GEORGIAN MATHEMATICAL JOURNAL: Vol. 2, No. 6, 1995, 559-576

THE FOURIER METHOD IN THREE-DIMENSIONAL BOUNDARY-CONTACT DYNAMIC PROBLEMS OF ELASTICITY

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ABSTRACT. The basic three-dimensonal boundary-contact dynamic problems are considered for a piecewise-homogeneous isotropic elastic medium bounded by several closed surfaces. Using the Fourier method, the considered problems are proved to be solvable under much weaker restrictions on the initial data of the problems as compared with other methods.

1. Two well-known methods – the Laplace transform and the Fourier method – are widely used in investigating dynamic problems. In the works by V. Kupradze and his pupils the Laplace transform method was used to prove the existence of classical solutions of the basic three-dimensional boundary and boundary-contact dynamic problems of elasticity. Based on some results from these works, in this paper we use the Fourier method to show that the basic three-dimensional boundary-contact dynamic problems of elasticity are solvable in the classical sense. We have succeeded in weakening considerably the restrictions imposed on the data of the problems as compared with the Laplace transform method. Detailed consideration is given to the second basic problem. The other problems are treated similarly.

2. Throughout the paper we shall use the following notation:

$x=(x_1,x_2,x_3)$, $y=(y_1,y_2,y_3)$ are points of $\mathbb{R}^3$;

$|x-y|=(\sum_{k=1}^3(x_k-y_k)^2)^{1/2}$ is the distance between the points x and y;

$D_0\subset\mathbb{R}^3$ is a finite domain bounded by closed surfaces $S_0,S_1,\dots,S_m$ of the class $\Lambda_2(\alpha)$, $0<\alpha\le 1$, [1]; note that S_0 covers all other S_k, while these latter surfaces do not cover each other and $S_i\cap S_k=\varnothing$, $i\neq k$, $i,k=\overline{0,m}$; the finite domain bounded by S_k, $k=\overline{1,m}$, is denoted by D_k, $\overline{D}_0=D_0\cup(\bigcup\limits_{k=0}^{m}S_k)$, $\overline{D}_k=D_k\cup S_k$, $k=\overline{1,m}$;

1991 *Mathematics Subject Classification.* 73B30, 73C25.

Key words and phrases. Fourier method, boundary-contact dynamic problems of elasticity, classical solutions, Green's tensor, matrix of fundamential solutions.

$L = (0,\ell)$, $\overline{L} = [0,\ell]$, $\Omega_k = D_k \times L$ is a cylinder in $\mathbb{R}^4$, $k = \overline{0,m}$, $\overline{\Omega}_k = \overline{D}_k \times \overline{L}$.

The three-component vector $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ is called regular in Ω_k ($x \in D_k$, $t \in L$) if

$$u_i(x,t) \in C^1(\overline{\Omega}_k) \cap C^2(\Omega_k), \quad i = 1,2,3.$$

The system of differential equations of dynamics of classical elasticity for a homogeneous isotropic medium is written as [1]

$$\mu\Delta u + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,u + F(x,t) = \rho\,\frac{\partial^2 u}{\partial t^2}, \tag{1}$$

where $u(x,t) = (u_1,u_2,u_3)$ is the displacement vector, Δ is the three-dimensional Laplace operator, $F(x,t)$ is the mass force vector, $\rho = \mathit{const} > 0$ is the medium density and t is time; λ and μ are the elastic Lamé constants satisfying the natural conditions

$$\mu > 0, \quad 3\lambda + 2\mu > 0.$$

In this paper we shall be concerned only with real vector-functions. Any three-dimensional vector $f = (f_1, f_2, f_3)$ with norm $|f| = \sqrt{\sum_{k=1}^3 f_k^2}$ is treated as a 3×1 one-column matrix: $f = \|f_k\|_{3\times 1}$;

the sign $[\cdot]^T$ denotes transposition;

if $A = \|A_{ij}\|_{3\times 3}$ is a 3×3 matrix, then $|A|^2 = \sum_{j,i=1}^3 A_{ij}^2$.

We introduce the matrix differential operator

$$A(\partial_x) = \|A_{ij}(\partial_x)\|_{3\times 3},$$

where

$$A_{ij}(\partial_x) = \delta_{ij}\mu\Delta + (\lambda+\mu)\,\frac{\partial^2}{\partial x_i \partial x_j},$$

where δ_{ij} is the Kronecker symbol. Now (1) can be rewritten in the vector-matrix form

$$A(\partial_x)u(x,t) - \rho\,\frac{\partial^2 u(x,t)}{\partial t^2} = -F(x,t). \tag{2}$$

The matrix-differential operator

$$T(\partial_x, n(x)) = \|T_{ij}(\partial_x, n(x))\|_{3\times 3},$$

where

$$T_{ij}(\partial_x, n(x)) = \lambda n_i(x)\,\frac{\partial}{\partial x_j} + \lambda n_j(x)\,\frac{\partial}{\partial x_i} + \mu\delta_{ij}\,\frac{\partial}{\partial n(x)},$$

$n(x)$ is an arbitrary unit vector at the point x (if $x \in S_k$, $k = \overline{0,m}$, then $n(x)$ is the normal unit vector external with respect to the domain D_0), is called the stress operator.

It will be assumed that the domains D_k, $k=\overline{0,r}$, are filled with homogeneous isotropic elastic media with the Lamé constants λ_k, μ_k and density ρ_k, while the other domains D_k, $k=\overline{r+1,m}$, are hollow inclusions.

When the operators $A(\partial_x)$ and $T(\partial_x, n(x))$ contain λ_k and μ_k instead of λ and μ, we shall write $\overset{k}{A}(\partial_x)$ and $\overset{k}{T}(\partial_x, n(x))$, respectively. Furthermore, it will be assumed without loss of generality that in (2) $\rho_k=1$, $k=\overline{1,r}$. We introduce the notation

$$
\begin{aligned}
u^+(z,t) &= \lim_{D_0 \ni x \to z \in S_k} u(x,t), \quad k=\overline{0,m},\\
u^-(z,t) &= \lim_{D_k \ni x \to z \in S_k} u(x,t), \quad k=\overline{1,r}.
\end{aligned}
$$

The notation $(T(\partial_z, n(z))u(z,t))^{\pm}$ has a similar meaning.

