GEORGIAN MATHEMATICAL JOURNAL: Vol. 2, No. 4, 1995, 419-424

GLOBAL DIMENSIONS OF SUBIDEALIZER RINGS

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ABSTRACT. Recently, there have been many results which show that the global dimension of certain rings can be computed using a proper subclass of the cyclic modules, e.g., the simple modules. In this paper we view calculating global dimensions in this fashion as a property of a ring and show that this is a property which transfers to the ring's idealizer and subidealizer ring.

1. INTRODUCTION

Let R be an associative ring with unity. The right global dimension of R is defined to be

$$\mathrm{r.gl.dim}\, R = \sup \mathrm{pd}(A) | A \text{ is a right } R\text{-module}$$

where $\mathrm{pd}(A) = \inf\{n \geq 0 : \mathrm{Ext}\, R^{n+1}(A, \_) = 0\}$ or $\mathrm{pd}(A) = \infty$ if $\mathrm{Ext}\, R^n(A, \_) \neq 0$ for all $n \geq 0$. Usually $\mathrm{r.gl.dim}\, R$ is computed using Auslander's classical result [1, Theorem 9.12], $\mathrm{r.gl.dim}\, R = \sup\{\mathrm{pd}(A) | A_R$ is cyclic$\}$.

Much work has been done to reduce the number of cyclic modules that have to be checked in order to compute the global dimension. For example, it is well known that $\mathrm{r.gl.dim}\, R = \sup\{\mathrm{pd}(A) | A_R$ is simple$\}$ when R is commutative noetherian. Similar results have been given in [2–8].

Teply and Koker [5,8] have exhibited rings with Krull dimension (in the sense of Gordon and Robson [9]) such that $\mathrm{r.gl.dim}\, R = \sup\{\mathrm{pd}(C) | C_R$ is β-critical cyclic, $\beta < \alpha\}$, where $\mathrm{r.K.dim}\, R = \alpha$. For convenience, a ring will be called *α-proper* if $\mathrm{r.K.dim}\, R \geq \alpha$ and $\mathrm{r.gl.dim}\, R = \sup\{\mathrm{pd}(C) | C_R$ is β-critical cyclic, $\beta < \alpha\}$. Most of the results in [5,8] are for rings where $\mathrm{r.K.dim}\, A = \alpha$ or $\alpha = 1$. Here, we treat α-proper as a property and show that this is a property that passes from a ring to its idealizers and subidealizers.

1991 *Mathematics Subject Classification.* 16A60, 16A55, 16A33.
Key words and phrases. Idealizer of an ideal, subidealizer of an ideal, α-proper ring, Krull dimension, global dimension.

1072-947X/95/0700-0419$07.50/0

Let T be a ring and let M be a right ideal of T. The idealizer, S, of M in T is the largest subring of T which contains M as a 2-sided ideal. Thus, $S = \{t \in T | tM \subset M\}$. For examples and basic facts about idealizers, see [10]. There has been much work done to compare the properties of a ring and its idealizers. For example, see [11–13]. In general, a ring may have little in common with its idealizers unless some assumptions are made. A right ideal, M,is called semimaximal if it is the intersection of finitely many maximal right ideals, and M is generative if $TM = T$. If M is semimaximal, then there exists a generative semimaximal right ideal $M^* \supset M$ which has the same idealizer in T as M. See [10, Proposition 4.19]. It is well known [11, Theorem 5] that if S is the idealizer of a semimaximal right ideal of T, then $\operatorname{r.gl.dim} S = \sup\{1, \operatorname{r.gl.dim} T\}$.

Assume that T is α-proper, $\alpha \geq 1$, and that S is the idealizer of a semimaximal right ideal M of T. By [12, Theorem 3] $\operatorname{r.K.dim} S = \operatorname{r.K.dim} T$. Let $q = \sup\{\operatorname{pd}(B) | B_S \text{ is } \beta\text{-critical cyclic}, \beta < \alpha\}$. By [14, proposition 2.1], $\operatorname{pd}(A_S) = \operatorname{pd}(A_T)$ for all right T-modules A. Let C be any β-critical cyclic T-module. In order to show that $\operatorname{r.gl.dim} T \leq q$, it will suffice to show that $\operatorname{pd}(C_T) \leq q$, and by the comment above, this can be achieved by showing that $\operatorname{pd}(C_S) \leq q$. By [12, Proposition 2], $\operatorname{K.dim} C_S = \operatorname{K.dim} C_T = \beta < \alpha$. In turn,every cyclic subfactor of C_S has Krull dimension α. By [5, Lemma 2.1], we have that $\operatorname{pd}(C_S) \leq q$, and thus, $\operatorname{r.gl.dim} T \leq q$.

