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THE MAXIMUM PRINCIPLE IN OPTIMAL CONTROL PROBLEMS WITH CONCENTRATED AND DISTRIBUTED DELAYS IN CONTROLS

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ABSTRACT. In the present work there has been posed and studied a general nonlinear optimal problem and a quasi-linear optimal problem with fixed time and free right end. It contains absolutely continuous monotone delays in phase coordinates and absolutely continuous monotone and distributed delays in controls. For these problems the necessary and, respectively, sufficient conditions of optimality in the form of the maximum principle have been proved.

1. Statement of the problem. The maximum principle. Let $O_1 \subset R^n$, $O_2 \subset R^r$ be open sets and $S = [-s_1, 0] \times \cdots \times [-s_k, 0]$, $s_k > \cdots > s_1 > 0$. Let an n-dimensional vector-function $f(t, x_1, \dots, x_s, u_1, \dots, u_{v+k})$ with fixed $t \in I = [0, T_0]$ be continuously differentiable with respect to $(x_1, \dots, x_s, u_1, \dots, u_{\nu+k}) \in O_1^s \times O_2^{\nu+k}$. For fixed $(x_1, \dots, x_s, u_1, \dots, u_{\nu+k}) \in O_1^s \times O_2^{\nu+k}$ let this function be measurable with respect to $t \in I$, like the matrix-functions f_{x_i}, $i = 1, \dots, s$, f_{u_j}, $j = 1, \dots, \nu + k$. For each pair of compacts $K \subset O_1$, $M \subset O_2$ let there exist a function $m(t) = m_{K,M}(t)$ summable on I such that

$$|f(t, x_1, \dots, x_s, u_1, \dots, u_{\nu+k})| + \sum_{i=1}^{s} |f_{x_i}| + \sum_{i=1}^{\nu+k} |f_{u_i}| \le m(t), \tag{1}$$
$$\forall\, (t, x_1, \dots, x_s, u_1, \dots, u_{\nu+k}) \in I \times K^s \times M^{\nu+k}.$$

Let now the functions $\tau_i(t)$, $i = 1, \dots, s$, $\theta_j(t)$, $j = 1, \dots, \nu$, $t \in I$, be absolutely continuous, satisfy the conditions $\tau_i(t) \le t$, $\dot{\tau}_i(t) > 0$, $i = 1, \dots, s$, $\theta_j(t) \le t$, $\dot{\theta}_j(t) > 0$, $j = 1, \dots, \nu$, and $\tau = \min(\tau_1(0), \dots, \tau_s(0))$, $\theta = \min(\theta_1(0), \dots, \theta_\nu(0), -s_k)$; $U_1 \subset O_2$, $N \subset O_1$ be convex sets and $\Omega_1 = \Omega(U_1) = \{u \in L_\infty | u(t) \in U_1,\ t \in [\theta, T_0]$, the closure of $u([\theta, T_0])$ is

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compact and belongs to $O_2\}$. Let L_∞ be a space of essentially bounded measurable functions $u:[\theta,T_0]\to R^r$ with the norm $\|u\|=\operatorname{ess\,sup}|u(t)|$, $t\in[\theta,T_0]$; $G=\{\varphi\in E_\varphi|\varphi(t)\in N,\ t\in[\tau,0]\}$, E_φ be the space of piecewise continuous functions $\varphi:[\tau,0]\to R^n$ with a finite number of discontinuity points of the first kind with the norm $\|\varphi\|=\sup|\varphi(t)|$, $t\in[\tau,0]$, and let $q^i(t,x_0,x_1)$, $(t,x_0,x_1)\in I\times O_1^2$, $i=0,\dots,l$, be continuously differentiable scalar functions.

Definition 1. We shall call the element $\zeta=(T,x_0,\varphi,u)\in I\times O_1\times G\times\Omega_1$ admissible if the corresponding solution $x(t)=x(t;\zeta)$ of the system

$$\dot x(t)=\int_S f\big(t,x(\tau_1(t)),\dots,x(\tau_s(t)),u(\theta_1(t)),\dots,u(\theta_\nu(t)),u(t+s_1),\dots \\ \dots,u(t+s_k)\big)dS,\ \ t\in[0,T]\subset I,\ u\in\Omega_1 \tag{2}$$

(here and in what follows dS stands for $ds_1\dots ds_k$), with the initial condition

$$x(t)=\varphi(t),\ \ t\in[\tau,0),\ \ x(0)=x_0, \tag{3}$$

is defined on the interval $[0,T]$ and satisfies the condition

$$q^i(T,x_0,x(T))=0,\ \ i=1,\dots,l. \tag{4}$$

We shall denote a set of admissible elements by Δ_1.

Definition 2. We call the element $\widetilde\zeta=(\widetilde T,\widetilde x_0,\widetilde\varphi,\widetilde u)\in\Delta_1$, $\widetilde T\in(0,T_0)$ optimal if there exists a number $\delta>0$ such that for each element $\zeta=(T,x_0,\varphi,u)\in\Delta_1$ satisfying the condition

$$|T-\widetilde T|+|x_0-\widetilde x_0|+\|\varphi-\widetilde\varphi\|+\|u-\widetilde u\|\le\delta,$$

the inequality

$$q^0(\widetilde T,\widetilde x_0,\widetilde x(\widetilde T))\le q^0(T,x_0,x(T)) \tag{5}$$

is fulfilled, where $\widetilde x(t)=x(t;\widetilde\zeta)$, $x(t)=x(t;\zeta)$.

The optimal problem consists in finding the optimal elements.

Theorem 1 (The necessary conditions of optimality). *Let $\widetilde\zeta\in\Delta_1$ be an optimal element and $t=\widetilde T$ be a Lebesgue point of the function*

$$\widetilde f(t)=\int_S f\big(t,\widetilde x(\tau_1(t)),\dots,\widetilde x(\tau_s(t)),\widetilde u(\theta_1(t)),\dots \\ \dots,\widetilde u(\theta_\nu(t)),\widetilde u(t+s_1),\dots,\widetilde u(t+s_k)\big)dS.$$

Then there exist a nonzero vector $\pi = (\pi_0, \dots, \pi_l)$, $\pi_0 \le 0$, *and a solution* $\psi(t)$, $t \in [0, \widetilde{T}]$ *of the system*

$$\dot{\psi}(t) = -\sum_{j=1}^{s} \psi(\gamma_j(t)) \widetilde{f}_{x_j}(\gamma_j(t)) \dot{\gamma}_j(t), \quad t \in [0, \widetilde{T}], \quad \psi(t) = 0, \quad t > \widetilde{T}, \tag{6}$$

such that the following conditions are fulfilled:

1. *the integral maximum principle:*

$$\sum_{j=1}^{s} \int_{\tau_j(0)}^{0} \psi(\gamma_j(t)) \widetilde{f}_{x_j}(\gamma_j(t)) \dot{\gamma}_j(t) \widetilde{\varphi}(t) dt \ge$$
$$\ge \sum_{j=1}^{s} \int_{\tau_j(0)}^{0} \psi(\gamma_j(t)) \widetilde{f}_{x_j}(\gamma_j(t)) \dot{\gamma}_j(t) \varphi(t) dt, \quad \forall \varphi \in G, \tag{7}$$

$$\int_0^{\widetilde{T}} \psi(t) \Big\{ \sum_{i=1}^{\nu} \widetilde{f}_{u_i}(t) \widetilde{u}(\theta_i(t)) + \sum_{i=1}^{k} \int_S \widetilde{f}_{u_{\nu+i}}(t, s_1, \dots, s_k) \times$$
$$\times \widetilde{u}(t + s_i) dS \Big\} dt \ge \int_0^{\widetilde{T}} \psi(t) \Big\{ \sum_{i=1}^{\nu} \widetilde{f}_{u_i}(t) u(\theta_i(t)) +$$
$$+ \sum_{i=1}^{k} \int_S \widetilde{f}_{u_{\nu+i}}(t, s_1, \dots, s_k) u(t + s_i) dS \Big\} dt, \quad \forall u \in \Omega_1; \tag{8}$$

2. *the conditions of transversality:*

$$\pi \widetilde{Q}_t = -\psi(\widetilde{T}) \widetilde{f}(\widetilde{T}), \quad \pi \widetilde{Q}_{x_0} = -\psi(0), \quad \pi \widetilde{Q}_{x_1} = \psi(\widetilde{T}). \tag{9}$$

Here $Q = (q^0, \dots, q^l)$, *the tilde sign over* Q *means that the corresponding gradient is calculated at the point* $(\widetilde{T}, \widetilde{x}_0, \widetilde{x}(\widetilde{T}))$, $\gamma_j(t)$ *is the inverse of the function* $\tau_j(t)$,

$$\widetilde{f}_{x_j}(t) = \int_S \widetilde{f}_{x_j}(t, s_1, \dots, s_k) dS,$$
$$\widetilde{f}_{x_j}(t, s_1, \dots, s_k) = f_{x_j}\big(t, \widetilde{x}(\tau_1(t)), \dots, \widetilde{x}(\tau_s(t)), \widetilde{u}(\theta_1(t)), \dots$$
$$\dots, \widetilde{u}(\theta_\nu(t)), \widetilde{u}(t + s_1), \dots, \widetilde{u}(t + s_k)\big).$$

Remark 1. If the rank of the matrix $(\widetilde{Q}_t, \widetilde{Q}_{x_0}, \widetilde{Q}_{x_1})$ is equal to $1 + l$, then $\psi(t) \not\equiv 0$.

Consider now the problem with a fixed time and a free right end:

$$\dot{x}(t) = \sum_{i=1}^{s} A_i(t) x(\tau_i(t)) + \int_S f\big(t, u(\theta_1(t)), \dots$$

$$\ldots, u(\theta_\nu(t)), u(t+s_1), \ldots, u(t+s_k)\big)dS, \tag{10}$$
$$x(t) = \varphi(t), \quad t \in [\tau, 0), \quad x(0) = x_0, \quad \varphi \in G,$$
$$I(\zeta) = \int_0^{T_0} \Big[g\big(t, x(\tau_1(t)), \ldots, x(\tau_s(t))\big) + \int_S f^0\big(t, u(\theta_1(t)), \ldots$$
$$\ldots, u(\theta_\nu(t)), u(t+s_1), \ldots, u(t+s_k)\big)dS\Big]dt,$$

where $A_i(t)$, $i = 1, \ldots, s$, $t \in I$, are the summable matrix-functions; the function $f(t, u_1, \ldots, u_{\nu+k})$ with fixed $t \in I$ is continuous with respect to $(u_1, \ldots, u_{\nu+k}) \in O_2^{\nu+k}$, and with fixed $(u_1, \ldots, u_{\nu+k}) \in O_2^{\nu+k}$ it is measurable with respect to $t \in I$; for each compact $M \subset O_2$ there exists a summable function $m_1(t) = m_M(t)$ such that

$$|f(t, u_1, \ldots, u_{\nu+k})| \le m_1(t), \quad \forall (t, u_1, \ldots, u_{\nu+k}) \in I \times M^{\nu+k};$$

the function $g(t, x_1, \ldots, x_s)$ with fixed $t \in I$ is continuously differentiable and convex with respect to $(x_1, \ldots, x_s) \in O_3^{ns}$, with fixed $(x_1, \ldots, x_s) \in O_3^{ns}$ it is measurable with respect to $t \in I$, and for each compact $K \subset O_3$ there exists a summable function $m_2(t) = m_k(t)$ such that

$$|g(t, x_1, \ldots, x_s)| \le m_2(t), \quad \forall (t, x_1, \ldots, x_s) \in I \times O_3^{ns};$$

$O_3 \subset R^n$ is a convex open set; the scalar function f^0 satisfies some conditions like f; moreover, $\Omega_2 = \Omega(U_2)$, where $U_3 \subset O_2$ is an arbitrary set and $x_0 \in R^n$ is a given point.

Definition 3. An element $\widetilde{\zeta} = (\widetilde{\varphi}, \widetilde{u}) \in \Delta_2 = G \times \Omega_2$ is called optimal if for each $\zeta \in \Delta_2$ the inequality

$$I(\widetilde{\zeta}) \le I(\zeta)$$

is fulfilled.

Theorem 2. *For an element $\widetilde{\zeta} \in \Delta_2$ to be optimal, it suffices to fulfill the following conditions:*

$$\sum_{j=1}^{s} \int_{\tau_j(0)}^{0} \big[\psi(\gamma_j(t))A_j(\gamma_j(t)) - \widetilde{g}_{x_j}(\gamma_j(t))\big]\dot{\gamma}_j(t)\widetilde{\varphi}(t)dt \ge$$
$$\ge \sum_{j=1}^{s} \int_{\tau_j(0)}^{0} \big[\psi(\gamma_j(t))A_j(\gamma_j(t)) - \widetilde{g}_{x_j}(\gamma_j(t))\big]\dot{\gamma}_j(t)\varphi(t)dt, \quad \forall \varphi \in G, \tag{11}$$
$$\int_0^{T_0} \big[\psi(t)f(t, \widetilde{u}(\cdot)) - f^0(t, \widetilde{u}(\cdot))\big]dt \ge \int_0^{T_0} \big[\psi(t)f(t, u(\cdot)) -$$
$$- f^0(t, u(\cdot))\big]dt, \quad \forall u \in \Omega_2. \tag{12}$$