3. Let us consider the following problem: In the cylinder Ω_k, $k=\overline{0,r}$, find regular vectors $\overset{k}{u}(x,t)$, $k=\overline{0,r}$, satisfying:

(1) the equations

$$
\forall (x,t)\in\Omega_k : \overset{k}{A}(\partial_x)\overset{k}{u}(x,t) - \frac{\partial^2 \overset{k}{u}(x,t)}{\partial t^2} = -\overset{k}{F}(x,t), \quad k=\overline{0,r};
$$

(2) the initial conditions

$$
\begin{aligned}
\forall x\in \overline{D}_k : &\lim_{t\to 0} \overset{k}{u}(x,t) = \overset{k}{\varphi}(x),\\
&\lim_{t\to 0} \frac{\partial \overset{k}{u}(x,t)}{\partial t} = \overset{k}{\psi}(x), \quad k=\overline{0,r};
\end{aligned}
$$

(3) the contact conditions

$$
\forall (z,t)\in S_k\times \overline{L} : \left.\begin{aligned}
&\overset{\circ}{u}{}^+(z,t) - \overset{k}{u}{}^-(z,t) = \overset{k}{\Phi}(z,t),\\
&\big(\overset{\circ}{T}\overset{\circ}{u}(z,t)\big)^+ - \big(\overset{k}{T}\overset{k}{u}(z,t)\big)^- = \overset{k}{\Psi}(z,t),
\end{aligned}\right\} \; k=\overline{1,r};
$$

(4) the boundary conditions

$$
\begin{gathered}
\forall (z,t)\in S_k\times \overline{L} : \big(\overset{\circ}{T}(\partial_z, n(z))\overset{\circ}{u}(z,t)\big)^+ = \overset{k}{f}(z,t),\\
k=0, r+1, \ldots, m.
\end{gathered}
$$

We shall refer to the above problem as $(\mathrm{II})_{F,\varphi,\psi,\Phi,\Psi,f}$. The given vector-functions $\overset{k}{F}$, $\overset{k}{\varphi}$, $\overset{k}{\psi}$, $\overset{k}{\Phi}$, $\overset{k}{\Psi}$, $\overset{k}{f}$ are assumed to satisfy the conditions:

1. $\overset{k}{F}(\cdot,\cdot) \in C^2(\overline{\Omega}_k)$ and third-order derivatives belong to the class $L_2(D_k)$, $k=\overline{0,r}$. Moreover,

$$\overset{\circ}{T}\overset{\circ}{F}\big|_{S_k}=0,\quad k=0,r+1,\dots,m,\quad t\in\overline{L},$$
$$\overset{\circ}{F}{}^{+}=\overset{k}{F}{}^{-},\ \ (\overset{\circ}{T}\overset{\circ}{F})^{+}=(\overset{k}{T}\overset{k}{F})^{-},\ \ (\overset{\circ}{A}\overset{\circ}{F})^{+}=(\overset{k}{A}\overset{k}{F})^{-},\ \ k=\overline{1,r},\ \ t\in\overline{L};$$

2. $\forall t\in\overline{L}:\dfrac{\partial^p}{\partial t^p}\overset{k}{\Phi}(\cdot,t)\in C^2(S_k),\ \ p=\overline{0,7},$

$$\forall z\in S_k:\overset{k}{\Phi}(z,\cdot)\in C^7(\overline{L}),\ \ \Big(\frac{\partial^m\overset{k}{\Phi}(z,t)}{\partial t^m}\Big)_{t=0}=0,\ \ m=\overline{0,5},\ \ k=\overline{1,r};$$

3. $\forall t\in\overline{L}:\dfrac{\partial^p}{\partial t^p}\overset{k}{\Psi}(\cdot,t)\in C^1(S_k),\ \ p=\overline{0,7},$

$$\forall z\in S_k:\overset{k}{\Psi}(z,\cdot)\in C^7(\overline{L}),\ \ \Big(\frac{\partial^m\overset{k}{\Psi}(z,t)}{\partial t^m}\Big)_{t=0}=0,\ \ m=\overline{0,5},\ \ k=\overline{1,r};$$

4. $\forall t\in\overline{L}:\dfrac{\partial^p}{\partial t^p}\overset{k}{f}(\cdot,t)\in C^1(S_k),\ \ p=\overline{0,7},$

$$\forall z\in S_k:\overset{k}{f}(z,\cdot)\in C^7(\overline{L}),\ \ \Big(\frac{\partial^m\overset{k}{f}(z,t)}{\partial t^m}\Big)_{t=0}=0,\ \ m=\overline{0,5},$$
$$k=0,r+1,\dots,m;$$

5. $\overset{k}{\varphi}\in C^3(\overline{D}_k)$ and fourth-order derivatives belong to class $L_2(D_k)$. Moreover,

$$\overset{\circ}{T}\overset{\circ}{\varphi}\big|_{S_k}=\overset{\circ}{T}\overset{\circ}{A}\overset{\circ}{\varphi}\big|_{S_k}=0,\quad k=0,r+1,\dots,m,$$
$$\overset{\circ}{\varphi}{}^{+}=\overset{k}{\varphi}{}^{-},\ \ (\overset{\circ}{T}\overset{\circ}{\varphi})^{+}=(\overset{k}{T}\overset{k}{\varphi})^{-},\ \ (\overset{\circ}{A}\overset{\circ}{\varphi})^{+}=(\overset{k}{A}\overset{k}{\varphi})^{-},$$
$$(\overset{\circ}{T}\overset{\circ}{A}\overset{\circ}{\varphi})^{+}=(\overset{k}{T}\overset{k}{A}\overset{k}{\varphi})^{-},\quad k=\overline{1,r}.$$

6. $\overset{k}{\psi}\in C^2(\overline{D}_k)$ and third-order derivatives belong to the class $L_2(D_k)$. Moreover,

$$\overset{\circ}{T}\overset{\circ}{\psi}\big|_{S_k}=0,\ \ k=0,r+1,\dots,m,\quad \overset{\circ}{\psi}{}^{+}=\overset{k}{\psi}{}^{-},\ \ (\overset{\circ}{T}\overset{\circ}{\psi})^{+}=(\overset{k}{T}\overset{k}{\psi})^{-},\ \ k=\overline{1,r}.$$

The symbol $\cdot|_{S_k}$ denotes the restriction to S_k. The uniqueness of a regular solution of the problem posed is proved in [2].