Now, $\operatorname{r.gl.dim} S = \sup\{1, \operatorname{r.gl.dim} T\}$. Since $\operatorname{r.K.dim}\ T \geq 1$, T is not semisimple, i.e., $\operatorname{r.gl.dim} T \geq 1$, and so, $\operatorname{r.gl.dim} S = \operatorname{r.gl.dim} T$. Therefore, $\operatorname{r.gl.dim} S \leq q$. It is clear that $\operatorname{r.gl.dim} S \geq q$. Hence, r.gl.dim $S = q$, that is, S is α-proper. We have proved the following result.

Proposition 1.1. *Let S be the idealizer of a semimaximal* (*generative*) *right ideal M of T with* $\operatorname{r.K.dim} T \geq \alpha \geq 1$. *If T is α-proper, then S is α-proper.*

We will say that a ring R is weak α-proper if $\operatorname{r.K.dim}\ R \geq \alpha$ and w.gl.dim $R = \sup\{\operatorname{wd}(C) | C_R \text{ is } \beta\text{-critical cyclic}, \beta < \alpha\}$. Using a similar argument gives the following.

Proposition 1.2. *Let S be the idealizer of a semimaximal* (*generative*) *right ideal M of T with* $\operatorname{r.K.dim} T \geq \alpha \geq 1$. *If T is weak α-proper, then S is weak α-proper.*

2. Subidealizers

Let T be a ring and let M be a right ideal of T. A subidealizer R of M in T is any subring of T which contains M as a 2-sided ideal. For example, if Z is any subring of the center of R, then $M + Z$ is a subidealizer of M in T. As with idealizers, much work has been done to compare properties of rings and their subidealizers. See, for example, [14–16].

It can be shown that if M is a semimaximal right ideal of T and S is the idealizer, then S/M is a semisimple ring. We say that a subidealizer is tame if R/M is a semisimple ring. Goodearl has shown [14] that for a tame subidealizer R in T,

$$\dim T \leq \dim R \leq \dim T + 1,$$

where dim stands for any of the homological dimensions. Here, we show that the property of α-proper transfers from rings to their subideals.

To begin, we need a result similar to [13, Proposition 2.11]. We relax the condition of $(T/M)_R$ having finite length to $\mathrm{K.dim}(T/M)_R < \mathrm{r.K.dim}\, R$.

Proposition 2.1. *Let T be a right noetherian ring with generative right ideal M. Let R be any tame subidealizer of M in T such that $\gamma = \mathrm{K.dim}(T/M)_R < \mathrm{r.K.dim}\, R = \alpha$ and $\alpha \geq 1$. Then $\mathrm{r.gl.dim}\, R = \sup\{1, m, \mathrm{r.gl.dim}\, T\}$, where $m = \sup\{\mathrm{pd}(A) | A_R$ is β-critical cyclic, $\beta \leq \gamma\}$*

Proof. Let $n = \mathrm{r.gl.dim}\, T$. It is clear that $\mathrm{r.gl.dim}\, R \geq n$, $\mathrm{r.gl.dim}\, R \geq m$, and that $\mathrm{r.K.dim}\, R \geq 1$ implies that $\mathrm{r.gl.dim}\, R \geq 1$. Thus, $\mathrm{r.gl.dim}\, R \geq \sup 1, m, n$.

Let $k = \sup 1, m, n$. We must show that r.g.dim $R \leq k$. It will suffice to show that $\mathrm{pd}(R/J)_R \leq k$ for all right ideals J. Since T is right noetherian, JT is a finitely generated right ideal of T, and thus, JT/J is an R-homomorphic image of a direct sum of finitely many copies of T/M. Therefore, $\mathrm{K.dim}(JT/J)_R \leq \mathrm{K.dim}(T/M)_R \leq \gamma$. Thus, [5, Lemma 2.1] implies that $\mathrm{pd}(JT/J)_R \leq m \leq k$. Consider the exact sequence of R-modules

