=2,3$,	32°	$Y_2=Y_3=Y_4=0$,	$Y_1 \neq 0$.

Note that the last case $Y_i = 0$, $i \in N_4$, is impossible in view of the assumptions of the theorem. Sometimes, if, for example, 1° is valid at $\bar{t}$ we shall write $1°(\bar{t})$.

We shall investigate how the initial conditions vary when t increases in $[t_0, b)$. First note that for $J_1 \subset [t_0, b)$, y is of

$$\left.\begin{array}{l} \text{Type III in } J_1 \text{ iff one of the cases } 1° - 4° \text{ holds in } J_1; \\ \text{Type IV in } J_1 \text{ iff one of the cases } 5° - 8° \text{ holds in } J_1; \\ \text{Type V in } J_1 \text{ iff one of the cases } 9° - 32° \text{ holds in } J_1 \end{array}\right\} \tag{12}$$

(see Remark 5, too).

Consider y in $J_1 = [t_0, \bar{b})$, $\bar{b} \le b$. Let $j,\ k \in \{1, 2, \dots, 32°\}$. The symbol $j°(t_0) \to k°(t_1)$ denotes that either $j°$ holds in J_1 (and y is one of Types III–V according to (12)) or $t_1,\ t_2 \in J_1$, $t_0 < t_2 < t_1$ exist such that $j°$ holds in $[t_0, t_2)$, $k°$ holds in $(t_2, t_1]$ and either $j°$ or $k°$ is valid at t_2. Generally, the notation $j°(t_0) \to \{k_1°, \dots, k_s°\}(t_1)$ denotes that $j°(t_0) \to k_e°(t_1)$ is valid for suitable $e \in \{1, \dots, s\}$. The following relations can be proved for y defined in $[t_0, b)$

$$\left.\begin{aligned}
&1°(t_0) \to 2°(t_1), 2°(t_0) \to 3°(t_1), 3°(t_0) \to 4°(t_1), 4°(t_0) \to 1°(t_1),\\
&5°(t_0) \to \{6°, 13°, 15°, 18°, 19°, 23°, 26°, 27°, 29°, 32°\}(t_1),\\
&6°(t_0) \to \{7°, 9°, 16°, 20°, 21°, 23°, 26°, 28°, 30°, 32°\}(t_1),\\
&7°(t_0) \to \{8°, 10°, 11°, 17°, 20°, 21°, 23°, 26°, 28°, 30°, 32°\}(t_1),\\
&8°(t_0) \to \{5°, 12°, 14°, 18°, 19°, 22°, 24°, 25°, 29°, 31°\}(t_1),\\
&9°(t_0) \to \{2°, 20°\}(t_1), 10°(t_0) \to \{3°, 17°\}(t_1),\\
&11°(t_0) \to \{1°, 25°\}(t_1), 12°(t_0) \to \{2°, 24°\}(t_1),\\
&13°(t_0) \to \{1°, 27°\}(t_1), 14°(t_0) \to \{4°, 19°\}(t_1),\\
&15°(t_0) \to \{3°, 26°\}(t_1), 16°(t_0) \to \{4°, 21°\}(t_1),\\
&17°(t_0) \to 1°(t_1), 18°(t_0) \to \{2°, 9°, 29°\}(t_1),\\
&19°(t_0) \to \{1°, 13°\}(t_1), 20°(t_0) \to \{3°, 10°\}(t_1),\\
&21°(t_0) \to 2°(t_1), 22°(t_0) \to \{3°, 15°, 31°\}(t_1),\\
&23°(t_0) \to \{1°, 11°, 32°\}(t_1), 24°(t_0) \to 4°(t_1),\\
&25°(t_0) \to \{2°, 12°\}(t_1), 26°(t_0) \to \{4°, 16°\}(t_1),\\
&27°(t_0) \to 3°(t_1), 28°(t_0) \to \{4°, 14°, 30°\}(t_1),\\
&29°(t_0) \to \{1°, 17°\}(t_1), 30°(t_0) \to \{3°, 27°\}(t_1),\\
&31°(t_0) \to \{2°, 21°\}(t_1), 32°(t_0) \to \{4°, 24°\}(t_1),\ t_1 \in (t_0, b).
\end{aligned}\right\} \tag{13}$$