Let $\overset{k}{u}{}^{(1)}(x,t)$ be a regular solution of problem $(\mathrm{II})_{0,0,0,\Phi,\Psi,f}$, and $\overset{k}{u}{}^{(2)}(x,t)$ be a regular solution of problem $(\mathrm{II})_{F,\varphi,\psi,0,0,0}$. Then, as one can easily verify, $\overset{k}{u}(x,t)=\overset{k}{u}{}^{(1)}(x,t)+\overset{k}{u}{}^{(2)}(x,t)$ will be a regular solution of problem $(\mathrm{II})_{F,\varphi,\psi,\Phi,\Psi,f}$. The existence of $\overset{k}{u}{}^{(1)}(x,t)$ under our assumptions immediately follows from the results of [2]. Therefore it remains for us to prove the existence of a regular solution of problem $(\mathrm{II})_{F,\varphi,\psi,0,0,0}$, which is what we are going to do below by means of the Fourier method.

4. Let $\varkappa_0$ be an arbitrary fixed positive integer, I be the 3×3 unit matrix, $D=\bigcup\limits_{k=0}^{r} D_k$.

We apply the term "Green tensor" of the second basic problem of the operator $\overset{k}{A}(\partial_x)-\varkappa_0^2 I$ to a 3×3 matrix $G(x,y,-\varkappa_0^2)=\overset{k}{G}(x,y,-\varkappa_0^2)$, $x\in D_k$, $y\in D$, $x\neq y$, $k=\overline{0,r}$, which satisfies the conditions:

(1) $\forall x\in D_k$, $\forall y\in D$, $x\neq y$:
$\overset{k}{A}(\partial_x)\overset{k}{G}(x,y,-\varkappa_0^2)-\varkappa_0^2\overset{k}{G}(x,y,-\varkappa_0^2)=0$, $k=\overline{0,r}$;

(2) $\forall z\in S_k$, $\forall y\in D$:
$\big(\overset{\circ}{G}(z,y,-\varkappa_0^2)\big)^+=\big(\overset{k}{G}(z,y,-\varkappa_0^2)\big)^-$, $k=\overline{1,r}$,
$\big(\overset{\circ}{T}(\partial_z,n(z))\overset{\circ}{G}(z,y,-\varkappa_0^2)\big)^+=\big(\overset{k}{T}(\partial_z,n(z))\overset{k}{G}(z,y,-\varkappa_0^2)\big)^-$, $k=\overline{1,r}$;

(3) $\forall z\in S_k$, $\forall y\in D$:
$\big(\overset{\circ}{T}(\partial_z,n(z))\overset{\circ}{G}(z,y,-\varkappa_0^2)\big)^+=0$, $k=0,r+1,\dots,m$;

(4) $\overset{k}{G}(x,y,-\varkappa_0^2)=\overset{k}{\Gamma}(x-y,-\varkappa_0^2)-\overset{k}{g}(x,y)$, $x\in D_k$, $y\in D$, $k=\overline{0,r}$,

where $\overset{k}{\Gamma}(x-y,-\varkappa_0^2)$ is the Kupradze matrix of fundamental solutions with the Lamé constants λ_k and μ_k [1], and $\overset{k}{g}(x,y)$ is the regular solution of the following problem:

$$
\begin{aligned}
&\forall x\in D_k,\ \forall y\in D: \overset{k}{A}(\partial_x)\overset{k}{g}(x,y)-\varkappa_0^2\overset{k}{g}(x,y)=0,\quad k=\overline{0,r},\\
&\forall z\in S_k,\ \forall y\in D: \big(\overset{\circ}{g}(z,y)\big)^+-\big(\overset{k}{g}(z,y)\big)^-=\\
&\qquad=\overset{\circ}{\Gamma}(z-y,-\varkappa_0^2)-\overset{k}{\Gamma}(z-y,-\varkappa_0^2),\\
&\qquad\big(\overset{\circ}{T}(\partial_z,n(z))\overset{\circ}{g}(x,y)\big)^+-\big(\overset{k}{T}(\partial_z,n(z))\overset{k}{g}(x,y)\big)^-=\\
&\qquad=\overset{\circ}{T}(\partial_z,n(z))\overset{\circ}{\Gamma}(z-y,-\varkappa_0^2)-\\
&\qquad-\overset{k}{T}(\partial_z,n(z))\overset{k}{\Gamma}(z-y,-\varkappa_0^2),\quad k=\overline{1,r},\\
&\forall z\in S_k,\ \forall y\in D: \big(\overset{\circ}{T}(\partial_z,n(z))\overset{\circ}{g}(x,y)\big)^+=\overset{\circ}{T}(\partial_z,n(z))\overset{k}{\Gamma}(z-y,-\varkappa_0^2),\\
&\qquad k=0,r+1,\dots,m.
\end{aligned}
$$

The solvability of this problem is given in [1], which provides the existence of $G(x,y,-\varkappa_0^2)$. By using the Green formula it is easy to prove [1] that $G(x,y,-\varkappa_0^2)$ possesses the symmetry property of the form

$$
G(x,y,-\varkappa_0^2)=G^T(y,x,-\varkappa_0^2). \tag{3}
$$

Moreover, we have the estimates [2]

$$\left.\begin{aligned} &\forall (x,y)\in D\times D : G_{mn}(x,y,-\varkappa_0^2)=O(|x-y|^{-1}),\\ &\qquad\qquad \frac{\partial}{\partial x_j}G_{mn}(x,y,-\varkappa_0^2)=O(|x-y|^{-2}),\\ &\qquad\qquad m,n,j=1,2,3, \end{aligned}\right\} \tag{4}$$

5. Consider the problem with eigenvalues

$$\left.\begin{aligned} &\forall x\in D_k : \overset{k}{A}(\partial_x)\overset{k}{w}(x)+\omega\overset{k}{w}(x)=0,\ \ k=\overline{0,r},\\ &\forall z\in S_k : \overset{\circ}{w}{}^{+}(z)=\overset{k}{w}{}^{-}(z),\ \ (\overset{\circ}{T}\overset{\circ}{w}(z))^{+}=(\overset{k}{T}\overset{k}{w}(z))^{-},\ \ k=\overline{1,r},\\ &\forall z\in S_k : (\overset{\circ}{T}\overset{\circ}{w}(z))^{+}=0,\ \ k=0,r+1,\dots,m. \end{aligned}\right\} \tag{5}$$

The eigenvector-function of problem (5) $w(x)=\overset{k}{w}(x)=(\overset{k}{w}_1(x),\overset{k}{w}_2(x),\overset{k}{w}_3(x))$, $x\in D_k$, $k=\overline{0,r}$, is regular if $\overset{k}{w}_i\in C^1(\overline{D}_k)\cap C^2(D_k)$, $i=1,2,3$, $k=\overline{0,r}$.