$$0 \to JT/J \to T/J \to T/JT \to 0. \tag{$*$}$$

By [14, Proposition 2.1], $\mathrm{pd}(T/JT)_R = \mathrm{pd}(T/JT)_T \leq n \leq k$. Applying $\mathrm{Ext}\, R(-, B)$ to $(*)$ gives

$$\mathrm{Ext}\, R^{k+1}(T/JT.B) \to \mathrm{Ext}\, R^{k+1}(T/J, B) \to \mathrm{Ext}\, R^{k+1}(JT/J, B)$$

exact. Now, $\mathrm{Ext}\, R^{k+1}(T/JT, B) = 0$ gives that $\mathrm{pd}(J/T)$. Applying $\mathrm{Ext}\, R(-, B)$ to

$$0 \to R/J \to T/J \to T/R \to 0$$

gives

$$\mathrm{Ext}\, R^{k+1}(T/J.B) \to \mathrm{Ext}\, R^{k+1}(R/J, B) \to \mathrm{Ext}\, R^{k+2}(T/R, B)$$

exact. Since T_R is projective, $\mathrm{Ext}\, R^{k+2}(T/R, B) = 0$ and so, $\mathrm{Ext}\, R^{k+1}$ $(R/J, B) = 0$. Hence, $\mathrm{pd}(R/J)_R \leq k$ as desired. $\square$

In order to obtain the transfer we are looking for, we will need two results concerning Krull dimension. The proof of Lemma 2.2 is straightforward and the proof of Lemma 2.3 is very similar to that of [15, Proposition 2.2] which is the same result for rings in which $(T/M)_R$ has finite length. Here we only require that $(T/M)_R$ has Krull dimension.

Lemma 2.2. *Let R be any subidealizer of a right ideal M of a right noetherian ring T such that $(T/M)_R$ has Krull dimension. Suppose that B is a finitely generated right T-module. If A is any R-submodule of B, then $(A/AM)_R$ has Krull dimension.*

Lemma 2.3. *Let R be a tame subidealizer of a right ideal M of a right noetherian ring T such that $(T/M)_R$ has Krull dimension. If B_T is finitely generated, then* $\mathrm{K.dim}(B_T) = \mathrm{K.dim}(B_R)$ *if either side exists.*

These can be applied to obtain the following result.

Theorem 2.4. *Let R be a tame subidealizer of a right generative ideal M of a right noetherian ring T. Assume that* $\mathrm{r.K.dim}\, R \geq \alpha \geq 1$ *and that* $\mathrm{K.dim}(T/M)_R < \alpha$ *if T is α-proper; then R is α-proper.*

Proof. Since T_T has Krull dimension, Lemma 2.3 implies that $\mathrm{K.dim}(T_R) = \mathrm{K.dim}(T_T)$. Since R_R is a submodule of T_R, $\mathrm{K.dim}(R_R) \leq \mathrm{K.dim}(T_R)$. However, $\mathrm{K.dim}(T_R) \leq \mathrm{r.K.dim}\, R = \mathrm{K.dim}(R_R)$. Therefore $\mathrm{r.K.dim}\, R = \mathrm{r.K.dim}\, T$.

Let $q = \sup\{\mathrm{pd}(A_R) | A$ is cyclic β-critical$\}$. Suppose that C is a β-critical T-module with $\beta < \alpha$. Then, [14, Proposition 2.1] implies that $\mathrm{pd}(C_R) = \mathrm{pd}(C_T)$ and Lemma 2.3 implies that $\mathrm{K.dim}(C_R) = \mathrm{K.dim}(C_T)$. By [5, Lemma 2.1], $\mathrm{pd}(C_R) \leq q$ and hence $\mathrm{r.gl.dim}\, T \leq q$.

Now, by Proposition 2.1, r.gl.dim $R = \sup\{1, m, \mathrm{r.gl.dim}\, T\}$, where m is $\sup\{\mathrm{pd}(B_R) | B$ is γ-critical cyclic, $\gamma \leq \mathrm{K.dim}(T/M)_R\}$. Since $\mathrm{r.K.dim}\, R \geq 1$, $\mathrm{r.gl.dim}\, R \geq 1$, and thus, $q \geq \mathrm{r.gl.dim}\, T \geq 1$. Clearly, $q \geq m$ and so $\mathrm{r.gl.dim}\, R \leq q$. Finally, $q \leq \mathrm{r.gl.dim}\, R$, implying that $\mathrm{r.gl.dim}\, R = q$, and hence, R is α-proper. $\square$

Specializing to α and $\mathrm{K.dim}(T/M_R) = 0$, Theorem 2.4 gives that if $\mathrm{r.gl.dim}\, T = \sup\{\mathrm{pd}(C_T) | C\}$ is simple, then $\mathrm{r.gl.dim}\, R = \sup\{\mathrm{pd}(C_R) | C$ is simple$\}$.

