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COMMUTATIVITY FOR A CERTAIN CLASS OF RINGS

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ABSTRACT. We discuss the commutativity of certain rings with unity 1 and one-sided s-unital rings under each of the following conditions: $x^r[x^s,y]=\pm[x,y^t]x^n$, $x^r[x^s,y]=\pm x^n[x,y^t]$, $x^r[x^s,y]=\pm[x,y^t]y^m$, and $x^r[x^s,y]=\pm y^m[x,y^t]$, where r, n, and m are non-negative integers and $t>1$, s are positive integers such that either s, t are relatively prime or $s[x,y]=0$ implies $[x,y]=0$. Further, we improve the result of [6, Theorem 3] and reprove several recent results.

Throughout the paper R will represent an associative ring (with or without unity 1). Let $C(R)$ denote the commutator ideal of R, $Z(R)$ the center of R, and H the heart of R. By $(GF(q))_2$ we mean the ring of 2×2 matrices over the Galois field $GF(q)$ with q elements. Set $e_{11}=\begin{pmatrix}1&0\\0&0\end{pmatrix}$, $e_{12}=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $e_{21}=\begin{pmatrix}0&0\\1&0\end{pmatrix}$, and $e_{22}=\begin{pmatrix}0&0\\0&1\end{pmatrix}$ in $(GF(p))_2$ for a prime p. Following [1], a ring R is said to be left (resp., right) s-unital, if $x\in Rx$ (resp., $x\in xR$) for each element x in R. Further, R is called s-unital if $x\in Rx\cap xR$ (see [2] and [3]). The symbol $[x,y]$ stands for the commutator $xy-yx$ for any x, $y\in R$. In some particular cases several authors [1–3, 5] studied the commutativity of rings satisfying the following conditions:

(c_1) For every $x,y\in R$ there holds $x^r[x^s,y]=\pm[x,y^t]x^n$ with integers $t>1$, $s\geq 1$, $n\geq 0$, $r\geq 0$.

(c_2) For every $x,y\in R$ there holds $x^r[x^s,y]=\pm[x,y^t]y^m$ with integers $t>1$, $s\geq 1$, $m\geq 0$, $r\geq 0$.

(c_3) For every $x,y\in R$ there holds $x^r[x^s,y]=\pm x^n[x,y^t]$ with integers $t>1$, $s\geq 1$, $n\geq 0$, $r\geq 0$.

(c_4) For every $x,y\in R$ there holds $x^r[x^s,y]=\pm y^m[x,y^t]$ with integers $t>1$, $s\geq 1$, $m\geq 0$, $r\geq 0$.

To develop the commutativity of a ring R satisfying one of the above conditions, we need some extra condition such as

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Q(s): For any positive integer s $s[x,y] = 0$ implies $[x,y] = 0$ for all $x, y \in R$,

or

Q(s,t): s and t are relatively prime integers.

To prove our results we need a few preliminary lemmas. We begin with the following well-known result [6, p. 221].

Lemma 1. *Let $x, y \in R$ and $[[x,y],x] = 0$. Then $[x^k, y] = kx^{k-1}[x,y]$ for any positive integer k.*

Lemma 2 ([7]). *Let R be a ring with unity 1, and let $f : R \to R$ be a function such that $f(x+1) = f(x)$ for every $x \in R$. If for some positive integer n we have $x^n f(x) = f(x)x^n = 0$ for all $x \in R$, then necessarily $f(x) = 0$.*

Lemma 3 ([8]). *Let f be a polynomial in the non-commuting indeterminates $x_1, x_2, \ldots, x_n$ with integer coefficients. Then the following statements are equivalent:*

(i) *$C(R)$ is a nil ideal for any ring R satisfying $f = 0$.*

(ii) *$(GF(p))_2$ fails to satisfy $f = 0$ for every prime p.*

The main results of the paper are:

Theorem 1. *Let R be a ring with 1 satisfying (c_1) together with either $Q(s,t)$ or $Q(s)$. Then R is commutative.*

Theorem 2. *Let R be a ring with 1 satisfying (c_2) together with either $Q(s,t)$ or $Q(s)$. Then R is commutative.*

Theorem 3. *Let R be a ring with 1 satisfying (c_3) together with either $Q(s,t)$ or $Q(s)$. Then R is commutative.*