By the known procedure [1] it can be shown that problem (5) is equivalent to a system of integral equations

$$w(x)=(\omega+\varkappa_0^2)\int_D G(x,y,-\varkappa_0^2)w(y)dy,\quad x\in D. \tag{6}$$

By virtue of (3) and (4) we see that (6) is an integral equation with a symmetric kernel of the class $L_2(D)$. By the Hilbert–Schmidt theorem there exists a countable system of eigenvalues $(\omega_n+\varkappa_0^2)_{n=1}^{\infty}$ and the corresponding orthonormal in D system of eigenvectors $(w^{(n)}(x))_{n=1}^{\infty}=(\overset{k}{w}{}^{(n)}(x))_{n=1}^{\infty}$, $x\in D_k$, $k=\overline{0,r}$, of equation (6). Hence, in turn, it follows that $(\omega_n)_{n=1}^{\infty}$ and $(\overset{k}{w}{}^{(n)}(x))_{n=1}^{\infty}$ are respectively the eigenvalues and eigenvectors of problem (5). It has been established [1] that all $\omega_n\geq 0$. Note that $\omega=0$ is a sixth-rank eigenvalue and the corresponding eigenvectors are the rigid displacement vectors

$$\chi^{(n)}(x)=(\chi_1^{(n)},\chi_2^{(n)},\chi_3^{(n)}),\quad n=\overline{1,6},\ \ x\in D.$$

In what follows we shall assume that $\omega_n=0$, $w^{(n)}(x)=\chi^{(n)}(x)$, $n=\overline{1,6}$. The properties of a volume potential [1] imply the regularity of eigenvectors.

Let us prove that the system $(w^{(n)}(x))_{n=1}^{\infty}$ is complete in $L_2(D)$. For this it is sufficient to show that if $f\in L_2(D)$ is an arbitrary vector then the conditions

$$\int_D f^T(x)w^{(n)}(x)\,dx=0,\quad n=1,2,\dots, \tag{7}$$

imply that $f(x)=0$ almost everywhere in D. We introduce the notation

$$h(x)=\int_D G(x,y,-\varkappa_0^2)f(y)\,dy.$$

By the Hilbert–Schmidt theorem $h(x)$ is expanded in an absolutely and uniformly convergent series

$$h(x)=\sum_{n=1}^{\infty}\frac{f_n w^{(n)}(x)}{\omega_n+\varkappa_0^2},$$

where

$$f_n=\int_D f^T(x)w^{(n)}(x)\,dx,\quad n=1,2,\dots .$$

By (7) all $f_n=0$ and therefore

$$h(x)\equiv 0. \tag{8}$$

On the other hand, we know [1] that

$$\big(\overset{k}{A}(\partial_x)-\varkappa_0^2 I\big)h(x)+f(x)=0 \tag{9}$$

almost for all $x\in D$.

By virtue of (8) and (9) we finally conclude that $f(x)=0$ almost everywhere in D.

6. Let $u(x)=\overset{k}{u}(x)$ and $v(x)=\overset{k}{v}(x)$, $x\in D_k$, $k=\overline{0,r}$, be arbitrary vectors belonging to the class $C^1(\overline{D}_k)$ and their second-order derivatives belong to the class $L_2(D_k)$. Then the following Green formulas are valid [1]:

$$\int_D\big(v^TAu+E(v,u)\big)dx=\int_{S_0}(\overset{\circ}{v}{}^T)^+(\overset{\circ}{T}\overset{\circ}{u})^+dS+$$
$$+\sum_{k=r+1}^{m}\int_{S_k}(\overset{\circ}{v}{}^T)^+(\overset{\circ}{T}\overset{\circ}{u})^+dS+\sum_{k=1}^{r}\int_{S_k}\big((\overset{\circ}{v}{}^T)^+(\overset{\circ}{T}\overset{\circ}{u})^+-(\overset{k}{v}{}^T)^+(\overset{k}{T}\overset{k}{u})^-\big)dS; \tag{10}$$

$$\int_D\big(v^TAu-u^TAv\big)dx=\int_{S_0}\big((\overset{\circ}{v}{}^T)^+(\overset{\circ}{T}\overset{\circ}{u})^+-(\overset{\circ}{u}{}^T)^+(\overset{\circ}{T}\overset{\circ}{v})^+\big)dS+$$
$$+\sum_{k=r+1}^{m}\int_{S_k}\big((\overset{\circ}{v}{}^T)^+(\overset{\circ}{T}\overset{\circ}{u})^+-(\overset{\circ}{u}{}^T)^+(\overset{\circ}{T}\overset{\circ}{v})^+\big)dS+$$
$$+\sum_{k=1}^{r}\Big(\int_{S_k}\big((\overset{\circ}{v}{}^T)^+(\overset{\circ}{T}\overset{\circ}{u})^+-(\overset{k}{v}{}^T)^-(\overset{k}{T}\overset{k}{u})^-\big)dS-$$
$$-\int_{S_k}\big((\overset{\circ}{u}{}^T)^+(\overset{\circ}{T}\overset{\circ}{v})^+-(\overset{k}{u}{}^T)^-(\overset{k}{T}\overset{k}{v})^-\big)dS\Big); \tag{11}$$

here

$$E(v,u)=\sum_{p,q=1}^{3}\Big(\mu\,\frac{\partial v_p}{\partial x_q}\,\frac{\partial u_p}{\partial x_q}+\lambda\,\frac{\partial v_p}{\partial x_p}\,\frac{\partial u_q}{\partial x_q}+\mu\,\frac{\partial v_p}{\partial x_q}\,\frac{\partial u_q}{\partial x_p}\Big). \tag{12}$$

On rewriting (12) as

$$E(v,u)=\frac{3\lambda+2\mu}{3}\operatorname{div} v\operatorname{div} u+\frac{\mu}{2}\sum_{p\neq q}\Big(\frac{\partial v_p}{\partial x_q}+\frac{\partial v_q}{\partial x_p}\Big)\Big(\frac{\partial u_p}{\partial x_q}+\frac{\partial u_q}{\partial x_p}\Big)+$$
$$+\frac{\mu}{3}\sum_{p,q}\Big(\frac{\partial v_p}{\partial x_p}-\frac{\partial v_q}{\partial x_q}\Big)\Big(\frac{\partial u_p}{\partial x_p}-\frac{\partial u_q}{\partial x_q}\Big)$$

we easily find that $E(v,u)=E(u,v)$ and $E(v,v)\geq 0$.

