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ON PROPER OSCILLATORY AND VANISHING AT INFINITY SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH A DEVIATING ARGUMENT

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Abstract. Sufficient conditions are found for the existence of multiparametrical families of proper oscillatory and vanishing-at-infinity solutions of the differential equation

$$u^{(n)}(t) = g\big(t, u(\tau_0(t)), \dots, u^{(m-1)}(\tau_{m-1}(t))\big),$$

where $n \geq 4$, m is the integer part of $\frac{n}{2}$, $g : R_+ \times R^m \to R$ is a function satisfying the local Carathéodory conditions, and $\tau_i : R_+ \to R$ $(i = 0, \dots, m-1)$ are measurable functions such that $\tau_i(t) \to +\infty$ for $t \to +\infty$ $(i = 0, \dots, m-1)$.

Introduction

In this paper we consider the differential equation

$$u^{(n)}(t) = g\big(t, u(\tau_0(t)), \dots, u^{(m-1)}(\tau_{m-1}(t))\big) \tag{0.1}$$

and its particular cases

$$u^{(n)}(t) = p(t)\big|u(\tau(t))\big|^{\lambda} \operatorname{sgn} u(\tau(t)), \tag{0.2}$$

$$u^{(n)}(t) = p(t)u(\tau(t)), \tag{0.3}$$

$$u^{(n)}(t) = \sum_{i=0}^{m-1} p_i(t)u^{(i)}(\tau_i(t)). \tag{0.4}$$

Throughout the paper it will be assumed that $n \geq 4$, m is integer part of the number $\frac{n}{2}$, $g : R_+ \times R^m \to R$ is a function satisfying the local Carathéodory conditions, $p : R_+ \to R$ and $p_i : R_+ \to R$ $(i = 0, \dots, m-1)$ are locally

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summable functions, while $\tau_i : R_+ \to R$ $(i = 0, \dots, m-1)$ and $\tau : R_+ \to R$ are measurable functions such that

$$\lim_{t\to+\infty} \tau_i(t) = +\infty \quad (i = 0, \dots, m-1) \tag{0.5}$$

and

$$\lim_{t\to+\infty} \tau(t) = +\infty. \tag{0.6}$$

Let $t_0 \in R_+$. A function $u : [t_0, +\infty[\to R$ is called **a solution of equation (0.1)** if it is locally absolutely continuous together with its derivatives up to order $n-1$ inclusive and if there exists an $m-1$ times continuously differentiable function $\overline{u} : R \to R$ such that $\overline{u}(t) = u(t)$ for $t \geq t_0$ and the equality

$$u^{(n)}(t) = g\big(t, \overline{u}(\tau_0(t)), \dots, \overline{u}^{(m-1)}(\tau_{m-1}(t))\big).$$

is fulfilled almost everywhere on $[t_0, +\infty[$.

A solution u of equation (0.1) determined on the interval $[t_0, +\infty[$ is called **proper** if it is not identically zero in anyone of the neighborhoods of $+\infty$ and is called **vanishing-at-infinity** if $u(t) \to 0$ for $t \to +\infty$.

A proper solution is called **oscillatory** if it has a sequence of zeros converging to $+\infty$, and **nonoscillatory** otherwise.

In the papers dealing with oscillatory properties of differential equations with deviating arguments it is always assumed a priori that the considered equation has proper solutions and sufficient conditions are established for these solutions to be oscillatory (see, for example, [1–9] and the references cited therein). However, the problem of the existence of proper solutions is far from being trivial and has not yet been investigated for a wide class of equations.[1]

Therefore the question as to the existence of at least one oscillatory solution of such equations remains open. We do not know, for example, of a single result on the existence of oscillatory solutions of equations like (0.1), (0.2), or (0.3) when

$$\tau_i(t) > t \ \ (i = 0, \dots, m-1), \quad \tau(t) > t \ \text{ for } \ t \geq t_0, \tag{0.7}$$

though such equations occur rather frequently in the oscillation theory. Further, it is not likewise clear for us whether (0.1), (0.2) or (0.3) have at least one proper solution vanishing-at-infinity. Hence this paper deals with these two open problems.

In §1 we prove, by means of the results of [10], theorems on the existence and uniqueness of two auxiliary boundary value problems with integral conditions for differential equations with a deviating argument. Using these

[1] Equations with a delay for which this problem is studied in [1] are an exception.

theorems and the oscillation theorems from [11], in §§2 and 3 we establish sufficient conditions for equations (0.1)–(0.4) to have multiparametric families of proper oscillatory and vanishing-at-infinity solutions.[2]

Throughout the paper the following notation will be used.

μ_i^k $(i = 0, 1, \dots; k = 2i, 2i+1, \dots)$ are the numbers given by the recurrent relations

$$\mu_0^{i+1} = \frac{1}{2}, \quad \mu_i^{2i} = 1, \quad \mu_{i+1}^k = \mu_{i+1}^{k-1} + \mu_i^{k-2} \quad (i = 0, 1, \dots; k = 2i+3, \dots).$$

m is the integer part of $\frac{n}{2}$; m_0 is the integer part of $\frac{n}{4}$;

$$\gamma_n = \sum_{j=0}^{m_0-1} \frac{n!}{(2m-2-4j)!}\, \mu_{m-1-2j}^n;$$

$$\gamma_{0n} = \frac{m-1}{4}\Big[\frac{(2m)!\gamma_n}{n!\mu_m^n} + \frac{(m-2)(4m^2-m-3)}{3} + 4\Big]^{m-1} - (-1)^m \frac{n!}{2}.$$

§ 1. Auxiliary Boundary Value Problems

For the differential equations

$$u^{(n)}(t) = h(t, u(t), u(\tau_0(t)), \dots, u^{(m-1)}(\tau_{m-1}(t))), \tag{1.1}$$

$$u^{(n)}(t) = \sum_{i=0}^{m-1} p_i(t) u^{(i)}(\tau_i(t)) + q(t) \tag{1.1$'$}$$

we consider the boundary value problems

$$u^{(i)}(0) = c_i \quad (i = 0, \dots, m-1), \quad \int_0^{+\infty} \big|u^{(m)}(t)\big|^2 dt < +\infty; \tag{1.2}$$

$$\begin{gathered} u^{(i)}(0) = c_i \quad (i = 0, \dots, m-1), \\ \int_0^{+\infty} t^{2j} \big|u^{(j)}(t)\big|^2 dt < +\infty \quad (j = 0, \dots, m), \end{gathered} \tag{1.3}$$

where $n \geq 4$, $c_i \in R$ $(i = 0, \dots, m-1)$,

$$h : R_+ \times R^{m+1} \to R \quad \text{satisfies the local Carathéodory conditions,} \tag{1.4}$$

$p_i : R_+ \to R$ $(i = 0, \dots, m-1)$ and $q : R_+ \to R$ are the locally summable functions, and $\tau_i : R_+ \to R_+$ $(i = 0, \dots, m-1)$ are measurable functions satisfying condition (0.5).

[2] When $\tau_i(t) \equiv t$ $(i = 0, \dots, m-1)$ and $\tau(t) \equiv t$, sufficient conditions for equations (0.1)–(0.4) to have proper oscillatory and vanishing-at-infinity solutions are obtained in [11–15].

Alongside with the notation listed in the Introduction we shall need in this section the following notation as well:

$$\tau_{0*}(t) = \min\{t, \tau_0(t)\}, \quad \tau_0^*(t) = \max\{t, \tau_0(t)\}.$$

L is the space of locally Lebesgue integrable functions $v : R_+ \to R$ with a topology of convergence in the mean on each segment from R_+.