Here

$$f(t,u(\cdot)) = \int_S f\big(t, u(\theta_1(t)), \dots, u(\theta_\nu(t)), u(t+s_1), \dots, u(t+s_k)\big)dS,$$

$f^0(t,u(\cdot))$ *is defined analogously;* $\widetilde{g}_{x_j}(t) = g_{x_j}\big(t, \widetilde{x}(\tau_1(t)), \dots, \widetilde{x}(\tau_s(t))\big)$ *and* $\psi(t)$ *is the solution of the system*

$$\dot{\psi}(t) = \sum_{j=1}^{s} \big[\psi_0(\gamma_j(t))\widetilde{g}_{x_j}(\gamma_j(t)) - \psi(\gamma_j(t))A_j(\gamma_j(t))\big]\dot{\gamma}_j(t), \quad t \in [0, T_0], \tag{13}$$

$$\psi(t) = 0, \quad t \geq T_0, \quad \psi_0(t) = 1, \quad t \in [0, T_0], \quad \psi_0(t) = 0, \quad t > T_0. \tag{14}$$

Remark 2. Theorems 1 and 2 are valid in the case where $\theta_j(t)$, $t \in I$, $j = 1, \dots, \nu$, are piecewise absolutely continuous functions (with a finite number of discontinuity points) satisfying the conditions $\theta_j(t) \leq t$ and $\dot{\theta}_j(t) > 0$.

2. Proof of Theorem 1.

2.1. *Continuous dependence and differentiability of the solution.* Let $\widetilde{\zeta} = (\widetilde{T}, \widetilde{x}_0, \widetilde{\varphi}, \widetilde{u}) \in \Delta_1$, $\widetilde{T} < T_0$ be an optimal element of the problem (2)–(5) and let $\widetilde{x}(t) = x(t; \widetilde{\zeta})$, $t \in [0, \widetilde{T}]$ be the corresponding solution of system (2).

Lemma 1. *For each* $\varepsilon > 0$ *there exists a number* $\delta = \delta(\varepsilon) > 0$ *such that for each* $\mu = (x_0, \varphi, u) \in O_1 \times G \times \Omega_1$ *satisfying the condition*

$$|x_0 - \widetilde{x}_0| + \|\varphi - \widetilde{\varphi}\| + \|u - \widetilde{u}\| \leq \delta, \tag{15}$$

the system

$$\begin{gathered} \dot{x}(t) = \int_S f\big(t, x(\tau_1(t)), \dots, x(\tau_s(t)), u(\theta_1(t)), \dots \\ \dots, u(\theta_\nu(t)), u(t+s_1), \dots, u(t+s_k)\big)dS, \\ x(t) = \varphi(t), \quad t \in [\tau, 0), \quad x(0) = x_0, \end{gathered} \tag{16}$$

has the solution $x(t;\mu)$ *which is defined on* $[0, \widetilde{T} + \delta] \subset I$. *In this case if* $\mu_i = (x_0^i, \varphi_i, u_i)$, $i = 1, 2$, *satisfy the condition* (15), *then*

$$|x(t;\mu_1) - x(t;\mu_2)| \leq \varepsilon, \quad t \in [0, \widetilde{T} + \delta].$$

Proof. We rewrite the system (16) in the form

$$\begin{gathered} \dot{x}(t) = \widetilde{f}\big(t, x(\tau_1(t)), \dots, x(\tau_s(t))\big) + \delta f\big(t, x(\tau_1(t)), \dots, x(\tau_s(t))\big), \\ x(t) = \varphi(t), \quad t \in [\tau, 0), \quad x(0) = x_0, \end{gathered}$$

where

$$\widetilde{f}(t, x_1, \dots, x_s) = \int_S f\big(t, x_1, \dots, x_s, \widetilde{u}(\theta_1(t)), \dots$$

$$\ldots,\widetilde{u}(\theta_\nu(t)),\widetilde{u}(t+s_1),\ldots,\widetilde{u}(t+s_k)\big)dS,$$

$$\delta f(t,x_1,\ldots,x_s)=\int_S\Big[f\big(t,x_1,\ldots,x_s,u(\theta_1(t)),\ldots$$

$$\ldots,u(\theta_\nu(t)),u(t+s_1),\ldots,u(t+s_k)\big)-f\big(t,x_1,\ldots,x_s,\widetilde{u}(\theta_1(t)),\ldots$$

$$\ldots,\widetilde{u}(\theta_\nu(t)),\widetilde{u}(t+s_1),\ldots,\widetilde{u}(t+s_k)\big)\Big]dS.$$

Let $M_0\subset O_2$ be the closure of the set $\widetilde{u}([\theta,T_0])$. Then there exists a number $\delta>0$ such that the functions $u\in V_\delta=\{u\in\Omega_1|\ \|u-\widetilde{u}\|\leq\delta\}$ for almost all $t\in[\theta,T_0]$ take values from the compact $M_1\subset O_2$ containing some neighborhood of the compact M_0.

Let $K_0\subset O_1$ be a compact containing some neighborhood of the set $\{\widetilde{x}(t)|t\in[\tau,\widetilde{T}]\}$.

We shall estimate $\delta f(t,x_1,\ldots,x_s)$ for each $(x_1,\ldots,x_s)\in K_0^s$, $u\in V_\delta$. It is not difficult to notice that

$$|\delta f|\leq\int_S\Big\{\int_0^1\Big|\frac{d}{d\xi}f\big(t,x_1,\ldots,x_s,\widetilde{u}(\theta_1(t))+\xi\delta u(\theta_1(t)),\ldots$$

$$\ldots,\widetilde{u}(\theta_\nu(t))+\xi\delta u(\theta_\nu(t)),\widetilde{u}(t+s_1)+\xi\delta u(t+s_1),\ldots$$

$$\ldots,\widetilde{u}(t+s_k)+\xi\delta u(t+s_k)\big)\Big|d\xi\Big\}dS,$$

where $\delta u(t)=u(t)-\widetilde{u}(t)$.

It is clear that $\widetilde{u}+\xi\delta u\in V_\delta$ and hence, $\widetilde{u}(t)+\xi\delta u(t)\in M_1$ for almost all $t\in[\theta,T_0]$.