We prove only the validity of

$$18°(t_0) \to \{2°, 9°, 29°\}(t_1)\,. \tag{14}$$

The other relations can be proved similarly. Thus suppose for simplicity that

$$Y_1(t_0) = Y_2(t_0) = 0\,, \quad Y_3(t_0) > 0, \quad Y_4(t_0) < 0$$

holds. Then, according to Lemma 1, Y_2 is nonincreasing in some right neighborhood of t_0 and Y_1, Y_2 are nondecreasing, Y_4 nonincreasing until $Y_2 \ge 0$. From this and according to Lemma 2 one of the following possibilities is valid:

(i) $Y_1 \equiv Y_2 \equiv 0$, $Y_3 > 0$, $Y_4 < 0$ in $[t_0, b)$ (i.e., y is of Type V);
(ii) $t_1, t_1 \in (t_0, b)$ exists such that $Y_1 \equiv Y_2 \equiv 0$, $Y_4 < 0$ in $[t_0, t_1]$, $Y_3(t) > 0$ in $[t_0, t_1)$, $Y_3(t_1) = 0$ (i.e., 29° holds at t_1);

(iii) $t_1, t_1 \in (t_0, b)$ exists such that $Y_1 \equiv Y_2 \equiv 0, Y_3 > 0, Y_4 < 0$ in $[t_0, t_1]$, $Y_1 = 0$, $Y_2 > 0$ in some right neighborhood J_2 of t_1 (i.e., $9°$ holds in J_2)

(iv) $t_1, t_1 \in (t_0, b)$ exists such that $Y_1 \equiv Y_2 \equiv 0$, $Y_3 > 0, Y_4 < 0$ on $[t_0, t_1]$, $Y_1 = 0$, $Y_2 > 0$, $Y_3 > 0$, $Y_4 < 0$ in some right neighborhood J_2 of t_1 (i.e., $2°$ holds in J_2).

From this we can conclude that (14) holds. Further, note that if the cases $1°, 2°, 3°, 4°$ are repeated, $1° \to 2° \to 3° \to 4° \to 1°$, we get just the solution of Type I; in the case $5° \to 6° \to 7° \to 8° \to 5°$, we get Type II.

Let y be either of Type $\mathrm{I}(s,\infty)$ or of Type $\mathrm{II}(s,\infty)$ on the interval $[t_0, \bar{b})$, $\bar{b} \le b$. We prove by an indirect method that $\bar{b} = b$. Suppose that $\bar{b} < b$. As y is oscillatory, $y_i(\bar{b}) = 0$, $i \in N_4$ is valid and $\bar{b}$ is a Z-point of y. The contradiction to the assumptions of the theorem proves that $\bar{b} = b$.

The statement of the theorem for the interval $[t_0, b)$ follows from this and from (13). The statement in $(a, t_0]$ can be proved similarly, or the fact that t_0 is arbitrary can be used.

(ii) (a) If $\tau, \tau \in J$, is a Z-point of y, then $y_i(\tau) = 0$, $i \in N_4$ and the only types which can fulfill these conditions are $\mathrm{I}(s,\infty)$, $\mathrm{II}(s,\infty)$, $s \in Z$. Thus the statement follows from (i). The case (b) can be proved similarly.

(iii) The statement follows directly from (i), (ii). □

Proof of Remark 2. It can be proved similarly to Lemma 1 that the following two statements hold:

(a) If $i \in \{2,3,4\}$, $Y_i(t) > 0$ (< 0) in I, then Y_{j-1} is increasing (decreasing) in I.