The following gives a similar result for non-noetherian rings.

Theorem 2.5. *Let R be a tame subidealizer of a generative right ideal M of a ring T such that* $\mathrm{r.K.dim}\, T \geq \alpha \geq 1$ *and that $(T/M)_R$ has finite length. Assume that* $\mathrm{Tor}_1\, R(A, T/R) = 0$ *for all α-critical right modules A. If T is α-proper, then R is α-proper.*

Proof. Note that [15, Theorem 2.5] implies that $\mathrm{r.K.dim}\, R \geq \alpha$. Let $q = \sup\{\mathrm{pd}(A_R) | A \text{ is } \beta\text{-critical cyclic}, \beta < \alpha\}$. Let C be any β-critical cyclic right T-module, $\beta < \alpha$. Then, [15, Proposition 2.2] implies that $\mathrm{K.dim}(C_R) = \beta$. Therefore, [5, Lemma 2.1] implies that $\mathrm{pd}(C_R) \leq q$. Next, $\mathrm{pd}(C_T) = \mathrm{pd}(C_R) \leq q$, and hence, $\mathrm{r.gl.dim}\, T \leq q$.

We must show that $\mathrm{r.gl.dim}\, R = q$. It is clear that $\mathrm{r.gl.dim}\, R \geq q$. We can therefore assume that $q < \infty$ and show that $\mathrm{r.gl.dim}\, R \leq q$. Using [5, Lemma 2.1], it will suffice to show that $\mathrm{pd}(X_R) \leq q$ for all β-critical right R-modules X, $\beta < \alpha$. If $\beta < \alpha$, then this is clear. Otherwise assume that X is an α-critical cyclic right R-module. The hypothesis implies that

$$0 \to X \to X \otimes_R T \to X \otimes_R T/R \to 0$$

is exact. Now, $\mathrm{pd}(X \otimes_R T/R)_R \leq \mathrm{r.gl.dim}\, R \leq \mathrm{r.gl.dim}\, T + 1$ and $\mathrm{pd}(X \otimes_R T\ )_R = \mathrm{pd}(X \otimes_R T)_T \leq \mathrm{r.gl.dim}\, T$. Let $p = \mathrm{r.gl.dim}\, T$. Then

$$\mathrm{Ext}\, R^{p+1}(X \otimes_R T, B) \to \mathrm{Ext}\, R^{p+1}(X, B) \to \mathrm{Ext}\, R^{k+2}(X \otimes_R T/R, B)$$

is exact. Since both the left and the right ends are 0, we have that $\mathrm{Ext}\, R^{p+1}(X, B) = 0$ for all right modules B, and hence, $\mathrm{pd}(X_R) \leq p \leq q$ as desired. $\square$

Using similar techniques to those used above, we can obtain results about the weak global dimension.

Proposition 2.6. *Let T be a ring with generative right ideal M. Let $R \neq T$ be any subidealizer of M in T such that* $\mathrm{r.K.dim}\, R \geq \alpha \geq 1$. *Suppose that* $\gamma = \mathrm{K.dim}(T/M)_R < \alpha$. *Then* $\mathrm{r.gl.dim}\, R = \sup\{1, m, \mathrm{r.gl.dim}\, T\}$, *where* $m = \sup\{\mathrm{wd}(A) | A_R \text{ is } \beta\text{-critical cyclic}, \beta \leq \gamma\}$.

This result can be compared to [13, Proposition 2.11]. The following result is a consequence of Proposition 2.6.

Theorem 2.7. *Let R be a subidealizer of a right generative ideal M of a ring T with* $\mathrm{r.K.dim}(T/M)_R < \alpha$ *such that* $\mathrm{K.dim}(T/M)_R < \alpha$. *If T is weak α-proper,then R is weak α-proper.*

Acknowledgement

The author would like to thank J. Shapiro for suggesting the name α-proper.

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(Received 28.03.1994)

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