Taking (1) into consideration, we obtain

$$|\delta f|\leq\int_S\Big\{\int_0^1\Big[\sum_{j=1}^{\nu}\big|f_{u_i}\big(t,x_1,\ldots,x_s,\widetilde{u}(\theta_1(t))+\xi\delta u(\theta_1(t)),\ldots$$

$$\ldots,\widetilde{u}(\theta_\nu(t))+\xi\delta u(\theta_\nu(t)),\widetilde{u}(t+s_1)+\xi\delta u(t+\xi_1),\ldots$$

$$\ldots,\widetilde{u}(t+s_k)+\xi\delta u(t+\xi_k)\big)\big||\delta u(\theta_j(t))|+$$

$$+\sum_{j=1}^{k}\big|f_{u_{\nu+j}}\big(t,x_1,\ldots,x_s,\widetilde{u}(\theta_1(t))+\xi\delta u(\theta_1(t)),\ldots,\widetilde{u}(\theta_\nu(t))+$$

$$+\xi\delta u(\theta_\nu(t)),\widetilde{u}(t+s_1)+\xi\delta u(t+s_1),\ldots,\widetilde{u}(t+s_k)+\xi\delta u(t+s_k)\big)\big|\times$$

$$\times|\delta u(t+s_j)|\Big]d\xi\Big\}dS\leq(s_1\cdots s_k)m_{k_0,M_0}(t)\|u-\widetilde{u}\|.$$

Thus for each $(x_1,\ldots,x_s)\in K_0^s$, $u\in V_\delta$, $t',t''\in I$ we have

$$\Big|\int_{t'}^{t''}\delta f(t,x_1,\ldots,x_s)dt\Big|\leq c_0\delta. \tag{17}$$

Now it is not difficult to see that

$$\int_I \Big\{ \big|\delta f(t, x_1, \ldots, x_s)\big| + \sum_{j=1}^{s} \big|\delta f_{x_j}\big| \Big\} dt \le c_1 = const. \tag{18}$$

The inequalities (17) and (18) allow us to conclude that Lemma 1 implies Theorem 2.8 (see [1], p.52). □

Now let us introduce the set $V_1 = \big\{\delta\mu = (\delta x_0, \delta\varphi, \delta u) | \delta x_0 \in O_1 - \widetilde{x}_0$, $\delta\varphi \in G - \widetilde{\varphi}$, $\delta u \in \Omega - \widetilde{u}$, $|\delta x_0| \le c = const$, $\|\delta\varphi\| \le c$, $\|\delta u\| \le c\big\}$.

Lemma 2. *There exist numbers $\varepsilon_0 > 0$, $\delta_0 > 0$ such that for each $\varepsilon \in [0, \varepsilon_0]$ and $\delta\mu \in V_1$ the solution $x(t; \varepsilon\delta\mu) = x(t; \widetilde{\mu} + \varepsilon\delta\mu)$, $\widetilde{\mu} = (\widetilde{x}_0, \widetilde{\varphi}, \widetilde{u})$, is defined on $[0, \widetilde{T} + \delta_0]$, and*

$$\lim_{\varepsilon \to 0} x(t; \varepsilon\delta\mu) = x(t; \widetilde{\mu}) \ \text{ is uniform over } \ (t, \delta\mu) \in [0, \widetilde{T} + \delta_0] \times V_1.$$

Moreover

$$x(t; \varepsilon\delta\mu) = \widetilde{x}(t) + \varepsilon\delta x(t; \delta\mu) + R(t; \varepsilon\delta\mu), \ \ t \in [0, \widetilde{T} + \delta_0], \tag{19}$$

where $\widetilde{x}(t) = x(t; \widetilde{\mu})$,

$$\begin{aligned} \delta x(t; \delta\mu) = {} & Y(0; t)\delta x_0 + \sum_{j=1}^{s} \int_{\tau_j(0)}^{0} Y(\gamma_j(\xi); t) \widetilde{f}_{x_j}(\gamma_j(\xi)) \times \\ & \times \dot{\gamma}_j(\xi)\delta\varphi(\xi) d\xi + \int_0^t Y(\xi; t) \Big\{ \sum_{j=1}^{\nu} \widetilde{f}_{u_j}(\xi) \delta u(\theta_j(\xi)) + \\ & + \sum_{j=1}^{k} \int_S \widetilde{f}_{u_{\nu+j}}(\xi, s_1, \ldots, s_k) \delta u(\xi + s_j) dS \Big\} d\xi, \end{aligned} \tag{20}$$

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} R(t; \varepsilon\delta\mu) = 0 \ \text{ is uniform over } \ (t, \delta\mu) \in [0, \widetilde{T} + \delta_0] \times V_1. \tag{21}$$

Here $Y(\xi; t)$, $\xi, t \in I$ is a matrix function satisfying both the matrix equation

$$\frac{\partial Y(\xi; t)}{\partial \xi} + \sum_{j=1}^{s} Y(\gamma_j(\xi); t) \widetilde{f}_{x_j}(\gamma_j(\xi)) \dot{\gamma}_j(\xi) = 0, \ \ 0 \le \xi \le t,$$

and the condition

$$Y(\xi; t) = \begin{cases} E, & \xi = t, \\ 0, & \xi > t. \end{cases}$$

Proof. The first part of the Lemma is a simple corollary of Lemma 1.

It is easily seen that the function $\Delta x(t) = x(t) - \widetilde{x}(t)$, where $x(t) = x(t;\varepsilon\delta\mu)$, satisfies the system

$$\frac{d}{dt}\Delta x(t) = \int_S f\big(t, x(\tau_1(t)),\dots,x(\tau_s(t)),\widetilde{u}(\theta_1(t)) + \\ +\varepsilon\delta u(\theta_1(t)),\dots,\widetilde{u}(\theta_\nu(t)) + \varepsilon\delta u(\theta_\nu(t)),\widetilde{u}(t+s_1) + \\ +\varepsilon\delta u(t+s_1),\dots,\widetilde{u}(t+s_k)+\varepsilon\delta u(t+s_k)\big)dS - \widetilde{f}(t), \tag{22}$$

and the initial condition

$$\Delta x(t) = \varepsilon\delta\varphi(t),\ \ t\in[\tau,0),\ \ \Delta x(0) = \varepsilon\delta x_0.$$