C^{n-1} is the topological space of $(n-1)$-times continuously differentiable real functions given on R_+. By the convergence of the sequence $(u_k)_{k=1}^{+\infty}$ of elements from this space we mean the uniform convergence of sequences $(u_k^{(i)})_{k=1}^{+\infty}$ $(i = 0, \dots, n-1)$ on each finite segment from R_+.

$$C_0^{n-1,m} = \Big\{u \in C^{n-1} : \int_0^{+\infty} |u^{(m)}(t)|^2 dt < +\infty\Big\};$$

$$C^{n-1,m} = \Big\{u \in C^{n-1} : \int_0^{+\infty} t^{2i} |u^{(i)}(t)|^2 dt < +\infty \quad (i = 0, \dots, m)\Big\};$$

$$\|u\|_{0,m} = \Big[\sum_{i=0}^{m-1} |u^{(i)}(0)|^2 + \int_0^{+\infty} |u^{(m)}(t)|^2 dt\Big]^{\frac{1}{2}};$$

$$\|u\|_m = \Big[\int_0^{+\infty} (1+t)^{2m} |u^{(m)}(t)|^2 dt\Big]^{\frac{1}{2}}.$$

Theorem 1.1. *Let on $R_+ \times R^{m+1}$ the conditions*

$$|h(t, x, x_0, x_1, \dots, x_{m-1}) - h(t, x, x, 0, \dots, 0)| \le$$
$$\le a_{10}(t)|x - x_0|^{\lambda_0} + \sum_{i=1}^{m-1} a_{1i}(t)|x_i|^{\lambda_i}, \tag{1.5}$$

$$(-1)^{n-m-1} h(t, x, x, 0, \dots, 0)x \ge -a(t), \tag{1.6}$$

be fulfilled, where $\lambda_i \in [0,1]$ $(i = 0, \dots, m-1)$, $a_{1i} : R_+ \to R_+$ $(i = 0, \dots, m-1)$, and $a : R_+ \to R_+$ are measurable functions such that

$$\int_0^{+\infty} (1+t)^{n-m-\frac{1}{2}} \Big[a_{10}(t)\big(1 + \tau_0^*(t)\big)^{(m-\frac{3}{2})\lambda_0} |\tau_0(t) - t|^{\lambda_0} +$$
$$+ \sum_{i=1}^{m-1} a_{1i}(t)\big(1 + \tau_i(t)\big)^{(m-i-\frac{1}{2})\lambda_i}\Big] dt < \mu_m^n, \tag{1.7}$$

$$\int_0^{+\infty} (1+t)^{n-2m} a(t) dt < +\infty. \tag{1.8}$$

Then problem (1.1), (1.2) *has at least one solution.*

Proof. Let $r = \sum_{i=0}^{m-1} |c_i|$. By (1.7) and (1.8) there is a positive number ε such that the functions

$$\begin{aligned} a_1(t) = (1+\varepsilon)(1+t)^{m-\frac{1}{2}}\Big[a_{10}(t)\big(1+\tau_0^*(t)\big)^{(m-\frac{3}{2})\lambda_0}|\tau_0(t)-t|^{\lambda_0} + \\ + \sum_{i=1}^{m-1} a_{1i}(t)\big(1+\tau_i(t)\big)^{(m-i-\frac{1}{2})\lambda_i}\Big], \\ a_2(t) = \Big(1+\frac{1}{\varepsilon}\Big)(1+r)^2 a_1(t) + a(t) \end{aligned} \tag{1.9}$$

will satisfy the inequalities

$$\int_0^{+\infty}(1+t)^{n-2m}a_1(t)dt<\mu_m^n, \quad \int_0^{+\infty}(1+t)^{n-2m}a_2(t)dt<+\infty. \tag{1.10}$$

For any $u \in C^{n-1}$ we set

$$\chi(u) = \begin{cases} 1 & \text{for} \quad \sum\limits_{i=0}^{m-1}|u^{(i)}(0)| \le r \\ r+1-\sum\limits_{i=0}^{m-1}|u^{(i)}(0)| & \text{for} \quad r<\sum\limits_{i=0}^{m-1}|u^{(i)}(0)| \le 1+r \\ 0 & \text{for} \quad \sum\limits_{i=0}^{m-1}|u^{(i)}(0)| > 1+r \end{cases},$$

$$f(u)(t) = \chi(u)h\big(t, u(t), u(\tau_0(t)), \dots, u^{(m-1)}(\tau_{m-1}(t))\big).$$

The operator $f : C^{n-1} \to L$ is continuous on account of (1.4). On the other hand, it is obvious that problem (1.1),(1.2) is solvable if and only if the functionally differential equation

$$u^{(n)}(t) = f(u)(t) \tag{1.11}$$

has at least one solution satisfying the boundary conditions (1.2).

Using Theorem 1.1 from [10], we shall prove below that problem (1.11), (1.2) is solvable.

If $u \in C^{n-1,m}$, then by (1.5) and (1.6) we obtain

$$\begin{aligned} (-1)^{n-m-1}u(t)f(u)(t) = \\ = (-1)^{n-m-1}\chi(u)\big[h(t,u(t),u(\tau_0(t)),\dots,u^{(m-1)}(\tau_{m-1}(t))) - \\ -h(t,u(t),u(t),0,\dots,0)\big]u(t) + \\ +(-1)^{n-m-1}\chi(u)h(t,u(t),u(t),0,\dots,0)u(t) \ge \\ \ge -a_{10}(t)\chi(u)|u(\tau_0(t))-u(t)|^{\lambda_0}|u(t)| - \end{aligned}$$

$$-\sum_{i=1}^{m-1} a_{1i}(t)\chi(u)|u^{(i)}(\tau_i(t))|^{\lambda_i}|u(t)| - a(t), \tag{1.12}$$

$$|f(u)(t)| \leq |h(t,u(t),u(t),0,\dots,0)| + \chi(u)a_{10}(t)|u(\tau_0(t)) - u(t)|^{\lambda_0} +$$
$$+\chi(u)\sum_{i=1}^{m-1} a_{1i}(t)|u(\tau_i(t))|^{\lambda_i} \leq$$
$$\leq b_0(t,|u(t)|) + \chi(u)\sum_{i=0}^{m-1} a_{1i}(t)|u^{(i)}(\tau_i(t))|, \tag{1.13}$$

where

$$b_0(t,x) = \sum_{i=0}^{m-1} a_{1i}(t) + a_{10}(t)x +$$
$$+\max\big\{|h(t,s,s,0,\cdots,0)| : 0 \leq s \leq x\big\}. \tag{1.14}$$

On the other hand, for an arbitrary $i \in \{0,\dots,m-1\}$ we have

$$|u^{(i)}(t)| = \Big|\sum_{j=i}^{m-1}\frac{t^{j-i}}{(j-i)!}u^{(j)}(0) + \frac{1}{(m-i-1)!}\int_0^t (t-s)^{m-i-1}u^{(m)}(s)ds\Big| \leq$$
$$\leq (1+t)^{m-1-i}\sum_{j=0}^{m-1}|u^{(j)}(0)| +$$
$$+\frac{1}{(m-i-1)!}\Big(\int_0^t (t-s)^{2m-2i-2}ds\Big)^{\frac12}\Big(\int_0^t |u^{(m)}(s)|^2ds\Big)^{\frac12} \leq$$
$$\leq (1+t)^{m-i-\frac12}\Big[\sum_{j=0}^{m-1}|u^{(j)}(0)| + \Big(\int_0^{+\infty}|u^{(m)}(s)|^2ds\Big)^{\frac12}\Big] \leq$$
$$\leq (1+t)^{m-i-\frac12}\Big[\sum_{i=0}^{m-1}|u^{(i)}(0)| + \|u\|_{0,m}\Big] \leq$$
$$\leq (1+t)^{m-i-\frac12}\Big[\Big(1+\frac{1}{\varepsilon}\Big)\Big(\sum_{i=0}^{m-1}|u^{(i)}(0)|\Big)^2 + (1+\varepsilon)\|u\|_{0,m}^2\Big]^{\frac12}. \tag{1.15}$$

Therefore

$$\chi(u)|u^{(i)}(\tau_i(t))|^{\lambda_i}|u(t)| \leq$$
$$\leq (1+\tau_i(t))^{(m-i-\frac12)\lambda_i}(1+t)^{m-\frac12}\Big[\Big(1+\frac{1}{\varepsilon}\Big)(1+r)^2 +$$
$$+(1+\varepsilon)\|u\|_{0,m}^2\Big]^{\frac{1+\lambda_i}{2}} \leq \big(1+\tau_i(t)\big)^{(m-i-\frac12)\lambda_i}(1+t)^{m-\frac12}\times$$

$$\times\Big[1+\Big(1+\frac{1}{\varepsilon}\Big)(1+r)^2+(1+\varepsilon)\|u\|_{0,m}^2\Big] \tag{1.16}$$

and

$$\begin{aligned}\chi(u)|u(\tau_0(t))-u(t)|^{\lambda_0}|u(t)| &= \chi(u)\Big|\int_t^{\tau_0(t)}u'(s)ds\Big|^{\lambda_0}|u(t)|\le\\ &\le(1+\tau_0^*(t))^{(m-\frac{3}{2})\lambda_0}|\tau_0(t)-t|^{\lambda_0}(1+t)^{m-\frac{1}{2}}\times\\ &\times\Big[1+\Big(1+\frac{1}{\varepsilon}\Big)(1+r)^2+(1+\varepsilon)\|u\|_{0,m}^2\Big].\end{aligned} \tag{1.17}$$

By (1.9) and (1.15)–(1.17) it follows from (1.12) and (1.13) that

$$\begin{gathered}(-1)^{n-m-1}u(t)f(u)(t)\ge -a_1(t)\|u\|_{0,m}^2-a_2(t),\\ \big|f(u)(t)\big|\le b\big(t,|u(t)|,\|u\|_{0,m}\big),\end{gathered} \tag{1.18}$$

where

$$b(t,x,y)=b_0(t,x)+$$
$$+\sum_{i=0}^{m-1}a_{1i}(t)\big(1+\tau_i(t)\big)^{m-i-\frac{1}{2}}\Big[\Big(1+\frac{1}{\varepsilon}\Big)(1+r)^2+(1+\varepsilon)y^2\Big]^{\frac{1}{2}}.$$