Theorem 1. *The inequality*

$$\sum_{n=7}^{\infty}\Phi_n^2\omega_n\leq\int_D E(\Phi,\Phi)\,dx, \tag{13}$$

where

$$\Phi_n=\int_D \Phi^T(x)w^{(n)}(x)\,dx,$$

holds for any vector $\Phi(x)=\overset{k}{\Phi}(x)$, $x\in D_k$, $k=\overline{0,r}$, *satisfying the conditions* $\overset{k}{\Phi}\in C^0(\overline{D}_k)$, $\frac{\partial\overset{k}{\Phi}}{\partial x_i}\in L_2(D_k)$, $i=1,2,3$, $k=\overline{0,r}$; $\forall z\in S_k:\overset{\circ}{\Phi}{}^+(z)=\overset{k}{\Phi}{}^-(z)$, $k=\overline{1,r}$.

In particular, the theorem implies the convergence of the series on the left-hand side of (13).

Proof. Applying (10) to the vectors $\Phi(x)$ and $w^{(n)}(x)$ we obtain

$$\int_D E(\Phi,w^{(n)})\,dx=\omega_n\Phi_n,\quad n=7,\dots. \tag{14}$$

In particular, by setting $\Phi=w^{(n)}$ in (14) we have

$$\int_D E(w^{(m)},w^{(n)})\,dx=\begin{cases}\omega_n & \text{for } m=n,\\ 0 & \text{for } m\neq n.\end{cases} \tag{15}$$

If we consider a nonnegative value

$$\mathcal{I}=\int_D E(v,v)\,dx\geq 0$$

and assume that

$$v(x)=\Phi(x)-\sum_{n=7}^{n_0}\Phi_n w^{(n)}(x),$$

then simple calculations will give

$$\mathcal{I} = \int_D E(\Phi,\Phi)\,dx + \sum_{m,n=7}^{n_0} \Phi_m \Phi_n \int_D E(w^{(m)}, w^{(n)})\,dx -$$
$$-2\sum_{n=7}^{n_0} \Phi_n \int_D E(\Phi, w^{(n)})\,dx \geq 0. \tag{16}$$

Taking into account of (14) and (15), from (16) we obtain

$$\sum_{n=7}^{n_0} \Phi_n^2 \omega_n \leq \int_D E(\Phi,\Phi)\,dx. \qquad \square$$

Theorem 2. *The inequality*

$$\sum_{n=7}^{\infty} \Phi_n^2 \omega_n^2 \leq \int_D |A\Phi|^2 dx \tag{17}$$

holds for any vector $\Phi(x) = \overset{k}{\Phi}(x)$, $x \in D_k$, $k = \overline{0,r}$, *satisfying the conditions*

$$\overset{k}{\Phi} \in C^1(\overline{D}_k), \quad \frac{\partial^2 \overset{k}{\Phi}}{\partial x_i \partial x_j} \in L_2(D_k), \quad i,j = 1,2,3, \quad k = \overline{0,r},$$
$$\forall z \in S_k : \overset{\circ}{\Phi}{}^+(z) = \overset{k}{\Phi}{}^-(z), \quad \big(\overset{\circ}{T}\overset{\circ}{\Phi}(z)\big)^+ = \big(\overset{k}{T}\overset{k}{\Phi}(z)\big)^-, \quad k = \overline{1,r},$$
$$\forall z \in S_k : \big(\overset{\circ}{T}\overset{\circ}{\Phi}(z)\big)^+ = 0, \quad k = 0, r+1, \dots, m.$$

In particular, the theorem implies the convergence of series on the left-hand side of (17).

Proof. Applying (11) to the vectors $\Phi(x)$ and $w^{(n)}(x)$ we obtain

$$\int_D \big(\Phi^T A w^{(n)} - w^{(n)\,T} A\Phi\big) dx = 0, \quad n = 7, \dots. \tag{18}$$

Taking into account that $Aw^{(n)} + \omega_n w^{(n)}$, from (18) we have

$$\int_D A\Phi w^{(n)} dx = -\omega_n \Phi_n.$$

Hence

$$(A\Phi)_n = -\omega_n \Phi_n. \tag{19}$$

On writing the Bessel inequality

$$\sum_{n=7}^{\infty} (A\Phi)_n^2 \leq \int_D |A\Phi|^2 dx \tag{20}$$

for $A(\partial_x)\Phi(x)$, we find that on account of (19), from (20) we obtain (17). $\square$

7. Because of the linearity of problem $(\mathrm{II})_{F,\varphi,\psi,0,0,0}$ its solution can be represented as the sum of solutions of problems $(\mathrm{II})_{0,\varphi,\psi,0,0,0}$ and $(\mathrm{II})_{F,0,0,0,0,0}$. By a formal application of the Fourier method to problem $(\mathrm{II})_{0,\varphi,\psi,0,0,0}$ we obtain

$$u(x,t)=\overset{k}{u}(x,t)=\sum_{n=1}^{6}\chi^{(n)}(x)(\varphi_n+t\psi_n)+$$
$$+\sum_{n=7}^{\infty}w^{(n)}(x)\Big(\varphi_n\cos\sqrt{\omega_n}\,t+\frac{\psi_n}{\sqrt{\omega_n}}\sin\sqrt{\omega_n}\,t\Big),\quad x\in D,\quad t\in L,$$

where

$$\varphi_n=\int_D\varphi^T(x)w^{(n)}(x)\,dx,\quad \psi_n=\int_D\psi^T(x)w^{(n)}(x)\,dx.$$

On expanding formally the solution of problem $(\mathrm{II})_{F,0,0,0,0,0}$ $u(x,t)$ in series with respect to the system $(w^{(n)}(x))_{n=1}^{\infty}$

$$u(x,t)=\sum_{n=1}^{\infty}u_n(t)w^{(n)}(x),\quad F(x,t)=\sum_{n=1}^{\infty}F_n(t)w^{(n)}(x),$$

we have

$$u(x,t)=\sum_{n=1}^{6}\chi^{(n)}(x)\int_0^t\Big(\int_0^t F_n(\tau)d\tau\Big)dt+$$
$$+\sum_{n=7}^{\infty}w^{(n)}(x)\,\frac{1}{\sqrt{\omega_n}}\int_0^t F_n(\tau)\sin\sqrt{\omega_n}(t-\tau)d\tau,\quad x\in D.$$