(b) If $Y_1(t) > 0$ (< 0) in I, then Y_4 is decreasing (increasing) in I.

The statement of the remark follows from this. □

Proof of Theorem 2. On the contrary, suppose that a Z-point $\tau \in J$ exists. Without loss of generality we can suppose that τ is such that a right neighborhood of τ exists in which y is not trivial (for a left neighborhood the proof is similar).

As $y_i(\tau) = 0$, $i \in N_4$, an interval $J_1 = [\tau, \tau + \delta]$, $\delta > 0$ exists such that

$$|y_i(t)| \le \varepsilon, t \in J_1,\ i \in N_4\,. \tag{15}$$

Let ε_1, δ_1 and $J_2 = [\tau, \tau + \delta_1]$ be such that $0 < \varepsilon_1 \le \bar{\varepsilon}$, $0 < \delta_1 \le \delta$

$$\left.\begin{aligned} &\varepsilon_1 K < 1, \quad \varepsilon_1 \max_{\substack{0 \le s \le \varepsilon \\ j \in N_4}} g_j(s) \le \varepsilon\,, \\ &\max_{j \in N_4} \int\limits_{J_2} a_j(t)\,dt \le \varepsilon_1\,. \end{aligned}\right\} \tag{16}$$

Then by the use of (6), (15) we have for $t \in J_2$ and $i \in N_4$

$$|y_i(t)| \le \int_\tau^t |f_i(t, y_1(t), \dots, y_4(t))|\, dt \le$$
$$\le \int_{J_2} a_i(t) dt g_i(\max_{s\in J_2} |y_{i+1}(s)|)\,.$$

From this, by the use of (16) we get

$$\max_{s\in J_2} |y_i(s)| \le \varepsilon_1 g_i(\max_{s\in J_2} |y_{i+1}(s)|)\,, \quad i \in N_4\,. \tag{17}$$

Denote $\nu = \max_{s\in J_2} |y_1(s)|$. As y is not trivial in J_2 and $g_i(0) = 0$, it follows from (17) that $\nu > 0$ must be valid.

Further, according to (7), (16), and (17) we have

$$\nu \le \varepsilon_1 g_1(\varepsilon_1 g_2(\varepsilon_1 g_3(\varepsilon_1 g_4(\nu)))) \le \varepsilon_1 g_1(\bar\varepsilon g_2(\bar\varepsilon g_3(\bar\varepsilon g_4(\nu)))) \leqq$$
$$\leqq \varepsilon_1 K \nu < \nu\,.$$

The contradiction proves the theorem. □

Proof of Theorem 3. (i) The statement is a consequence of Theorem 2 and (5).

(ii) Let $\tau \in J$ be a Z-point such that no Z-point exists in some right neighborhood J_1 of τ.

Then according to Theorem 1, (ii) there exists a right neighborhood J_2 of τ, $J_2 \subset J_1$ such that y is either of Type I$(-\infty, s)$ or of Type II$(-\infty, s)$, $s \in \{Z, \infty\}$ in J_2. We prove by an indirect method that Type II is impossible. Thus, suppose that y is of Type II $(-\infty, s)$ in J. Let $\alpha > \tau$, $\alpha \in J_2$.

Put

$$F = Ay_4y_1 + By_2y_3 + Cy_2^2 + Dy_1y_3 + Ey_1y_2 + Gy_1^2, \tag{18}$$

where

$$A(t) = -\int_\tau^t \frac{1}{a_3(s)} \int_s^\alpha \frac{E(v)}{a_2(v)}\, dv\, ds\,, \quad B(t) = -\frac{a_3(t)}{a_1(t)} A(t)\,,$$
$$C(t) = -\frac{a_2(t)}{a_1(t)} \int_t^\alpha \frac{E(v)}{a_2(v)}\, dv \,-\, \frac{a_2(t)}{2}\Big(\frac{a_3(t)}{a_1(t)}\Big)' \times$$
$$\times \int_\tau^t \frac{1}{a_3(s)} \int_s^\alpha \frac{E(v)}{a_2(v)}\, dv\, ds\,,$$

$$D(t)=\int_t^{\alpha}\frac{E(v)}{a_2(v)}\,dv\,;E(t)=(v-\tau+\sigma)^{\frac{1}{2}}\,;\quad G(t)=-\frac{a_1(t)}{4E(t)}\,.$$