Moreover, the functions a_1 and a_2 satisfy inequalities (1.10) and b the condition

$$\lim_{\substack{t\to 0\\ y\to+\infty}}\Big(y^{-2}\int_0^t b(s,x,y)ds\Big)=0 \quad\text{for}\quad x\in R_+. \tag{1.19}$$

Thus all the conditions of Theorem 1.1 from [10] are fulfilled, thereby guaranteeing the solvability of problem (1.11),(1.2). □

Similarly to Theorem 1.1 we prove

Theorem 1.1′. *Let on $R_+\times R^{m+1}$ the conditions*

$$\big|h(t,x,x_0,x_1,...,x_{m-1})-h(t,x,x,x_1,...,x_{m-1})\big|\le a_1(t)|x-x_0|^{\lambda_0}, \tag{1.5′}$$
$$(-1)^{n-m-1}h(t,x,x,x_1,\dots,x_{m-1})x\ge -a(t), \tag{1.6′}$$

be fulfilled, where $\lambda_0\in[0,1]$, $a_1:R_+\to R_+$ and $a:R_+\to R_+$ are measurable functions such that

$$\int_0^{+\infty}(1+t)^{n-m-\frac{1}{2}}\big(1+\tau_0^*(t)\big)^{(m-\frac{3}{2})\lambda_0}\big|\tau_0(t)-t\big|^{\lambda_0}a_1(t)dt<\mu_m^n \tag{1.7′}$$

and inequality (1.8) *is fulfilled. Besides, let for some* $t_0 > 0$ *on the set* $[0, t_0] \times R^{m+1}$ *the inequality*

$$\big|h(t, x, x, x_1, \dots, x_{m-1})\big| \le b_0\big(t, |x|\big) \sum_{i=1}^{m-1} (1 + x_i^2)$$

hold, where $b_0 : [0, t_0] \times R_+ \to R_+$ *is the function summable with respect to the first argument and nondecreasing with respect to the second. Then problem* (1.1), (1.2) *has at least one solution.*

Theorem 1.2. *Let on* $R_+ \times R^m$ *the conditions*

$$|h(t,x,\overline{x}_0,...,\overline{x}_{m-1})-h(t,x,x_0,...,x_{m-1})| \le \sum_{i=0}^{m-1} a_{1i}(t)|\overline{x}_i - x_i|, \tag{1.20}$$

$$(-1)^{n-m-1}\big[h(t, \overline{x}, x_0, \dots, x_{m-1}) - \\ -h(t, x, x_0, \dots, x_{m-1})\big](\overline{x} - x) \ge a_{00}(t)(\overline{x} - x)^2, \tag{1.21}$$

$$(-1)^{n-m-1}\big[h(t, x, \overline{x}_0, x_1, \dots, x_{m-1}) - \\ -h(t, x, x_0, x_1, \dots, x_{m-1})\big](\overline{x}_0 - x_0) \ge a_{01}(t)(\overline{x}_0 - x_0)^2, \tag{1.22}$$

be fulfilled, where $a_{1i} : R_+ \to R_+$ $(i = 0, \cdots, m-1)$ *and* $a_{0j} : R_+ \to R$ $(j = 0, 1)$ *are measurable functions satisfying inequality* (1.7) *for* $\lambda_i = 1$ $(i = 0, \dots, m-1)$ *and*

$$a_0(t) = a_{00}(t) + a_{01}(t) \ge 0 \quad for \quad t > 0. \tag{1.23}$$

Then problem (1.1), (1.2) *has at most one solution. If, however, in addition to* (1.7) *and* (1.20)–(1.23) *we have the conditions*

$$h^2(t, 0, \cdots, 0) \le l(t)a_0(t) \quad for \quad t > 0, \\ \int_0^{+\infty} (1+t)^{n-2m}\, l(t)dt < +\infty\ , \tag{1.24}$$

then problem (1.1), (1.2) *has one and only one solution.*

Proof. First we shall prove the uniqueness of the solution. Let u and $\overline{u}$ be two arbitrary solutions of problem (1.1), (1.2). It is assumed that $v(t) = \overline{u}(t) - u(t)$,

$$\Delta_0(t) = h\big(t, \overline{u}(t), \overline{u}(\tau_0(t)), \dots, \overline{u}^{(m-1)}(\tau_{m-1}(t))\big) - \\ -h\big(t, u(t), \overline{u}(\tau_0(t)), \dots, \overline{u}^{(m-1)}(\tau_{m-1}(t))\big), \tag{1.25}$$

$$\Delta_1(t) = h\big(t, u(t), \overline{u}(\tau_0(t)), \overline{u}'(\tau_1(t)), \dots, \overline{u}^{(m-1)}(\tau_{m-1}(t))\big) - \\ -h\big(t, u(t), u(\tau_0(t)), \overline{u}'(\tau_1(t)), \dots, \overline{u}^{(m-1)}(\tau_{m-1}(t))\big), \tag{1.26}$$

$$\Delta(t) = h\big(t, u(t), u(\tau_0(t)), \overline{u}'(\tau_1(t)), \dots, \overline{u}^{(m-1)}(\tau_{m-1}(t))\big) -$$

$$-h\big(t,u(t),u(\tau_0(t)),u'(\tau_1(t)),\dots,u^{(m-1)}(\tau_{m-1}(t))\big), \tag{1.27}$$

$$l_1(t)=\begin{cases}\frac{\Delta_1(t)}{v(\tau_0(t))} & \text{for} \quad v(\tau_0(t))\neq 0\\ 0 & \text{for} \quad v(\tau_0(t))=0\end{cases}. \tag{1.28}$$

It is clear that

$$v^{(i)}(0)=0 \quad (i=0,\dots,m-1), \quad v\in C_0^{n-1,m}.$$

Therefore

$$\big|v^{(i)}(t)\big|\leq (1+t)^{m-i-\frac12}\|v\|_{0,m} \quad (i=0,\dots,m-1),$$
$$\big|v(\tau_0(t))-v(t)\big|\leq \big(1+\tau_0^*(t)\big)^{m-\frac32}\big|\tau_0(t)-t\big|\|v\|_{0,m}.$$

On the other hand, on account of (1.20)–(1.23) and (1.25)–(1.28) we have

$$\begin{gathered}(-1)^{n-m-1}\Delta_0(t)v(t)\geq a_{00}(t)v^2(t), \quad (-1)^{n-m-1}l_1(t)\geq a_{01}(t),\\ |l_1(t)|\leq a_{10}(t), \quad |\Delta(t)|\leq \sum_{i=1}^{m-1}a_{1i}(t)|v^{(i)}(\tau_i(t))|\end{gathered} \tag{1.29}$$

and

$$\begin{gathered}(-1)^{n-m-1}v(t)v^{(n)}(t)=(-1)^{n-m-1}\Delta_0(t)v(t)+(-1)^{n-m-1}l_1(t)v^2(t)+\\ +(-1)^{n-m-1}l_1(t)\big[v(\tau_0(t))-v(t)\big]v(t)++(-1)^{n-m-1}\Delta(t)v(t)\geq\\ \geq a_0(t)v^2(t)-|l_1(t)||v(\tau_0(t))-v(t)||v(t)|-|\Delta(t)||v(t)|\geq\\ \geq -a_{10}(t)|v(\tau_0(t))-v(t)||v(t)|-\sum_{i=1}^{m-1}a_{1i}(t)|v^{(i)}(\tau_i(t))||v(t)|.\end{gathered}$$

Therefore

$$(-1)^{n-m}(1+t)^{n-2m}v(t)v^{(n)}(t)\leq (1+t)^{n-2m}\overline{a}(t)\|v\|_{0,m}, \tag{1.30}$$

where

$$\overline{a}(t)=(1+t)^{m-\frac12}\Big[a_{10}(t)\big(1+\tau_0^*(t)\big)^{m-\frac32}|\tau_0(t)-t|+\sum_{i=1}^{m-1}a_{1i}(t)\big(1+\tau_i(t)\big)^{m-i-\frac12}\Big].$$

On integrating inequality (1.30) from 0 to t and applying Lemmas 4.1 and 4.4 from [11], we obtain

$$\mu_m^n\int_0^t\big|v^{(m)}(s)\big|^2ds\leq w(t)+\|v\|_{0,m}^2\int_0^t(1+s)^{n-2m}\overline{a}(s)ds,$$

where

$$w(t) = (n-2m)\sum_{i=0}^{n-m-1}(-1)^{n-m-i}(i+1)v^{(i)}(t)v^{(n-2-i)}(t) -$$
$$-(1+t)^{n-2m}\sum_{i=0}^{n-m-1}(-1)^{n-m-i}v^{(i)}(t)v^{(n-1-i)}(t);$$

moreover,

$$\liminf_{t\to+\infty}|w(t)| = 0.$$

It is therefore clear that

$$\mu_m^n\|v\|_{0,m}^2 \le \|v\|_{0,m}^2\int_0^{+\infty}(1+t)^{n-2m}\overline{a}(s)ds.$$

Hence by (1.7) we find that $\|v\|_{0,m} = 0$. Thus problem (1.1), (1.2) has at most one solution.