Thus to solve problem $(\mathrm{II})_{F,\varphi,\psi,0,0,0}$ we formally obtain the representation

$$\begin{aligned}u(x,t)=&\sum_{n=1}^{6}\chi^{(n)}(x)(\varphi_n+t\psi_n)+\\&+\sum_{n=7}^{\infty}w^{(n)}(x)\Big(\varphi_n\cos\sqrt{\omega_n}\,t+\frac{\psi_n}{\sqrt{\omega_n}}\sin\sqrt{\omega_n}\,t\Big)+\\&+\sum_{n=1}^{6}\chi^{(n)}(x)\int_0^t\Big(\int_0^t F_n(\tau)d\tau\Big)dt+\\&+\sum_{n=7}^{\infty}w^{(n)}(x)\,\frac{1}{\sqrt{\omega_n}}\int_0^t F_n(\tau)\sin\sqrt{\omega_n}(t-\tau)d\tau.\end{aligned}\tag{21}$$

Apply the Green formula (11) to $\varphi(x)$ and $w^{(n)}(x)$. Taking into account the boundary and contact conditions for these vectors we have

$$\int_D (A\varphi)^T w^{(n)} dx = -\omega_n \int_D \varphi^T w^{(n)} dx,$$

i.e.,

$$(A\varphi)_n = -\omega_n \varphi_n, \quad n = 7, 8, \dots . \tag{22}$$

Applying now (11) to the vectors $A\varphi(x)$ and $w^{(n)}(x)$ we obtain

$$(A^2\varphi)_n = -\omega_n (A\varphi)_n, \quad n = 7, 8, \dots .$$

Hence on account of (22)

$$\varphi_n = \frac{(A^2\varphi)_n}{\omega_n^2}. \tag{23}$$

Similarly, applying successively (11) to the vectors $\psi(x)$ and $w^{(n)}(x)$, $F(x,t)$ and $w^{(n)}(x)$, we have

$$\psi_n = -\frac{(A\psi)_n}{\omega_n}, \quad F_n(t) = -\frac{(AF)_n}{\omega_n}. \tag{24}$$

By virtue of (23) and (24) series (21) takes the form

$$\begin{aligned} u(x,t) = &\sum_{n=1}^{6} \chi^{(n)}(x)(\varphi_n + t\psi_n) + \sum_{n=1}^{6} \chi^{(n)}(x) \int_0^t \Big(\int_0^t F_n(\tau) d\tau \Big) dt + \\ &+ \sum_{n=7}^{\infty} \frac{w^{(n)}(x)}{\omega_n^2} (A^2\varphi)_n \cos \sqrt{\omega_n}\, t - \sum_{n=7}^{\infty} \frac{w^{(n)}(x)}{\omega_n^{3/2}} (A\psi)_n \sin \sqrt{\omega_n}\, t - \\ &- \sum_{n=7}^{\infty} \frac{w^{(n)}(x)}{\omega_n^{3/2}} \int_0^t (AF)_n(\tau) \sin \sqrt{\omega_n}\,(t-\tau) d\tau. \end{aligned} \tag{25}$$

Now to substantiate the Fourier method we have to prove that (25) actually provides a regular solution of problem $(\mathrm{II})_{F,\varphi,\psi,0,0,0}$. For this it is necessary to show that the series of (25) as well as the series obtained by a single term-by-term differentiation of these series converge uniformly in the closed cylinder, while the series obtained by a double term-by-term differentiation of these series converges uniformly inside the cylinder $\Omega = D \times L$.

8. First we consider the series

$$\sum_{n=7}^{\infty} \frac{w^{(n)}(x)}{\omega_n^2}\,(A^2\varphi)_n \cos\sqrt{\omega_n}\,t. \tag{26}$$

Estimating (26) by applying the Cauchy–Buniakovski inequality we obtain

$$\Big|\sum_{n=m}^{m+p} \frac{w^{(n)}(x)}{\omega_n^2}\,(A^2\varphi)_n \cos\sqrt{\omega_n}\,t\Big| \leq$$
$$\leq \Big[\sum_{n=m}^{m+p} \frac{|w^{(n)}(x)|^2}{\omega_n^4}\cdot\sum_{n=m}^{m+p}(A^2\varphi)_n^2\Big]^{1/2}. \tag{27}$$

Since $A^2\varphi \in L_2(D)$, by virtue of the Bessel inequality

$$\sum_{n=7}^{\infty}(A^2\varphi)_n^2 \leq \int_D |A^2\varphi|^2 dx.$$

Therefore for any $\varepsilon > 0$ there exists a number $N(\varepsilon)$ such that for $m \geq N(\varepsilon)$ and any natural p we have

$$\sum_{n=m}^{m+p}(A^2\varphi)_n^2 < \varepsilon. \tag{28}$$

By (28) it follows from (27) that in order to prove the uniform convergence of series (26) it is sufficient to establish in $\overline{\Omega}$ that the sum of the series

$$\sum_{n=7}^{\infty} \frac{|w^{(n)}(x)|^2}{\omega_n^4} \tag{29}$$

exists and is uniformly bounded in $\overline{D}$.

From (6) we have

$$\int_D G(x,y,-\varkappa_0^2)w^{(n)}(y)\,dy = \frac{w^{(n)}(x)}{\omega_n+\varkappa_0^2},$$

i.e.,

$$\big(G(x,y,-\varkappa_0^2)\big)_n = \frac{w^{(n)}(x)}{\omega_n+\varkappa_0^2}.$$

Therefore the Bessel inequality gives

$$\sum_{n=7}^{\infty} \frac{|w^{(n)}(x)|^2}{(\omega_n+\varkappa_0^2)^2} \leq \int_D |G(x,y,-\varkappa_0^2)|^2 dy. \tag{30}$$