The number $\sigma>0$ is chosen in a manner such that

$$G'(t)=\frac{-a_1'(t)}{4E(t)}+\frac{a_1(t)}{F(t)^3}\geq 0\,,$$

$$C'(t)+\frac{E(t)}{a_1(t)}=2\frac{E(t)}{a_1(t)}+\Big[\frac{a_2'(t)}{a_1(t)}+\frac{a_1'(t)a_2(t)}{2a_3(t)}-\frac{a_2(t)}{a_1(t)}\Big(\frac{a_3(t)}{a_1(t)}\Big)'\Big]\times \tag{19}$$
$$\times\int_t^{\alpha}\frac{E(s)}{a_2(s)}\,ds-\frac{1}{2}\Big(a_2(t)\Big(\frac{a_3(t)}{a_1(t)}\Big)'\Big)'\int_{\tau}^{t}\frac{1}{a_3(s)}\int_s^t\frac{E(v)}{a_2(v)}\,dv\,ds\geq 0\,,$$

$t\in J_3$, holds, where $J_3,\ J_3=[\tau,\bar t]$, $\tau<\bar t\leq\alpha$, is a suitable interval.

From this

$$F'=Ay_4'y_1+\frac{B}{a_2}y_3^2+(C'+\frac{E}{a_1})y_2^2+G'y_1^2$$

and according to (4), (5), (19) and $y_i(\tau)=0,\ \ i\in N_4$, we have

$$F'(t)\geq 0,\quad F(t)\geq 0\quad\text{on}\quad J_3\,. \tag{20}$$

Further, let β, $\beta\in(\tau,\bar t)$, be an arbitrary zero of y_2. Then according to the definition of Type II and (18) we have

$$A(t)<0,\quad D(t)>0,\quad G(t)<0\quad\text{on}\quad(\tau,\bar t)\,,$$
$$y_4(\beta)y_1(\beta)>0,\quad y_1(\beta)y_3(\beta)<0\,.$$

Thus $F(\beta)<0$ and the contradiction to (20) proves that y is of Type I $(-\infty,s)$ in J_2.

Let $\tau\in J$ be a Z-point such that no Z-point exists in some left neighborhood of τ. Then we can prove similarly that y is of Type II (s,∞), $s\in\{Z,-\infty\}$, in some left neighborhood of τ. Also, the transformation of the independent variable can be used for $x=\tau-t$.

The above-mentioned results and Theorem 1, (i) show that there exists at most one maximal interval of Z-points and the statement is a consequence of Theorem 1, (i).

(iii) The intervals mentioned in the definition of Property W may occur only in Type V. Thus the statement follows from the case (ii). □

Proof of Remark 4. This can be done similarly to that of Theorem 3, (iii).

References

1. M. Bartušek, The asymptotic behavior of oscillatory solutions of an equation of the fourth order. *Arch. Math.* **21**(1985), No. 2, 93 -104.

2. ——, On oscillatory solutions of differential inequalities. *Czech Math. J.* **42(117)**(1992), 45-52.

3. I. T. Kiguradze and T. A. Chanturia, Asymptotic properties of solutions of nonautonomous ordinary differential equations. (Russian) *Nauka, Moscow,* 1990.

4. T. A. Chanturia, On singular solutions of nonlinear system of ordinary differential equations. *Colloquia Math. Soc. János Bolyai* 15,*Differential Equations, Keszethely (Hungary)* 1975, 107-119.

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