We can rewrite the system (22) in the form

$$\frac{d}{dt}\Delta x(t) = \sum_{j=1}^{s}\widetilde{f}_{x_j}(t)\Delta x(\tau_j(t)) + \varepsilon\sum_{j=1}^{\nu}\widetilde{f}_{u_j}(t)\delta u(\theta_j(t)) + \\ +\varepsilon\sum_{j=1}^{k}\int_S f_{u_{\nu+j}}(t,s_1,\dots,s_k)\delta u(t+s_j)dS + \Gamma(t;\varepsilon\delta\mu), \tag{23}$$

where

$$\Gamma(t;\varepsilon\delta\mu) = \int_S \Big[f\big(t,x(\tau_1(t)),\dots,x(\tau_s(t)),\widetilde{u}(\theta_1(t)) + \varepsilon\delta u(\theta_1(t)),\dots \\ \dots,\widetilde{u}(\theta_\nu(t)) + \varepsilon\delta u(\theta_\nu(t)),\widetilde{u}(t+s_1)+\varepsilon\delta u(t+s_1),\dots,\widetilde{u}(t+s_k)) + \\ +\varepsilon\delta u(t+s_k)\big) - \widetilde{f}(t) - \sum_{j=1}^{s}\widetilde{f}_{x_j}(t)\Delta x(\tau_j(t)) - \varepsilon\sum_{j=1}^{\nu} f_{u_j}(t)\times \\ \times\delta u(\theta_j(t)) - \varepsilon\sum_{j=1}^{k}\widetilde{f}_{u_{\nu+j}}(t,s_1,\dots,s_k)\delta u(t+s_j)\Big]dS.$$

By means of the Cauchy formula the solution of the system (23) can be represented in the form

$$\Delta x(t) = \varepsilon\Big\{Y(0;t)\delta x_0 + \sum_{j=1}^{s}\int_{\tau_j(0)}^{0} Y(\gamma_j(\xi);t)\widetilde{f}_{x_j}(\gamma_j(\xi))\times \\ \times\dot{\gamma}_j(\xi)\delta\varphi(\xi)d\xi + \int_0^t Y(\xi;t)\Big[\sum_{j=1}^{\nu}\widetilde{f}_{u_j}(\xi)\delta u(\theta_j(\xi)) + \\ +\sum_{j=1}^{k}\int_S \widetilde{f}_{u_{\nu+j}}(\xi;s_1,\dots,s_k)\delta u(\xi+s_j)dS\Big]d\xi\Big\} +$$

$$+\int_0^t Y(\xi;t)\Gamma(\xi)d\xi = \varepsilon\delta x(t;\varepsilon\delta\mu) + R(t;\varepsilon\delta\mu).$$

Now we shall estimate R. We have

$$|R(t;\varepsilon\delta\mu)| \le \|Y\| \int_0^{\widetilde{T}+\delta_0} |\Gamma(t;\varepsilon\delta\mu)|dt \le$$

$$\le \|Y\| \int_0^{\widetilde{T}+\delta_0} \Big\{ \int_S \Big[\int_0^1 \Big| \frac{d}{d\xi} f\big(t, \widetilde{x}(\tau_1(t)) + \xi\Delta x(\tau_1(t)), \dots$$
$$\dots, \widetilde{x}(\tau_s(t)) + \xi\Delta x(\tau_s(t)), \widetilde{u}(\theta_1(t)) + \varepsilon\xi\delta u(\theta_1(t)), \dots$$
$$\dots, \widetilde{u}(\theta_\nu(t)) + \varepsilon\xi\delta u(\theta_\nu(t)), \widetilde{u}(t+s_1) + \varepsilon\xi\delta u(t+s_1), \dots$$
$$\dots, \widetilde{u}(t+s_k) + \varepsilon\xi\delta u(t+s_k)\big) - \sum_{j=1}^{s} \widetilde{f}_{x_j}(t)\Delta x(\tau_j(t)) -$$
$$-\varepsilon\Big(\sum_{j=1}^{\nu} \widetilde{f}_{u_j}(t)\delta u(\theta_j(t)) + \sum_{j=1}^{k} \widetilde{f}_{u_{\nu+j}}(t, s_1, \dots, s_k)\delta u(t+s_j)\Big)\Big| d\xi\Big] dS\Big\} dt,$$

where

$$\Delta x(t) = x(t) - \widetilde{x}(t), \quad \|Y\| = \max_{\xi, t\in[0,\widetilde{T}+\delta_0]} |Y(\xi;t)|.$$

Differentiating the integrand with respect to ξ and grouping corresponding terms we get

$$|R(t;\varepsilon\delta\mu)| \le \varepsilon\sigma_1(\varepsilon\delta\mu) + \|\Delta x\|\sigma_2(\varepsilon\delta\mu)$$

with

$$\|\Delta x\| = \max_{0\le t\le \widetilde{T}+\delta_0} |\Delta x(t)|,$$

$$\sigma_1(\varepsilon\delta\mu) = c\|Y\| \int_0^{\widetilde{T}+\delta_0} \Big\{ \int_S \Big[\int_0^1 \Big[\sum_{j=1}^{\nu} \Big| f_{u_j}\big(t, \widetilde{x}(\tau_1(t)) + \xi\Delta x(\tau_1(t)), \dots$$
$$\dots, \widetilde{x}(\tau_s(t)) + \xi\Delta x(\tau_s(t)), \widetilde{u}(\theta_1(t)) + \varepsilon\xi\delta u(\theta_1(t)), \dots, \widetilde{u}(\theta_\nu(t)) +$$
$$+\varepsilon\xi\delta u(\theta_\nu(t)), \widetilde{u}(t+s_1) + \varepsilon\xi\delta u(t+s_1), \dots, \widetilde{u}(t+s_k) + \varepsilon\xi\delta u(t+s_k)) -$$
$$-\widetilde{f}_{u_j}(t)\Big| + \sum_{j=1}^{k} \Big| \widetilde{f}_{u_{\nu+j}}(t, \widetilde{x}(\tau_1(t)) + \xi\Delta x(\tau_1(t)), \dots, \widetilde{x}(\tau_s(t)) +$$
$$+\xi\Delta x(\tau_s(t)), \widetilde{u}(\theta_1(t)) + \varepsilon\xi\delta u(\theta_1(t)), \dots, \widetilde{u}(\theta_\nu(t)) + \varepsilon\xi\delta u(\theta_\nu(t)), \widetilde{u}(t+s_1) +$$
$$+\varepsilon\xi\delta u(t+s_1), \dots, \widetilde{u}(t+s_k) + \varepsilon\xi\delta u(t+s_k)) - \widetilde{f}_{u_{\nu+j}}(t)\Big|\Big] d\xi\Big] dS\Big\} dt,$$

$$\sigma_2(\varepsilon\delta\mu) = \|Y\| \int_0^{\widetilde{T}+\delta_0} \Big\{ \int_S \Big[\int_0^1 \sum_{j=1}^s \Big| f_{x_j}\big(t, \widetilde{x}(\tau_1(t)) +$$
$$+\xi\Delta x(\tau_1(t)), \dots, \widetilde{x}(\tau_s(t)) + \xi\Delta x(\tau_s(t)), \widetilde{u}(\theta_1(t)) + \varepsilon\xi\delta u(\theta_1(t)), \dots$$
$$\dots, \widetilde{u}(\theta_\nu(t)) + \varepsilon\xi\delta u(\theta_\nu(t)), \widetilde{u}(t+s_1) + \varepsilon\xi\delta u(t+s_1), \dots$$
$$\dots, \widetilde{u}(t+s_k) + \varepsilon\xi\delta u(t+s_k)\big) - \widetilde{f}_{x_j}(t)\Big| d\xi\Big] dS\Big\} dt.$$

It is not difficult to see that the convergence $\lim_{\varepsilon\to 0}\sigma_i(\varepsilon\delta\mu) = 0$, $i = 1, 2$, is uniform for $\delta\mu \in V_1$.