By (4) it follows from (30) that the sum of the series

$$\sum_{n=7}^{\infty} \frac{|w^{(n)}(x)|^2}{(\omega_n + \varkappa_0^2)^2} \tag{31}$$

exists and is bounded in $\overline{D}$. Since

$$\lim_{n\to\infty} \Big(1 + \frac{\varkappa_0^2}{\omega_n}\Big) = 1, \quad n \ge 7,$$

we have the inequality

$$\begin{aligned} \frac{|w^{(n)}(x)|^2}{\omega_n^2} &= \frac{|w^{(n)}(x)|^2}{(\omega_n + \varkappa_0^2)^2} \cdot \frac{(\omega_n + \varkappa_0^2)^2}{\omega_n^2} = \frac{|w^{(n)}(x)|^2}{(\omega_n + \varkappa_0^2)^2} \Big(1 + \frac{\varkappa_0^2}{\omega_n}\Big)^2 \le \\ &\le M \, \frac{|w^{(n)}(x)|^2}{(\omega_n + \varkappa_0^2)^2}, \quad n \ge 7, \end{aligned} \tag{32}$$

where M is constant. The uniform boundedness of the sum of series (31) and inequality (32) imply that the sum of the series

$$\sum_{n=7}^{\infty} \frac{|w^{(n)}(x)|^2}{\omega_n^2} \tag{33}$$

exists and is uniformly bounded in $\overline{D}$. The same conclusion is even more valid for series (29).

Now we shall show that the series obtained by a single term-by-term differentiation of (26) are uniformly convergent in $\overline{\Omega}$.

Differentiating term-by-term series (26) with respect to t we obtain

$$-\sum_{n=7}^{\infty} \frac{w^{(n)}(x)}{\omega_n^{3/2}} \, (A^2\varphi)_n \sin \sqrt{\omega_n}\, t. \tag{34}$$

The remainder of series (34) is estimated as

$$\begin{aligned} &\Big| \sum_{n=m}^{m+p} \frac{w^{(n)}(x)}{\omega_n^{3/2}} \, (A^2\varphi)_n \sin \sqrt{\omega_n}\, t \Big| \le \\ &\le \Big[\sum_{n=m}^{m+p} \frac{|w^{(n)}(x)|^2}{\omega_n^3} \cdot \sum_{n=m}^{m+p} (A^2\varphi)_n^2 \Big]^{1/2}. \end{aligned}$$

The existence and uniform convergence of the sum of series (33) in $\overline{D}$ imply that the sum of the series

$$\sum_{n=7}^{\infty} \frac{|w^{(n)}(x)|}{\omega_n^3}$$

exists and is uniformly convergent in $\overline{D}$.

Hence, as above, we conclude that series (34) is uniformly convergent in $\overline{\Omega}$.

Now differentiating term-by-term series (26) with respect to x_i, $i=1,2,3$, we have

$$\sum_{n=7}^{\infty} \frac{\frac{\partial w^{(n)}(x)}{\partial x_i}}{\omega_n^2}\,(A^2\varphi)_n \cos\sqrt{\omega_n}\,t. \tag{35}$$

The remainder of series (35) is estimated as

$$\left|\sum_{n=m}^{m+p} \frac{\frac{\partial w^{(n)}(x)}{\partial x_i}}{\omega_n^2}\,(A^2\varphi)_n \cos\sqrt{\omega_n}\,t\right| \le$$
$$\le \left[\sum_{n=m}^{m+p} \frac{\left|\frac{\partial w^{(n)}(x)}{\partial x_i}\right|^2}{\omega_n^4} \sum_{n=m}^{m+p} (A^2\varphi)_n^2\right]^{1/2}.$$

By virtue of (28) in order to prove that series (35) is uniformly convergent in $\overline{\Omega}$ it is sufficient to prove that the sum of the series

$$\sum_{n=7}^{\infty} \frac{\left|\frac{\partial w^{(n)}(x)}{\partial x_i}\right|^2}{\omega_n^4} \tag{36}$$

exists and is uniformly bounded in $\overline{D}$.

Let $G^{(2)}(x,y,-\varkappa_0^2)$ be the iterated kernel of $G(x,y,-\varkappa_0^2)$. We have

$$G^{(2)}(x,y,-\varkappa_0^2) = \int_D G(x,z,-\varkappa_0^2)G(z,y,-\varkappa_0^2)dz, \quad x\neq y. \tag{37}$$

Differentiate (37) with respect to x_i. On the right-hand side of (37) differention can be performed under the integral sign, since we shall now show that the integral obtained by a formal differentiation is uniformly convergent for any $x\neq y$:

$$\frac{\partial G^{(2)}(x,y,-\varkappa_0^2)}{\partial x_i} = \int_D \frac{\partial G(x,z,-\varkappa_0^2)}{\partial x_i}\,G(z,y,-\varkappa_0^2)dz, \ \ x\neq y.$$

By virtue of (4) and the theorem on the composition of two kernels [3] we obtain

$$\forall (x,y)\in D\times D : \left|\frac{\partial G^{(2)}(x,y,-\varkappa_0^2)}{\partial x_i}\right| \le c_1\big|\ln|x-y|\big| + c_2.$$

Thus

$$\frac{\partial G^{(2)}(x,y,-\varkappa_0^2)}{\partial x_i} \in L_2(D), \quad i=1,2,3.$$

It is likewise clear that

$$w^{(n)}(x) = \omega_n^2 \int_D G^{(2)}(x, y, -\varkappa_0^2) w^{(n)}(y)\, dy. \tag{38}$$

Therefore the Bessel inequality gives

$$\sum_{n=7}^{\infty} \frac{\left|\frac{\partial w^{(n)}(x)}{\partial x_i}\right|^2}{\omega_n^4} \le \int_D \left|\frac{\partial G^{(2)}(x, y, -\varkappa_0^2)}{\partial x_i}\right|^2 dy.$$

Hence we conclude that the sum of series (26) exists and is uniformly convergent in $\overline{D}$.

We shall now prove that the series obtained by a double term-by-term differentiation of (26) are uniformly convergent in the cylinder $\overline{\Omega}'_k = \overline{D}'_k \times \overline{L}$, where $\overline{D}'_k \subset D_k$, $k = \overline{0, r}$, is an arbitrary strictly internal closed subdomain.

A double term-by-term differentiation of series (26) with respect to t leads to

$$-\sum_{n=7}^{\infty} \frac{w^{(n)}(x)}{\omega_n} (A^2\varphi)_n \cos\sqrt{\omega_n}\, t. \tag{39}$$

On estimating the remainder of series (39) as above, by virtue of (28) and the uniform boundedness of the sum of series (33) we conclude that series (39) is uniformly convergent in $\overline{\Omega}$.