To complete the proof of the theorem it remains to show that if in addition to (1.7) and (1.20)–(1.23) condition (1.24) is fulfilled, too, then problem (1.1), (1.2) is solvable.

By virtue of (1.21)–(1.24)

$$(-1)^{n-m-1}h(t,x,x,0,\dots,0)x =$$
$$= (-1)^{n-m-1}\big[h(t,x,x,0,\dots,0) - h(t,0,x,\dots,0)\big]x +$$
$$+(-1)^{n-m-1}\big[h(t,0,x,\dots,0)-h(t,0,\dots,0)\big]x+(-1)^{n-m-1}h(t,0,\dots,0)x \ge$$
$$\ge a_0(t)x^2 - l^{\frac12}(t)a_0^{\frac12}(t)|x| \ge -a(t),$$

where $a(t) = \frac14 l(t)$ satisfies condition (1.8). Thus all the conditions of Theorem 1.1 are fulfilled, thereby guaranteeing the solvability of problem (1.1), (1.2). $\square$

When $h(t,x,x_0,x_1,\dots,x_{m-1}) = \sum_{i=0}^{m-1}p_i(t)x_i + q(t)$ Theorem 1.2 implies

Corollary 1.1. *Let* $(-1)^{n-m-1}p_0(t)\ge 0$ *for* $t\in R_+$,

$$\int_0^{+\infty}(1+t)^{n-m-\frac12}\Big[|p_0(t)|\big(1+\tau_0^*(t)\big)^{m-\frac32}|\tau_0(t)-t| +$$
$$+\sum_{i=1}^{m-1}|p_i(t)|(1+\tau_i(t))^{m-i-\frac12}\Big]dt < \mu_m^n,$$

$$q^2(t)\le l(t)|p_0(t)| \quad for \quad t\in R_+, \quad \int_0^{+\infty}(1+t)^{n-2m}l(t)dt < +\infty.$$

Then problem (1.1′), (1.2) *has one and only one solution.*

Theorem 1.3. *Let on* $R_+ \times R^{m+1}$ *condition* (1.5) *and*

$$(-1)^{n-m-1} h(t,x,x,0,\dots,0)x \geq \gamma(1+t)^{-n}x^2 - a_2(t) \tag{1.31}$$

be fulfilled, where $\lambda_i \in [0,1]$ $(i=0,\dots,m-1)$, γ *is a positive constant,* $a_{1i} : R_+ \to R_+$ $(i=0,\dots,m-1)$, *and* $a_2 : R_+ \to R_+$ *are measurable functions such that*

$$\delta = \frac{n!}{(2m)!}\mu_m^n - \int_0^{+\infty} (1+t)^{n-\frac{1}{2}} \big[a_{10}(t)\big(1+\tau_{0*}(t)\big)^{-\frac{3}{2}\lambda_0} |\tau_0(t)-t|^{\lambda_0} + \\ + \sum_{i=1}^{m-1} a_{1i}(t)(1+\tau_i(t))^{-(i+\frac{1}{2})\lambda_i}\big]dt > 0, \tag{1.32}$$

$$\int_0^{+\infty} (1+t)^n a_2(t)dt < +\infty, \tag{1.33}$$

$$\gamma > \frac{m-1}{4}\gamma_n\Big[\frac{\gamma_n}{\delta} + \frac{(m-2)(4m^2-m-3)}{3} + 4\Big]^{m-1} - (-1)^m\frac{n!}{2}. \tag{1.34}$$

Then problem (1.1), (1.3) *has at least one solution.*

Proof. Problem (1.1), (1.3) is equivalent to problem (1.11), (1.3), where $f(u)(t) = h(t,u(t),u(\tau_0(t)),\dots,u^{(m-1)}(\tau_{m-1}(t)))$. Using Theorem 1.3 from [10], we shall prove that problem (1.11), (1.2) is solvable. First of all we would like to note that the operator $f : C^{n-1} \to L$ is continuous on account of (1.4). On the other hand, for any $u \in C^{n-1,m}$ inequalities (1.5) and (1.31) imply

$$(-1)^{n-m-1}u(t)f(u)(t) = \\ = (-1)^{n-m-1}\big[h(t,u(t),u(\tau_0(t)),\dots,u^{(m-1)}(\tau_{m-1}(t))) - \\ -h(t,u(t),u(t),0,\dots,0)\big]u(t) + (-1)^{n-m-1}h(t,u(t),u(t),0,\dots,0)u(t) \geq \\ \geq -a_{10}(t)|u(\tau_0(t)) - u(t)|^{\lambda_0}|u(t)| - \sum_{i=1}^{m-1} a_{1i}(t)|u^{(i)}(\tau_i(t))|^{\lambda_i}|u(t)| + \\ +\gamma(1+t)^{-n}|u(t)|^2 - a_2(t). \tag{1.35}$$

However, for any $u \in C^{n-1,m}$ and $i \in \{0,\dots,m-1\}$ we have the representation

$$u^{(i)}(t) = \frac{1}{(m-1-i)!}\int_{+\infty}^t (t-s)^{m-1-i}u^{(m)}(s)ds,$$

Therefore

$$|u^{(i)}(t)| \leq \int_t^{+\infty} (1+s)^{m-1-i}|u^{(m)}(s)|ds \leq$$

$$\leq \Big[\int_t^{+\infty}(1+s)^{-2-2i}ds\Big]^{\frac12}\Big[\int_t^{+\infty}(1+s)^{2m}|u^{(m)}(s)|^2ds\Big]^{\frac12}\leq$$
$$\leq (1+t)^{-i-\frac12}\|u\|_m \quad (i=0,\dots,m-1)\,, \tag{1.36}$$
$$|u(\tau_0(t))-u(t)|^{\lambda_0}|u(t)|=\Big|\int_t^{\tau_0(t)}u'(s)ds\Big|^{\lambda_0}|u(t)|\leq$$
$$\leq (1+\tau_{0*}(t))^{-\frac32\lambda_0}(1+t)^{-\frac12}|\tau_0(t)-t|^{\lambda_0}\|u\|_m^{1+\lambda_0}\leq$$
$$\leq \big(1+\tau_{0*}(t)\big)^{-\frac32\lambda_0}(1+t)^{-\frac12}\big|\tau_0(t)-t\big|^{\lambda_0}\big(1+\|u\|_m^2\big). \tag{1.37}$$

On account of (1.36) and (1.37) inequality (1.35) implies

$$(-1)^{n-m-1}u(t)f(u)(t)\geq\gamma(1+t)^{-n}|u(t)|^2-a_1(t)\|u_m\|^2-\widetilde a_2(t),$$

where

$$a_1(t)=(1+t)^{-\frac12}\Big[a_{10}(t)\big(1+\tau_{0*}(t)\big)^{-\frac32\lambda_0}|\tau_0(t)-t|^{\lambda_0}+$$
$$+\sum_{i=1}^{m-1}a_{1i}(t)\big(1+\tau_i(t)\big)^{-(i+\frac12)\lambda_i}\Big],\quad \widetilde a_2(t)=a_1(t)+a_2(t). \tag{1.38}$$

Moreover, by virtue of (1.5), (1.14), and (1.36), inequality (1.18) holds, where

$$b(t,x,y)=b_0(t,x)+\sum_{i=0}^{m-1}a_{1i}(t)\big(1+\tau_i(t)\big)^{-i-\frac12}y \tag{1.39}$$

and b_0 is the function given by equality (1.14). On the other hand, by (1.32), (1.38) it is obvious that

$$\delta=\frac{n!}{(2m)!}\mu_m^n-\int_0^{+\infty}(1+t)^na_1(t)dt>0 \tag{1.40}$$

and the function b satisfies condition (1.19).