Now,

$$\|\Delta x\| \le \varepsilon\|\delta x(\delta\mu)\| + \|R(\varepsilon\delta\mu)\| \le$$
$$\le \varepsilon\big(\|\delta x(\delta\mu)\| + \sigma_1(\varepsilon\delta\mu)\big) + \sigma_2(\varepsilon\delta\mu)\|\Delta x\|.$$

Here

$$\|\delta x(\delta\mu)\| = \max_{0\le t\le \widetilde{T}+\delta_0} |\delta x(t;\delta\mu)|, \quad \|R(\varepsilon\delta\mu)\| = \max_{0\le t\le \widetilde{T}+\delta_0} |R(t;\varepsilon\delta\mu)|.$$

Hence

$$\|\Delta x\| \le \frac{\varepsilon\big(\|\delta x(\delta\mu)\| + \sigma_1(\varepsilon\delta\mu)\big)}{1 - \sigma_2(\varepsilon\delta\mu)} = \varepsilon\alpha(\varepsilon\delta\mu),$$

where the value $\alpha(\varepsilon\delta\mu)$ is bounded when $\varepsilon \to 0$ uniformly for $\delta\mu \in V_1$.

Thus

$$\|R(\varepsilon\delta\mu)\|/\varepsilon \le \sigma_1(\varepsilon\delta\mu) + \alpha(\varepsilon\delta\mu)\sigma_2(\varepsilon\delta\mu),$$

whence follows (21). □

Let D_0 be a set of points $\zeta = (T, x_0, \varphi, u) \in I \times O_1 \times G \times \Omega_1$ defined by the following condition: the system

$$\dot{x}(t) = \int_S f\big(t, x(\tau_1(t)), \dots, x(\tau_s(t)), u(\theta_1(t)), \dots$$
$$\dots, u(\theta_\nu(t)), u(t+s_1), \dots, u(t+s_k)\big) dS,$$
$$x(t) = \varphi(t), \quad t \in [\tau, 0), \quad x(0) = x_0,$$

has the solution $x(t) = x(t;\zeta)$ which is defined on $[0, T]$.

Thus the mapping

$$L : D_0 \to R^{1+n} \tag{24}$$

is given on D_0 by the formula $L(\zeta) = (T, x_0, x(T))$. It follows from Lemma 1 that the set D_0 is open in the topology induced from the space $E_\zeta = R^1 \times R^n \times E\varphi \times L_\infty$.

Lemma 3. *The mapping* (24) *is differentiable at the point* $\widetilde{\zeta} = (\widetilde{T}, \widetilde{x}_0, \widetilde{\varphi}, \widetilde{u})$. *Namely,*

$$\begin{gathered} dL_{\widetilde{\zeta}}(\delta\zeta) = (\delta T, \delta x_0, \delta x_1), \\ \textit{where } \delta x_1 = \widetilde{f}(\widetilde{T})\delta T + \delta x(\widetilde{T}; \delta\mu), \ \delta\zeta = (\delta T, \delta\mu) \in E_\zeta - \widetilde{\zeta}. \end{gathered} \tag{25}$$

Proof. Let $V = \{\delta\zeta = (\delta T, \delta\mu) |\ |\delta T| \le c,\ \delta\mu \in V_1\}$. With small enough $\varepsilon > 0$ we have $\varepsilon\delta T \le \delta_0$ for each $\delta\zeta \in V$. Therefore, using the representation (19) and the fact that $t = \widetilde{T}$ is the Lebesgue point of the function $\widetilde{f}(t)$, we have

$$\begin{aligned} x(\widetilde{T} + \varepsilon\delta T; \varepsilon\delta\mu) &= \widetilde{x}(\widetilde{T} + \varepsilon\delta T) + \varepsilon\delta x(\widetilde{T} + \varepsilon\delta T; \delta\mu) + \\ +R(\widetilde{T} + \varepsilon\delta T; \varepsilon\delta\mu) &= \widetilde{x}(\widetilde{T}) + \int_{\widetilde{T}}^{\widetilde{T}+\varepsilon\delta T} \widetilde{f}(t)dt + \varepsilon\delta x(\widetilde{T}; \delta\mu) + \\ &+\varepsilon\big(\delta x(\widetilde{T} + \varepsilon\delta T; \delta\mu) - \delta x(\widetilde{T}; \delta\mu)\big) + R(\widetilde{T} + \varepsilon\delta T; \delta\mu) = \\ &= \widetilde{x}(\widetilde{T}) + \varepsilon\delta x_1 + o_1(\varepsilon\delta\zeta), \end{aligned}$$

where $\lim_{\varepsilon\to 0} o_1(\varepsilon\delta\zeta)/\varepsilon = 0$ uniformly for $\delta\zeta \in V$.

Therefore

$$\begin{gathered} L(\widetilde{\zeta} + \varepsilon\delta\zeta) - L(\widetilde{\zeta}) = \big(\widetilde{T} + \varepsilon\delta T, \widetilde{x}_0 + \varepsilon\delta x_0, x(\widetilde{T} + \varepsilon\delta T)) - \\ -(\widetilde{T}, \widetilde{x}_0, \widetilde{x}(\widetilde{T})\big) = (\delta T, \delta x_0, \delta x_1)\cdot\varepsilon + o(\varepsilon\delta\zeta) = dL_{\widetilde{\zeta}}(\delta\zeta)\varepsilon + o(\varepsilon\delta\zeta), \end{gathered}$$

where $o(\varepsilon\delta\zeta) = \big(0, 0, o_1(\varepsilon\delta\zeta)\big)$. $\square$

2.2. *Deduction of the maximum principle.* Consider the linear topological space $E_z = R^1 \times E_\zeta$ of the points $z = (\sigma, \zeta)$.