A double term-by-term differentiation of series (26) with respect to x gives

$$\sum_{n=7}^{\infty} \frac{\frac{\partial^2 w^{(n)}(x)}{\partial x_i \partial x_j}}{\omega_n^2} (A^2\varphi)_n \cos\sqrt{\omega_n}\, t, \quad i, j = 1, 2, 3. \tag{40}$$

On estimating the remainder of series (40) as above, by virtue of (28) in order to prove the uniform convergence of series (40) in $\overline{\Omega}' = \overline{D}' \times \overline{L}$, $\overline{D}' = \bigcup\limits_{k=0}^{r} \overline{D}'_k$, it is sufficient to show that the sum of the series

$$\sum_{n=7}^{\infty} \frac{\left|\frac{\partial^2 w^{(n)}(x)}{\partial x_i \partial x_j}\right|^2}{\omega_n^4} \tag{41}$$

exists and is uniformly bounded in $\overline{D}'$. Using some results from [4] we obtain the estimate

$$\forall (x, y) \in \overline{D}' \times D : \left|\frac{\partial^2 G^{(2)}(x, y, -\varkappa_0^2)}{\partial x_i \partial x_j}\right| \le \frac{c}{|x - y|}, \quad i, j = 1, 2, 3.$$

By virtue of (38) the above estimate allows us to write the Bessel inequality

$$\sum_{n=7}^{\infty} \frac{\left|\frac{\partial^2 w^{(n)}(x)}{\partial x_i \partial x_j}\right|^2}{\omega_n^4} \leq \int_D \left|\frac{\partial^2 G^{(2)}(x,y,-\varkappa_0^2)}{\partial x_i \partial x_j}\right|^2 dy, \quad x \in \overline{D}',$$

implying that the sum of series (41) exists and is uniformly bounded in $\overline{D}'$.

9. We proceed to investigating the second series of (25), having rewritten it as

$$\sum_{n=7}^{\infty} \frac{w^{(n)}(x)}{\omega_n^2} \left((A\psi)_n \sqrt{\omega_n}\right) \sin \sqrt{\omega_n}\, t. \tag{42}$$

Comparing series (42) and (26) one can easily note that they have the same structure. The only difference is that the cosine is replaced by the sine, and $(A^2\varphi)_n$ by $(A\psi)_n\sqrt{\omega_n}$. In investigating series (26) we used the fact that the series

$$\sum_{n=7}^{\infty} (A^2\varphi)_n^2$$

is convergent. Due to the restrictions imposed on ψ the convergence of the series

$$\sum_{n=7}^{\infty} (A\psi)_n^2 \omega_n$$

immediately follows from Theorem 1 proved above. Therefore the above-described investigation scheme is applicable to series (42).

10. Finally, we shall investigate the third series of (25), having rewritten it as

$$\sum_{n=7}^{\infty} \frac{w^{(n)}(x)}{\omega_n^2} \int_0^t (AF)_n(\tau)\sqrt{\omega_n} \sin \sqrt{\omega_n}\,(t-\tau) d\tau.$$

As is clear from the foregoing discussion, we are to show the convergence of the series

$$\sum_{n=7}^{\infty} \int_0^e \left((AF)_n(\tau)\right)^2 \omega_n \, d\tau.$$

The latter assertion immediately follows from Theorem 1 and the well-known theorem on limit passage under the Lebesgue integral sign.

It remains for us to prove that the Fourier series of the vector-function $F(x,t)$

$$\sum_{n=1}^{\infty} F_n(t) w^{(n)}(x) \tag{43}$$

is uniformly convergent in the closed cylinder $\overline{\Omega}$. Consider the series

$$\sum_{n=7}^{\infty} w^{(n)}(x) \int_0^t \frac{dF_n(\tau)}{d\tau}\, d\tau. \tag{44}$$

Estimate the remainder of series (44) by the Cauchy–Buniakovski inequality

$$\left| \sum_{n=m}^{m+p} w^{(n)}(x) \int_0^t \frac{dF_n(\tau)}{d\tau}\, d\tau \right| \leq$$
$$\leq \left[\sum_{n=m}^{m+p} \frac{|w^{(n)}(x)|^2}{\omega_n^2} \sum_{n=m}^{m+p} \int_0^e \left| \frac{dF_n(\tau)}{d\tau} \right|^2 \omega_n^2\, d\tau \right]^{1/2}.$$

Applying Theorem 2 for the vector-function $\frac{\partial F(x,t)}{\partial t}$ and the theorem on limit passage under the integral sign we can state that the series

$$\sum_{n=7}^{\infty} \int_0^e \left| \frac{dF_n(\tau)}{d\tau} \right|^2 \omega_n^2\, d\tau$$

is convergent. Hence, taking into account the uniform boundedness of the sum of series (33), we find that series (44) is uniformly convergent in $\overline{\Omega}$. From (44) we have

$$\sum_{n=7}^{\infty} w^{(n)}(x) \int_0^t \frac{dF_n(\tau)}{d\tau}\, d\tau =$$
$$= \sum_{n=7}^{\infty} F_n(t) w^{(n)}(x) - \sum_{n=7}^{\infty} F_n(0) w^{(n)}(x).$$

Now it is clear that in order to prove the uniform convergence of series (43) in $\overline{\Omega}$ it is sufficient to show that the series

$$\sum_{n=7}^{\infty} F_n(0) w^{(n)}(x) \tag{45}$$

is uniformly convergent in $\overline{D}$. We estimate the remainder of series (45) as follows:

$$\left| \sum_{n=m}^{m+p} F_n(0) w^{(n)}(x) \right| \leq \left[\sum_{n=m}^{m+p} \frac{|w^{(n)}(x)|^2}{\omega_n^2} \sum_{n=m}^{m+p} F_n^2(0) \omega_n^2 \right]^{1/2}.$$

Applying Theorem 2 for $F(x,0)$, we immediately obtain the convergence of the series

$$\sum_{n=7}^{\infty} F_n^2(0) \omega_n^2.$$

Now taking into account the uniform boundedness of the sum of series (33), we obtain the uniform convergence of series (45) in $\overline{D}$, which completes the substantiation of the Fourier method for the considered problem.

Thus, the main result of this paper can be formulated as the following

Theorem 3. *If F, φ, ψ satisfy the conditions of Section 3, then the series (25) is a regular solution of the problem $(\mathrm{II})_{F,\varphi,\psi,0,0,0}$.*

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(Received 16.03.1994)

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