Thus all the conditions of Theorem 1.3 from [10] are satisfied, thereby guaranteeing the solvability of problem (1.11), (1.3). □

Similarly to Theorem 1.3 we prove

Theorem 1.3′. *Let on $R_+\times R^{m+1}$ the conditions (1.5′) and*

$$(-1)^{n-m-1}h(t,x,x,x_1,\dots,x_{m-1})x\geq\gamma(1+t)^{-n}x^2-a_2(t)$$

be fulfilled, where $\lambda_0\in[0,1]$, γ is a positive constant and $a_i:R_+\to R_+$ $(i=1,2)$ are measurable functions such that

$$\delta=\frac{n!}{(2m)!}\mu_m^n-\int_0^{+\infty}(1+t)^{n-\frac12}\big(1+\tau_{0*}(t)\big)^{-\frac32\lambda_0}\big|\tau_0(t)-t\big|^{\lambda_0}a_1(t)dt>0$$

and inequalities (1.33) *and* (1.34) *are fulfilled. Moreover, let for some* $t_0 > 0$ *on* $[0, t_0] \times R^{m+1}$ *the inequality*

$$\big|h(t, x, x, x_1, \dots, x_{m-1})\big| \le b_0\big(t, |x|\big) \sum_{i=1}^{m-1} (1 + x_i^2)$$

hold, where $b_0 : [0, t_0] \times R_+ \to R_+$ *is a function summable with respect to the first argument and nondecreasing with respect to the second. Then problem* (1.1), (1.3) *has at least one solution.*

Theorem 1.4. *Let on* $R_+ \times R^{m+1}$ *conditions* $(1.20) - (1.22)$ *be fulfilled, where* $a_{1i} : R_+ \to R_+$ $(i = 0, \dots, m-1)$ *and* $a_{0j} : R_+ \to R$ $(j = 0, 1)$ *are measurable functions, and there exists a positive number* γ *such that*

$$a_0(t) = a_{00}(t) + a_{01}(t) > \gamma(1 + t)^{-n} \quad \textit{for} \quad t \in R_+ \tag{1.41}$$

and inequalities (1.32) *and* (1.34) *hold for* $\lambda_i = 1$ $(i = 0, \dots, m-1)$*. Then problem* (1.1), (1.3) *has at least one solution. If in addition to* $(1.20) - (1.22)$, (1.32), (1.34), *and* (1.41) *the condition*

$$\int_0^{+\infty} (1 + t)^n \, \frac{h^2(t, 0, \dots, 0)}{a_0(t)} dt < +\infty \tag{1.42}$$

is fulfilled, too, then problem (1.1), (1.3) *has one and only one solution.*

Proof. As noted above, problem (1.1), (1.3) is equivalent to problem (1.11), (1.3), where $f(u)(t) = h(t, u(t), u(\tau_0(t)), \dots, u^{(m-1)}(\tau_{m-1}(t)))$. Let us show that (1.11), (1.3) has at most one solution.

Let u and $\overline{u}$ be arbitrary functions from $C^{n-1,m}$ and $v(t) = \overline{u}(t) - u(t)$. Then the representation

$$\begin{aligned} &(-1)^{n-m-1}\big(\overline{u}(t) - u(t)\big)\big(f(\overline{u})(t) - f(u)(t)\big) = \\ &= (-1)^{n-m-1} l_1(t) v^2(t) + (-1)^{n-m-1} l_1(t)\big[v(\tau_0(t)) - v(t)\big] v(t) + \\ &+ (-1)^{n-m-1} \Delta_0(t) v(t) + (-1)^{n-m-1} \Delta(t) v(t) \end{aligned} \tag{1.43}$$

is valid, where Δ_0, Δ, and l_1 are functions given by equalities (1.25)–(1.28).

Inequalities (1.29) are fulfilled by (1.20)–(1.22). On the other hand,

$$|v^{(i)}(t)| \le (1 + t)^{-i-\frac{1}{2}} \|v\|_m = (1 + t)^{-i-\frac{1}{2}} \|\overline{u} - u\|_m \quad (i = 0, \dots, m-1),$$

$$\big|v(\tau_0(t)) - v(t)\big| = \Big|\int_t^{\tau_0(t)} v'(s) ds\Big| \le \big(1 + \tau_{0*}(t)\big)^{-\frac{3}{2}} |\tau_0(t) - t| \|\overline{u} - u\|_m .$$

Therefore (1.41) and (1.43) imply

$$\begin{aligned} &(-1)^{n-m-1}\big(\overline{u}(t) - u(t)\big)\big(f(\overline{u})(t) - f(u)(t)\big) \ge \\ &\ge \gamma(1 + t)^{-n}\big(\overline{u}(t) - u(t)\big)^2 - a_1(t)\|\overline{u} - u\|_m^2, \end{aligned}$$

where a_1 is the function given by equality (1.38) for $\lambda_i = 1$ $(i = 0,\dots,m-1)$ and satisfying condition (1.40). Therefore by Theorem 1.3 from [10] problem (1.1), (1.3) has at most one solution.

Now let condition (1.42) be fulfilled. Without loss of generality it can be assumed that the inequality $(1-\varepsilon)a_0(t) > \gamma(1+t)^{-n}$, where ε is a positive constant, holds instead of (1.41). Then (1.21) and (1.22) imply

$$(-1)^{n-m-1}h(t,x,x,0,\dots,0)x \geq a_0(t)x^2 - |h(t,0,\dots,0)||x| \geq$$
$$\geq \gamma(1+t)^{-n}x^2 + \varepsilon a_0(t)x^2 - 2\varepsilon^{\frac{1}{2}}a_0^{\frac{1}{2}}(t)|x|a_2^{\frac{1}{2}}(t) \geq \gamma(1+t)^{-n}x^2 - a_2(t),$$

where $a_2(t) = \frac{h^2(t,0,\dots,0)}{4\varepsilon a_0(t)}$. Moreover, since on account of (1.42) condition (1.33) is satisfied, by Theorem 1.3 problem (1.1), (1.3) is solvable. $\square$

The proven theorem immediately implies

Corollary 1.2. *Let* $(-1)^{n-m-1}p_0(t) > \gamma(1+t)^{-n}$ *for* $t \in R_+$,

$$\delta = \frac{n!}{(2m)!}\mu_m^n - \int_0^{+\infty}(1+t)^{n-\frac{1}{2}}\Big[|p_0(t)|\big(1+\tau_{0*}(t)\big)^{-\frac{3}{2}}|\tau_0(t)-t| + \sum_{i=1}^{m-1}|p_i(t)|(1+\tau_i(t))^{-i-\frac{1}{2}}\Big]dt > 0,$$

$$\int_0^{+\infty}(1+t)^n\frac{q^2(t)}{|p_0(t)|}dt < +\infty,$$

where γ *is a positive constant satisfying inequality* (1.34). *Then problem* (1.1′), (1.3) *has one and only one solution.*

§ 2. Oscillatory Solutions

2.1. Equations with Property O_m.

We introduce

Definition 2.1. Equation (0.1) has property O_m if each proper solution $u : [t_0,+\infty[\to R$ of this equation, satisfying the condition

$$\int_{t_0}^{+\infty}\big|u^{(m)}(t)\big|^2dt < +\infty, \tag{2.1}$$

is oscillatory when m is even, and either oscilatory or satisfying, on some interval $[t^*,+\infty[\subset[t_0,+\infty[$, the inequalities

$$(-1)^iu^{(i)}(t)u(t) > 0 \quad (i = 0,\dots,n-1) \tag{2.2}$$

when m is odd.

Before we proceed to formulating the theorem on equation (0.1) having property O_m we shall give the following auxiliary statement.

Lemma 2.1. *Let the function* $u : [t_0, +\infty[\to R$ *be locally absolutely continuous together with its derivatives up to order* $n-1$ *inclusive and satisfy the inequalities*

$$u(t) \neq 0, \quad \operatorname{mes}\big\{s \in [t, +\infty[: u^{(n)}(s) \neq 0\big\} > 0 \quad for \quad t \geq t_0, \tag{2.3}$$

$$(-1)^{n-m-1} u^{(n)}(t) u(t) \geq 0 \quad for \quad t \geq t_0. \tag{2.4}$$

Then there are $k \in \{0, \dots, n\}$ *and* $t^* \in [t_0, +\infty[$ *such that* $k+m$ *is odd and*

$$\begin{aligned} &u^{(i)}(t) u(t) > 0 \quad (i = 0, \dots, k-1), \\ &(-1)^{i-k} u^{(i)}(t) u(t) > 0 \quad (i = k, \dots, n-1) \quad for \quad t \geq t^*. \end{aligned} \tag{2.5}$$

Moreover, if $k = 0$*, then* $t^* = t_0$ *and therefore*

$$(-1)^i u^{(i)} u(t) > 0 \quad (i = 0, \dots, n-1) \quad for \quad t \geq t_0. \tag{2.6}$$

The above lemma immediately follows from Lemma 1.1 in the monograph [11].