Let D be a subset of this space,

$$D = \big\{z | z = (\sigma, \zeta) | \sigma \in R^1,\ \ \zeta \in D_0\big\}.$$

The set D is open (the more so, finitely open) (see [2,3]). Define on D the mapping $P : D \to R^{1+l}$ by the formula

$$P(z) = Q(L(\zeta)) + (\sigma, 0, \dots, 0) = \big(q^o(L(\zeta)), \dots, q^l(L(\zeta))\big) + (\sigma, 0, \dots, 0).$$

The differential of the mapping P at the point $\widetilde{\zeta} = (0, \widetilde{T}, \widetilde{x}_0, \widetilde{\varphi}, \widetilde{u})$ has the form

$$\begin{gathered} dP_{\widetilde{z}}(\delta z) = \widetilde{\sigma} + \frac{\partial\widetilde{Q}}{\partial t}\delta T + \frac{\partial\widetilde{Q}}{\partial x_0}\delta x_0 + \frac{\partial\widetilde{Q}}{\partial x_1}\delta x_1, \\ \widetilde{\sigma} = (\delta\sigma, 0, \dots, 0),\ \ \delta z \in E_z - \widetilde{z}. \end{gathered}$$

Using the expressions (20) and (25), we obtain

$$
\begin{aligned}
dP_{\widetilde{z}}(\delta z) = \widetilde{\sigma} + \Big(\frac{\partial \widetilde{Q}}{\partial t} + \frac{\partial \widetilde{Q}}{\partial x_1}\widetilde{f}(\widetilde{T})\Big)\delta T + \Big(\frac{\partial \widetilde{Q}}{\partial x_0} + \frac{\partial \widetilde{Q}}{\partial x_1} Y(0;\widetilde{T})\Big)\delta x_0 + \\
+\sum_{j=1}^{s} \frac{\partial \widetilde{Q}}{\partial x_1}\int_{\tau_j(0)}^{0} Y(\gamma_j(t);\widetilde{T})\widetilde{f}_{x_j}(\gamma_j(t))\dot{\gamma}_j(t)\delta\varphi(t)dt + \\
+\frac{\partial \widetilde{Q}}{\partial x_1}\int_0^{\widetilde{T}} Y(t;\widetilde{T})\Big[\sum_{j=1}^{\nu}\widetilde{f}_{u_j}(t)\delta u(Q_j(t)) + \\
+\sum_{j=1}^{k}\int_S \widetilde{f}_{u_{\nu+j}}(t;s_1,\dots,s_k)\delta u(t+s_j)dS\Big]dt.
\end{aligned}
$$

In the space E_z let us define a filter by the following elements:

$$
W_{\widetilde{z}} = (R^1_+ \cap V_0) \times V_{\widetilde{T}} \times V_{\widetilde{x}_0} \times (G \cap V_{\widetilde{\varphi}}) \times (\Omega_1 \cap V_{\widetilde{u}}),
$$

where V_0, $V_{\widetilde{T}}$, $V_{\widetilde{x}_0}$, $V_{\widetilde{\varphi}}$, $V_{\widetilde{u}}$ are arbitrary convex neighborhoods of the points $0 \in R^1$, $\widetilde{T} \in (0, T_0)$, $\widetilde{x}_0 \in O_1$, $\widetilde{\varphi} \in G$, $\widetilde{u} \in \Omega_1$, correspondingly, $R^1_+ = [0,\infty)$.

The filter Φ is convex (the more so, quasi-convex, (see [2,3]). It is evident that $[\Phi] = \Phi$, where $[\Phi]$ is the filter whose elements are the convex hulls of the elements of the filter Φ. It follows from Lemma 1 that the mapping P is continuous on Φ (see [2,3]).

The optimality of the element $\widetilde{\zeta}$ is equivalent to its extremality and the latter implies criticality of the mapping P on Φ, which is proved by the well-known method (see [2,3]).

Thus all the premises for the necessary criticality condition are fulfilled [2,3].

Therefore there exist a nonzero vector $\pi = (\pi_0, \dots, \pi_l)$ and an element

$$
\widetilde{W}_{\widetilde{z}} = (R^1_+ \cap \widetilde{V}_0) \times \widetilde{V}_{\widetilde{T}} \times \widetilde{V}_{\widetilde{x}_0} \times (G \cap \widetilde{V}_{\widetilde{\varphi}}) \times (\Omega_1 \cap \widetilde{V}_{\widetilde{u}})
$$

of the filter Φ such that

$$
\pi dP_{\widetilde{z}}(\delta z) \le 0, \ \ \forall\, \delta z \in K(\widetilde{W}_{\widetilde{z}} - \widetilde{z}), \tag{26}
$$

where $K(W)$ is a cone stretched on the set W. It is easily seen that the condition $\delta z \in K(\widetilde{W}_{\widetilde{z}} - \widetilde{z})$ is equivalent to the condition $\delta\sigma \in R^1_+$, $\delta T \in R^1 - \widetilde{T}$, $\delta x_0 \in R^n - \widetilde{x_0}$, $\delta\varphi \in K(\widetilde{V}_{\widetilde{\varphi}} - \widetilde{\varphi}) \supset G - \widetilde{\varphi}$, $\delta u \in K(\widetilde{V}_{\widetilde{u}} - \widetilde{u}) \supset \Omega_1 - \widetilde{u}$.

Using the expression for the differential $dP_{\widetilde{z}}(\delta z)$, assuming that $\delta u = 0$, $\delta\varphi = 0$, and taking into consideration that $\delta\sigma \in R^1_+$, δT, δx_0 may take arbitrary values, we obtain for $\pi_0 \le 0$

$$\begin{cases} \pi\Big(\frac{\partial \widetilde{Q}}{\partial t} + \frac{\partial \widetilde{Q}}{\partial x_1}\widetilde{f}(\widetilde{T})\Big) = 0, \\ \pi\Big(\frac{\partial \widetilde{Q}}{\partial x_0} + \frac{\partial \widetilde{Q}}{\partial x_1}Y(0;\widetilde{T})\Big) = 0. \end{cases} \tag{27}$$

Assuming that in (26) $\delta t_1 = 0$, $\delta x_0 = 0$, $\delta u = 0$, $\delta\sigma = 0$ we obtain

$$\sum_{j=1}^{s} \pi\frac{\partial \widetilde{Q}}{\partial x_1}\int_{\tau_j(0)}^{0} Y(\gamma_j(t);\widetilde{T})\widetilde{f}_{x_j}(\gamma_j(t))\dot{\gamma}_j(t)\delta\varphi(t)dt \le 0, \quad \delta\varphi \in G - \widetilde{\varphi}. \tag{28}$$

If we assume $\delta t_1 = 0$, $\delta x_0 = \delta\varphi = 0$, $\delta\sigma = 0$ we shall get

$$\pi\frac{\partial \widetilde{Q}}{\partial x_1}\int_0^{\widetilde{T}} Y(t;\widetilde{T})\Big[\sum_{j=1}^{\nu}\widetilde{f}_{u_j}(t)\delta u(\theta_j(t)) + \\ + \int_S \sum_{j=1}^{k}\widetilde{f}_{u_{\nu+j}}(t;s_1,\dots,s_k)\delta u(t+s_j)dS\Big]dt \le 0. \tag{29}$$