For an arbitrary $\varepsilon > 0$ and an arbitrary positive $\lambda \neq 1$ we set

$$D_\varepsilon(\tau_0, \dots, \tau_{m-1}) =$$

$$= \Big\{(t, x_0, \dots, x_{m-1}) : t \geq \frac{1}{\varepsilon}, |x_i| \leq \varepsilon [\tau_i(t)]^{m-\frac{1}{2}-i} \; (i = 0, \dots, m-1)\Big\},$$

$$\sigma(\lambda) = \begin{cases} n - m + (m-1)\lambda & \text{for} \quad 0 < \lambda < 1 \\ n - 1 & \text{for} \quad \lambda > 1 \quad \text{and} \quad m \quad \text{is even} \\ n + \lambda - 2 & \text{for} \quad \lambda > 1 \quad \text{and} \quad m \quad \text{is odd} \end{cases}.$$

Theorem 2.1. *Let for some* $\varepsilon > 0$

$$\tau_i(t) \geq t \quad for \quad t \geq \varepsilon^{-1} \quad (i = 0, \dots, m-1) \tag{2.7}$$

and on the set $D_\varepsilon(\tau_0, \dots, \tau_{m-1})$ *the inequality*

$$(-1)^{n-m-1} g(t, x_0, \dots, x_{m-1}) \operatorname{sgn} x_0 \geq p_0(t) |x_0|^\lambda \tag{2.8}$$

hold, where $\lambda \neq 1$ *is a positive constant and* $p_0 : R_+ \to R_+$ *is a locally summable function such that*

$$\int_0^{+\infty} t^{\sigma(\lambda)} p_0(t) dt = +\infty. \tag{2.9}$$

Then equation (0.1) *has property* O_m.

Proof. Assume the contrary, i.e., that equation (0.1) has no property O_m. Then there is a proper nonoscillatory solution $u : [t_0, +\infty[\to R$ of this equation satisfying condition (2.1). Moreover, if m is odd, then on each interval $[t^*, +\infty[\subset [t_0, +\infty[$ at least one of inequalities (2.2) does not hold.

By condition (2.1) it can be assumed without loss of generality that $t_0 \geq \varepsilon^{-1}$, $u(t) \neq 0$ and $(t, u(\tau_0(t)), \dots, u^{(m-1)}(\tau_{m-1}(t))) \in D_\varepsilon(\tau_0, \dots, \tau_{m-1})$ for $t \geq t_0$. Then inequalities (2.3) and (2.4) are fulfilled on account of (2.7)–(2.9). By Lemma 2.1 there is $t^* \geq t_0$ such that we have

$$u'(t)u(t) > 0 \quad \text{for} \quad t \geq t^*, \tag{2.10}$$

but if m is odd, then

$$u'(t)u(t) > 0, \quad u''(t)u(t) > 0 \quad \text{for} \quad t \geq t^*. \tag{2.11}$$

Let $g_0(t) = g(t, u(\tau_0(t)), \dots, u^{(m-1)}(\tau_{m-1}(t)))|u(t)|^{-\lambda} \operatorname{sgn} u(t)$. Then

$$u^{(n)}(t) = g_0(t)\big|u(t)\big|^\lambda \operatorname{sgn} u(t). \tag{2.12}$$

On the other hand, due to (2.8) and the fact that the function u has a constant sign we have

$$(-1)^{n-m-1} g_0(t) \geq \eta(t) p_0(t) \quad \text{for} \quad t \geq t_0, \tag{2.13}$$

where $\eta(t) = |u(\tau_0(t))|^\lambda |u(t)|^{-\lambda}$. Moreover, by (2.7) and (2.10) we have $\eta(t) \geq 1$ for $t \geq t^*$. Therefore (2.9) and (2.13) imply

$$(-1)^{n-m-1} g_0(t) \geq 0 \quad \text{for} \quad t \geq t_0, \quad \int_0^{+\infty} t^{\sigma(\lambda)} \big|g_0(t)\big| dt = +\infty. \tag{2.14}$$

By virtue of condition (2.14) and Theorems 15.1, 15.2, and 15.4 from the monograph [11] we conclude that for the even m (odd m), equation (2.12) has no proper nonoscillatory solution satisfying condition (2.1) (conditions (2.1) and (2.11)). The obtained contradiction proves the theorem. □

Quite similarly, using Theorems 1.6 and 1.7 from [11] we shall prove

Theorem 2.2. *Let inequalities* (2.7) *be fulfilled for some* $\varepsilon > 0$ *and on the set* $D_\varepsilon(\tau_0, \dots, \tau_{m-1})$ *the condition*

$$(-1)^{n-m-1} g(t, x_0, \dots, x_{m-1}) \operatorname{sgn} x_0 \geq p_0(t)|x_0| \tag{2.15}$$

hold, where $p_0 : R_+ \to R_+$ *is a locally summable function such that*

$$\limsup_{t \to +\infty} \Big(t \int_t^{+\infty} s^{n-2} p_0(s) ds\Big) > (n-1)!\,. \tag{2.16}$$

Then equation (0.1) *has property* O_m.

2.2. Theorem on the Existence of Proper Oscillatory Solutions of Equation (0.1).

Theorem 2.3. *Let for some* $\varepsilon > 0$

$$\tau_i(t) \geq t + \Delta(t) \quad \text{for} \quad t \geq \varepsilon^{-1} \quad (i = 0, \dots, m-1) \tag{2.17}$$

and on the $D_\varepsilon(\tau_0, \dots, \tau_{m-1})$ *the conditions*

$$(-1)^{n-m-1} g(t, x_0, \dots, x_{m-1}) x_0 \geq 0, \tag{2.18}$$

$$\big|g(t, x, x_1, ..., x_{m-1}) - g(t, x_0, x_1, ..., x_{m-1})\big| \leq l(t) |x - x_0|^{\lambda_0} \tag{2.19}$$

hold, where $\lambda_0 \in [0, 1]$, $\Delta : R_+ \to]0, +\infty[$ *is a continuous function and* $l : R_+ \to R_+$ *is a measurable function such that*

$$\int_{\varepsilon^{-1}}^{+\infty} (1+t)^{n-m-\frac{1}{2}} \big(1+\tau_0(t)\big)^{(m-\frac{3}{2})\lambda_0} \big(\tau_0(t) - t\big)^{\lambda_0} l(t) dt < +\infty. \tag{2.20}$$

Moreover, let equation (0.1) *have property* O_m. *Then for the even* m *(odd* m*) this equation has an* m*-parametric (*$(m-1)$*-parametric) family of proper oscillatory solutions.*

Proof. Choose $t_0 \geq \frac{1}{\varepsilon}$ such that

$$\int_{t_0}^{+\infty} (1+t)^{n-m-\frac{1}{2}} \big(1+\tau_0(t)\big)^{(m-\frac{3}{2})\lambda_0} \big(\tau_0(t) - t\big)^{\lambda_0} l(t) dt < \mu_m^n. \tag{2.21}$$

It can be assumed without loss of generality that $\tau_i(t) = t$ for $0 \leq t \leq t_0$ $(i = 0, \dots, m-1)$. We set

$$\chi_i(t, x) = \begin{cases} x & \text{for} \quad |x| \leq \varepsilon[\tau_i(t)]^{m-\frac{1}{2}-i} \\ \varepsilon[\tau_i(t)]^{m-\frac{1}{2}-i} \operatorname{sgn} x & \text{for} \quad |x| > \varepsilon[\tau_i(t)]^{m-\frac{1}{2}-i} \end{cases},$$

$$h(t, x, x_0, x_1, \dots, x_{m-1}) =$$

$$= \begin{cases} 0 & \text{for} \quad 0 \leq t \leq t_0 \\ g\big(t, \chi_0(t, x_0), \dots, \chi_{m-1}(t, x_{m-1})\big) & \text{for} \quad t \geq t_0 \end{cases} \tag{2.22}$$

and for any $c_0, \dots, c_{m-1}$ which are not simultaneously equal to zero we consider problem (1.1), (1.2).

Due to (2.17)–(2.19), (2.21), and (2.22), conditions (1.5′), (1.6′), and (1.7′) are fulfilled with $a_1(t) = 0$ for $a \leq t \leq t_0$, $a_1(t) = l(t)$ for $t \geq t_0$, and $a(t) = 0$ for $t \geq 0$.

By Theorem 1.1′, problem (1.1), (1.2) has a solution u. From (2.17), (2.18), and (2.22) it follows that u is a proper solution. On the other hand, by condition (2.1) there is $t^* \geq t_0$ such that $(t, u(\tau_0(t)), \dots, u^{(m-1)}(\tau_{m-1}(t))) \in D_\varepsilon(\tau_0, \dots, \tau_{m-1})$ for $t \geq t^*$. Hence due to (2.22) it is obvious that u is a solution of equation (0.1) on $[t^*, +\infty[$.

However, by our assumption equation (0.1) has property O_m. Therefore, when m is even, u is the oscillatory solution, and when m is odd, it is either oscillatory or satisfies inequalities (2.2) on the interval $[t^*,+\infty[$. If u satisfies (2.2), then by (1.2), (2.18), and (2.22) we shall have

$$(-1)^i c_i c_j > 0 \quad (i=0,\dots,m-1). \tag{2.23}$$

Thus if at least one of inequalities (2.23) is not fulfilled, say, $c_{m-1}=0$, then u will be an oscillatory solution. We have thereby shown that when m is even (m is odd), to arbitrary numbers $c_0,\dots,c_{m-2}$ ($c_0,\dots,c_{m-1}$), which are not simultaneously equal to zero, there corresponds at least one oscillatory solution of equation (0.1). □

By Theorems 2.1 and 2.2, Theorem 2.3 gives rise to the following propositions.

Corollary 2.1. *Let inequalities* (2.17) *be fulfilled for some* $\varepsilon>0$ *and on the set* $D_\varepsilon(\tau_0,\dots,\tau_{m-1})$ *conditions* (2.8) *and* (2.19) *hold, where* $\lambda\neq 1$ *is a positive constant* $\lambda_0\in[0,1]$, $\Delta: R_+\to]0,+\infty[$ *is a continuous function, while* $p_0: R_+\to R_+$ *and* $l_0: R_+\to R_+$ *are locally summable functions satisfying conditions* (2.9) *and* (2.20). *Then for the even* m *(odd* m*) equation* (0.1) *has a* m*-parametric* ($(m-1)$*-parametric*) *family of proper oscillatory solutions.*

Corollary 2.2. *Let inequalities* (2.17) *be fulfilled for some* $\varepsilon>0$ *and on the set* $D_\varepsilon(\tau_0,\dots,\tau_{m-1})$ *conditions* (2.15) *and* (2.19) *hold, where* $\lambda_0\in[0,1]$, $\Delta: R_+\to]0,+\infty[$ *is a continuous function, while* $p_0: R_+\to R_+$ *and* $l: R_+\to R_+$ *are locally integrable functions satisfying conditions* (2.16) *and* (2.20). *Then for even* m *(odd* m*) equation* (0.1) *has an* m*-parametric* ($(m-1)$*-parametric*) *family of proper oscillatory solutions.*

2.3. Sufficient Conditions for the Existence of Proper Oscillatory Solutions of Equations (0.2) and (0.3). Conditions 2.1 and 2.2 imply the following propositions.

Corollary 2.3. *Let for some* $t_0>0$ *the inequalities*

$$\tau(t)\geq t+\Delta(t), \quad (-1)^{n-m-1}p(t)\geq 0 \quad \text{for} \quad t\geq t_0 \tag{2.24}$$

hold, where $\Delta:[t_0,+\infty[\to]0,+\infty[$ *is a continuous function. Moreover, let*

$$\int_{t_0}^{+\infty} t^{\sigma(\lambda)}|p(t)|dt=+\infty$$

and

$$\int_{t_0}^{+\infty}(1+t)^{n-m-\frac12}\big(1+\tau(t)\big)^{(m-\frac12)\lambda-\lambda_0}\big(\tau_0(t)-t\big)^{\lambda_0}|p(t)|dt<+\infty,$$

where $\lambda_0 = \lambda$ for $0 < \lambda < 1$ and $\lambda_0 = 1$ for $\lambda > 1$. Then for even m (odd m) equation (0.2) *has an m-parametric ($(m-1)$-parametric) family of proper oscillatory solutions.*

Corollary 2.4. *Let for some $t_0 > 0$ inequalities* (2.24) *hold, where $\Delta : [t_0, +\infty[\to]0, +\infty[$ is a continuous function. Moreover, let*

$$\limsup_{t\to+\infty}\Big(t\int_t^{+\infty} s^{n-2}|p(s)|ds\Big) > (n-1)!$$

and

$$\int_{t_0}^{+\infty}(1+t)^{n-m-\frac{1}{2}}\big(1+\tau(t)\big)^{m-\frac{3}{2}}\big(\tau(t)-t\big)|p(t)|dt < +\infty.$$

Then for even m (odd m) equation (0.3) *has an m-parametric ($(m-1)$-parametric) family of proper oscillatory solutions.*

§ 3. Vanishing-at-Infinity Solutions

3.1. Existence Theorem for Equation (0.1). For any $s \in R$ and $\varepsilon > 0$ we set

$$[s]_+ = \frac{1}{2}\big(|s| + s\big),$$

$$D_\varepsilon^*(\tau_1, \dots, \tau_{m-1}) = \Big\{(t, x_0, x_1, \dots, x_{m-1}) : t \geq \frac{1}{\varepsilon},$$
$$|x_0| \leq \varepsilon t^{-\frac{1}{2}}, |x_i| \leq [\tau_i(t)]^{-i-\frac{1}{2}} (i = 0, \dots, m-1)\Big\}.$$

Theorem 3.1. *Let for some $\varepsilon > 0$*

$$\tau_i(t) \geq t + \Delta(t) \quad \text{for} \quad t \geq \varepsilon^{-1} \quad (i = 0, \dots, m-1) \tag{3.1}$$

and on the set $D_\varepsilon^*(\tau_1, \dots, \tau_{m-1})$ the inequalities

$$(-1)^{n-m-1} g(t, x_0, \dots, x_{m-1}) x_0 \geq \big[\gamma(1+t)^{-n} x_0^2 - l_0(t)\big]_+, \tag{3.2}$$

$$\big|g(t, x, x_1, \dots, x_{m-1}) - g(t, x_0, x_1, \dots, x_{m-1})\big| \leq l(t)|x - x_0|^{\lambda_0} \tag{3.3}$$

hold, where

$$\gamma > \gamma_{0n}, \tag{3.4}$$

$\lambda_0 \in [0, 1]$, $\Delta : R_+ \to]0, +\infty[$ is a continuous function, and $l : R_+ \to R_+$ are measurable functions such that

$$\int_{\varepsilon^{-1}}^{+\infty} t^n l_0(t) dt < +\infty, \quad \int_{\varepsilon^{-1}}^{+\infty} t^{n-\frac{1}{2}-\frac{3}{2}\lambda_0}\big(\tau_0(t) - t\big)^{\lambda_0} l(t) dt < +\infty. \tag{3.5}$$

Then for even m (odd m) equation (0.1) *has an m-parametric ($(m-1)$-parametric) family of vanishing-at-infinity proper oscillatory solutions.*

Proof. By the definition of γ_{0n} and condition (3.5) there is $t_0 > \frac{1}{\varepsilon}$ such that

$$\delta = \frac{n!}{(2m)!}\mu_m^n - \int_{t_0}^{+\infty} (1+t)^{n-\frac{1}{2}-\frac{3}{2}\lambda_0}\big(\tau_0(t)-t\big)^{\lambda_0} l(t)dt > 0 \qquad (3.6)$$

and inequality (1.34) is fulfilled. It can be assumed without loss of generality that $\tau_i(t) = t$ for $0 \le t \le t_0$ $(i = 0, \dots, m-1)$.

Let

$$\chi_0(t,x) = \begin{cases} x & \text{for} \quad |x| \le \varepsilon t^{-\frac{1}{2}} \\ \varepsilon t^{-\frac{1}{2}} \operatorname{sgn} x & \text{for} \quad |x| > \varepsilon t^{-\frac{1}{2}} \end{cases},$$

$$\chi(t,x) = \begin{cases} 1 & \text{for} \quad x = 0 \\ \frac{\chi_0(t,x)}{x} & \text{for} \quad x \ne 0 \end{cases}.$$

If $i \in \{1, \dots, m-1\}$, then

$$\chi_i(t,x) = \begin{cases} x & \text{for} \quad |x| \le \varepsilon[\tau_i(t)]^{-\frac{1}{2}-i} \\ \varepsilon[\tau_i(t)]^{-\frac{1}{2}-i} \operatorname{sgn} x & \text{for} \quad |x| > \varepsilon[\tau_i(t)]^{-\frac{1}{2}-i} \end{cases}.$$

We set

$$h(t,x,x_0,\dots,x_{m-1}) = \gamma(1+t)^{-n}x \quad \text{for} \quad 0 \le t \le t_0, \qquad (3.7)$$

$$h(t,x,x_0,\dots,x_{m-1}) = \gamma(1+t)^{-n}x +$$
$$+\chi(t,x)\big[g(t,\chi_0(t,x_1),\dots,\chi_{m-1}(t,x_{m-1}))-\gamma(1+t)^{-n}\chi_0(t,x)\big] \qquad (3.8)$$
$$\text{for} \quad t > t_0.$$

Let $c_0, \dots, c_{m-1}$ be arbitrary numbers which are not simultaneously equal to zero. Moreover, if m is odd, then $c_{m-1} = 0$. We shall consider problem (1.1), (1.3).