Let us introduce the notation

$$\psi(t) = \pi\frac{\partial \widetilde{Q}}{\partial x_1}Y(t;\widetilde{T}), \quad t \in [0,\widetilde{T}]. \tag{30}$$

It is clear that $\psi(t)$ satisfies the system (6) and the condition

$$\psi(\widetilde{T}) = \pi\frac{\partial \widetilde{Q}}{\partial x_1}. \tag{31}$$

We can now rewrite the condition (27) in the form

$$\begin{cases} \pi\frac{\partial \widetilde{Q}}{\partial t} = -\psi(\widetilde{T})\widetilde{f}(\widetilde{T}), \\ \pi\frac{\partial \widetilde{Q}}{\partial x_0} = -\psi(0). \end{cases} \tag{32}$$

Equations (31) and (32) give the conditions of transversality (9). Taking into account (30), we can rewrite inequalities (28), (29) in the form (7), (8). Thus Theorem 1 is completely proved. □

3. Proof of Theorem 2. Let for $\widetilde{\zeta} = (\widetilde{\varphi},\widetilde{u}) \in \Delta_2$ the conditions (11), (12) be fulfilled. Introducing the notation $\Delta x(t) = x(t;\zeta) - x(x;\widetilde{\zeta})$, $\zeta \in \Delta_2$ and taking into account (10), (13), and (14), we obtain

$$0 = \psi(T_0)\Delta x(t_0) - \psi(0)\Delta x(0) = \int_0^{T_0}\frac{d}{dt}(\psi(t)\Delta x(t))dt = \\ = \int_0^{T_0}\big[\dot{\psi}(t)\Delta x(t) + \psi(t)\frac{d}{dt}\Delta x(t)\big]dt =$$

$$=\int_0^{T_0}\Big\{\sum_{j=1}^{s}\psi_0(\gamma_j(t))\widetilde{g}_{x_j}(\gamma_j(t))\dot{\gamma}_j(t)\Delta x(t)-\sum_{j=1}^{s}\psi(\gamma_j(t))A_j(\gamma_j(t))\dot{\gamma}_j(t)\Delta x(t)+$$

$$+\psi(t)\sum_{j=1}^{s}A_j(t)\Delta x(\tau_j(t))+\psi(t)\big[f(t,u(\cdot))-f(t,\widetilde{u}(\cdot))\big]\Big\}dt.$$

Further, performing elementary transformations, in view of (14) we get

$$0=\psi(T_0)\Delta x(t_0)-\psi(0)\Delta x(0)=\sum_{j=1}^{s}\int_0^{\tau_j(T_0)}\widetilde{g}_{x_j}(\gamma_j(t))\dot{\gamma}_j(t)\Delta x(t)dt-$$

$$-\sum_{j=1}^{s}\int_0^{\tau_j(T_0)}\psi(\gamma_j(t))A_j(\gamma_j(t))\dot{\gamma}_j(t)\Delta x(t)dt+$$

$$+\sum_{j=1}^{s}\int_{\tau_j(0)}^{\tau_j(T_0)}\psi(\gamma_j(t))\times A_j(\gamma_j(t))\dot{\gamma}_j(t)\Delta x(t)dt+$$

$$+\int_0^{T_0}\psi(t)\big[f(t,u(\cdot))-f(t,\widetilde{u}(\cdot))\big]dt=\sum_{j=1}^{s}\int_{\gamma_j(0)}^{T_0}\widetilde{g}_{x_j}(t)\Delta x(\tau_j(t))dt+$$

$$+\sum_{j=1}^{s}\int_{\tau_j(0)}^{0}\psi(\gamma_j(t))A_j(\gamma_j(t))\dot{\gamma}_j(t)\Delta x(t)dt+\sum_{j=1}^{s}\int_0^{\gamma_j(0)}\widetilde{g}_{x_j}(t)\Delta x(\tau_j(t))dt-$$

$$-\sum_{j=1}^{s}\int_{\tau_j(0)}^{0}\widetilde{g}_{x_j}(\gamma_j(t))\dot{\gamma}_j(t)\Delta x(t)dt+\int_0^{T_0}\psi(t)\big[f(t,u(\cdot))-f(t,\widetilde{u}(\cdot))\big]dt=$$

$$=\sum_{j=1}^{s}\int_0^{T_0}\widetilde{g}_{x_j}(t)\Delta x(\tau_j(t))dt+\sum_{j=1}^{s}\int_{\tau_j(0)}^{0}\Big[\psi(\gamma_j(t))A_j(\gamma_j(t))-$$

$$-\widetilde{g}_{x_j}(\gamma_j(t))\Big]\dot{\gamma}_j(t)\Delta x(t)dt+\int_0^{T_0}\psi(t)\big[f(t,u(\cdot))-f(t,\widetilde{u}(\cdot))\big]dt.$$

Further,

$$I(\widetilde{\zeta})-I(\zeta)=\int_0^{T_0}\Big[g\big(t,\widetilde{x}(\tau_1(t)),\dots,\widetilde{x}(\tau_s(t))\big)-g\big(t,x(\tau_1(t)),\dots$$

$$\dots,x(\tau_s(t))\big)+f^0(t,\widetilde{u}(\cdot))-f^0(t,u(\cdot))\Big]dt+\psi(T_0)\Delta x(T_0)-$$

$$-\psi(0)\Delta x(0)=\int_0^{T_0}\Big[g\big(t,\widetilde{x}(\tau_1(t)),\dots,\widetilde{x}(\tau_s(t))\big)-g\big(t,x(\tau_1(t)),\dots$$

$$\dots,x(\tau_s(t))\big)+\sum_{j=1}^{s}\widetilde{g}_{x_j}(t)\Delta x(\tau_j(t))\Big]dt+\sum_{j=1}^{s}\int_{\tau_j(0)}^{0}\Big[-\widetilde{g}_{x_j}(\gamma_j(t))+$$

$$+\psi(\gamma_j(t))A_j(\gamma_j(t))\Big]\dot{\gamma}_j(t)(\varphi(t)-\widetilde{\varphi}(t))dt+\int_0^{T_0}\Big\{-f^0(t,u(\cdot))+ \\ +\psi(t)f(t,u(\cdot))-\big[-f^0(t,\widetilde{u}(\cdot))+\psi(t)f(t,\widetilde{u}(\cdot))\big]\Big\}dt. \tag{33}$$

Due to the convexity of the function g, by virtue of the inequalities (11), (12), it follows from (33) that $I(\widetilde{\zeta})-I(\zeta)\leq 0$ for $\zeta\in\Delta_2$. $\square$

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