By virtue of (3.1)–(3.8) all the conditions of Theorem 1.3′ are fulfilled, where $a_i(t) = 0$ for $0 \le t \le t_0$ $(i = 1, 2)$, $a_1(t) = l(t)$ and $a_2(t) = l_0(t)$ for $t \ge t_0$, $b_0(t,x) \equiv \gamma(1+t)^{-n}x$. Therefore problem (1.1), (1.3) has a solution u which due to (3.1), (3.2), (3.7), and (3.8) is proper and satisfies the inequalities

$$\begin{gathered}(-1)^{n-m-1}u^{(n)}(t)u\big(\tau_0(t)\big) \ge 0, \\ \operatorname{mes}\big\{s \in [t,+\infty[: u^{(n)}(s) \ne 0\big\} > 0 \quad \text{for} \quad t \ge 0.\end{gathered} \qquad (3.9)$$

On the other hand, by Lemma 4.5 from [11],

$$\lim_{t\to+\infty} \big(t^{t+\frac{1}{2}}u^{(i)}(t)\big) = 0 \quad (i = 0, \dots, m-1). \qquad (3.10)$$

By Lemma 2.1 it follows from (3.9) and (3.10) that for even m the solution u is oscillatory and for odd m it is either oscillatory or satisfies the inequalities

$$(-1)^i u^{(i)}(t)u(t) > 0 \quad \text{for} \quad t \geq 0 \quad (i = 0, \dots, n-1).$$

The latter assertion, however, can be discarded because when m is odd, then $u^{(m-1)}(0) = c_{m-1} = 0$. Therefore u is an oscillatory solution for odd m as well.

By (3.10) there is $t^* > t_0$ such that $(t, u(\tau_0(t)), \dots, u^{(m-1)}(\tau_{m-1}(t))) \in D^*_\varepsilon(\tau_1, \dots, \tau_{m-1})$ for $t > t^*$. Hence on account of (3.8) it is clear that u is a solution of equation (0.1) on $[t^*, +\infty[$. We have thereby shown that when m is even (m is odd), to arbitrary numbers $c_0, \dots, c_{m-1}$ ($c_0, \dots, c_{m-2}$) which are not simultaneously zero, there corresponds at least one vanishing-at-infinity proper oscillatory solution of equation (0.1). □

3.2. Corollaries for Equation (0.2).

Corollary 3.1. *Let $\lambda > 1$ and the conditions*

$$\tau(t) \geq t + \Delta(t), \quad (-1)^{n-m-1}p(t) > 0, \tag{3.11}$$

$$\begin{gathered} \int_{t_0}^{+\infty} \left|t^n p(t)\right|^{-\frac{2}{\lambda-1}} dt < +\infty, \\ \int_{t_0}^{+\infty} t^{n-\frac{3+\lambda}{2}} \big(\tau(t) - t\big)|p(t)| dt < +\infty \end{gathered} \tag{3.12}$$

be fulfilled for some $t_0 > 0$ and a continuous function $\Delta : [t_0, +\infty[\to]0, +\infty[$. Then for the even m (odd m) equation (0.2) has the m-parametric ($(m-1)$-parametric) family of proper oscillatory solutions.

Proof. Let γ be an arbitrary positive number satisfying inequality (3.4). Then by the Young inequality we obtain

$$|p(t)||x_0|^{\lambda+1} \geq \gamma x_0^2 - l_0(t) \quad \text{for} \quad t \geq t_0, \tag{3.13}$$

where

$$l_0(t) = \gamma^{\frac{\lambda+1}{\lambda-1}}(1+t)^{-\frac{n(n+1)}{\lambda-1}}|p(t)|^{-\frac{2}{\lambda-1}}.$$

We set $\varepsilon = \frac{1}{t_0}$, $\tau_0(t) = \tau(t)$, $\tau_i(t) = t + \Delta(t)$ $(i = 1, \dots, m-1)$, and $g(t, x_0, \dots, x_{m-1}) = p(t)|x_0|^\lambda \operatorname{sgn} x_0$. By (3.11)–(3.13) inequalities (3.1) are now fulfilled and on the set $D^*_\varepsilon(\tau_1, \dots, \tau_{m-1})$ conditions (3.2) and (3.3) hold, where $\lambda_0 = 1$ and $l(t) = \lambda t^{-\frac{\lambda-1}{2}}|p(t)|$. Moreover, l_0 and l satisfy conditions (3.5). Thus all the conditions of Theorem 3.1 are fulfilled. □

The propositions below are proved quite similarly.

Corollary 3.2. *Let $0 < \lambda < 1$ and the conditions*

$$\tau(t) \geq t + \Delta(t), \quad (-1)^{n-m-1} t^{n+\frac{1-\lambda}{2}} p(t) \geq \eta \quad for \quad t \geq t_0,$$
$$\int_{t_0}^{+\infty} t^{n-\frac{1+3\lambda}{2}} \big|\tau(t) - t\big|^{\lambda} |p(t)| dt < +\infty$$

hold for some $t_0 > 0$, $\eta > 0$ and a continuous function $\Delta : [t_0, +\infty[\to]0, +\infty[$. Then for even m (odd m) equation (0.2) has an m-parametric ($(m-1)$-parametric) family of vanishing-at-infinity proper oscillatory solutions.

3.3. Biernacki's Problem for Equations (0.3) and (0.4). By $Z^{(n)}(p;\tau)$ and $Z^{(n)}(p_0, \dots, p_{m-1}; \tau_0, \dots, \tau_{m-1})$ we denote respectively the spaces of vanishing-at-infinity solutions of equations (0.3) and (0.4), and by $\dim Z$ we denote the dimension of the space Z. For the case $\tau(t) \equiv t$ we set $Z^{(n)}(p) = Z^{(n)}(p;\tau)$. M. Biernacki [12] showed that if p is continuously differentiable and $p(t) \downarrow -\infty$ for $t \to +\infty$, then $\dim Z^{(4)}(p) \geq 1$, and he put forward the hypothesis that the inequality $\dim Z^{(4)}(p) \geq 2$ holds under the same restrictions on p. This hypothesis was later substantiated by M. Švec [13]. More exactly, he proved a more general proposition: if p is continuous and for some $t_0 > 0$ and $\eta > 0$ satisfies the inequality $p(t) \leq -\eta$ for $t \geq t_0$, then $\dim Z^{(4)}(p) \geq 2$. The question about dimension of the space of vanishing-at-infnity solutions of linear homogeneous differential equations of an arbitrary order was initially treated in [14]. [3] In particular, it is shown there that if p is locally summable and $(-1)^{n-m-1} t^n p(t) \to +\infty$ for $t \to +\infty$, then $\dim Z^{(n)}(p) \geq m$. The problem of dimensions of the spaces $Z^{(n)}(p;\tau)$ and $Z^{(n)}(p_0, \dots, p_{m-1}; \tau_0, \dots, \tau_{m-1})$ has never been studied for the cases $\tau(t) \not\equiv 0$ and $\tau_i(t) \not\equiv t$ $(i = 0, \dots, m-1)$.

Theorem 3.2. *If*

$$\liminf_{t \to +\infty} \big[(-1)^{n-m-1} t^n p_0(t)\big] > \gamma_{0n}, \quad \int_0^{+\infty} t^{n-\frac{1}{2}} \widetilde{p}(t) dt < +\infty, \tag{3.14}$$

where $\widetilde{p}(t) = (1 + \tau_{0*}(t))^{-\frac{3}{2}} |\tau_0(t) - t||p_0(t)| + \sum_{i=1}^{m-1} (1 + |\tau_i(t)|)^{-i-\frac{1}{2}} |p_i(t)|$ and $\tau_{0*}(t) = \min\{t, |\tau_0(t)|\}$, then

$$\dim Z^{(n)}(p_0, \dots, p_{m-1}; \tau_0, \dots, \tau_{m-1}) \geq m. \tag{3.15}$$

Proof. By (0.5) and (3.14) there are positive numbers t_0 and γ such that $\tau_i(t) > 0$ $(i = 0, \dots, m-1)$, $(-1)^{n-m-1} p_0(t) > \gamma(1+t)^{-n}$ for $t \geq t_0$,

$$\delta = \frac{n!}{(2m)!} \mu_m^n - \int_{t_0}^{+\infty} t^{n-\frac{1}{2}} \widetilde{p}(t) dt > 0$$

[3] See also §§4 and 5 of [11] where a detailed account of the results connected with this problem is given.

and inequality (1.34) holds. It can be assumed without loss of generality that $p_0(t) = 2\gamma(1+t)^{-n}$, $p_i(t) = 0$ $(i = 1, \dots, m-1)$, and $\tau_i(t) = t$ $(i = 0, \dots, m-1)$ for $0 \le t \le t_0$. Now, obviously, all the conditions of Corollary 1.2 will be fulfilled. Therefore problem (0.2), (1.3) has one and only one solution for any $c_0, \dots, c_{m-1}$. However, as mentioned above, this solution is vanishing at infinity, and therefore inequality (3.15) is valid. □

The theorem proved for equation (0.3) gives rise to

Corollary 3.3. *If*

$$\liminf_{t\to+\infty} \left[(-1)^{n-m-1} t^n p(t)\right] > \gamma_{0n}, \quad \int_0^{+\infty} t^{n-\frac{1}{2}} \widetilde{p}(t)dt < +\infty,$$

where $\widetilde{p}(t) = (1+\tau_*(t))^{-\frac{3}{2}} |\tau(t) - t||p(t)|$ *and* $\tau_*(t) = \min\{t, |\tau(t)|\}$, *then*

$$\dim Z^{(n)}(p;\tau) \ge m.$$

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