Theorem 4. *Let R be a ring with 1 satisfying (c_4) such that either $Q(s,t)$ or $Q(s)$ holds. Then R is commutative.*

Remark 1. The well-known Grassmann algebra rules out the possibility of $t = 1$ in the above theorems. Moreover, if we drop the restriction that R has unity 1 in the above theorems, then the ring R may be poorly non-commutative. Indeed, the following example demonstrates this constraint: Let D_k be the ring of all $k \times k$ matrices over a division ring D, and let $A_k = \{(a_{ij} \in D_k | a_{ij} = 0, j \geq i\}$. Then A_k is a non-commutative ring for any positive integer $k > 2$. But A_3 satisfies (c_1), (c_2), (c_3), and (c_4) for all positive integers s, t and non-negative integers m, n, and r.

According to [9], let f be a polynomial in two non-commuting indeterminates with integral coefficients. Now write f in the form

$$f(x,y) = \sum_{r=1}^{d} \sum_{i=0}^{r} f_{ri}(x,y),$$

where f_{ri} denotes the sum of all terms of f with degree i in x and $r-i$ in y. Let s_{ri} denote the sum of the coefficients of f_{ti}. Then we note that if

$$s_{ri} = 0 \quad \text{for all} \quad r \quad \text{and} \quad i, \tag{I}$$

then all commutative rings satisfy $f = 0$. The converse is also true as proved by Kezlan [9]. For this we take a transcendental extension field of rationals and use the fact that the polynomial

$$f(X^d, X^{d+1}) = \sum_{r=1}^{d} \sum_{i=0}^{r} s_{ri} X^{(d+1)r-i}$$

in one indeterminate X vanishes on it.

Thus if f is to be equivalent with the commutativity it must at least satisfy (I), and so we may write

$$f(x,y) = m[x,y] + \sum_{r=3}^{d} \sum_{i=1}^{r-1} f_{ri}(x,y)$$

for some integer m. Moreover, if m is divisible by a prime p, then the ring of strictly upper triangular 3×3 matrices over any field of characteristic p satisfies the identity, and so we assume that

$$m = \pm 1. \tag{II}$$

Let us consider the condition

$$f_{r1} = 0 \quad \text{for all} \quad r. \tag{III}$$

In [9] Kezlan proved the following

Theorem. *If f satisfies* (I), (II), *and* (III), *then an arbitrary ring R is commutative if and only if it satisfies the identity $f = 0$.*

Also, it should be remarked that (I) in the theorem could be replaced by

$$f_{r,r-1} = 0 \quad \text{for all} \quad r. \tag{IV}$$

An example was given in [9] to show that we must assume either (III) or (IV) or some other condition concerning the terms linear in x or in y. So (I) and (II) alone are not enough.

Further, in [9] Kezlan proved that for a polynomial $f(x,y)$ the identity $f(x,y) = 0$ is equivalent with the commutativity for all rings if $f(x,y) =$

$\pm[x,y] + \sum_{r=3}^{d} \sum_{i=1}^{r-1} f_{ri}(x,y)$, where f_{ri} denotes the sum of all terms of $f(x,y)$ with degree i in x and $r-i$ in y, where s_{si} denotes the sum of coefficients of f_{ri}. It was also shown that under certain restrictions on the terms linear in one variable or the other, the polynomial identity $f(x,y) = 0$ is indeed equivalent with the commutativity.

The following fact plays an important role in the proof of our results.

Lemma 4 ([9]). *Let R be a ring satisfying the polynomial identity $f(x,y) = \pm[x,y]$, where each homogeneous component of $f(x,y)$ has integer coefficients whose sum is zero and where $f(x,y)$ has no linear terms either in x or in y. Then R is commutative.*

Now we prove

Proposition 1. *Let R be a ring with $p[x,y] = 0$ for all x, y in R, p a prime, and let s be a positive integer not divisible by p. Suppose that R satisfies a polynomial identity of the form $f(x,y) = \pm m[x,y]$, where m is any non-negative integer and $f(x,y)$ satisfies the same condition as in Lemma 4. Then R is commutative.*

Proof. Let $qp + mn = 1$. By hypothesis, we have

$$f(x,y) = \pm m[x,y] \quad \text{for all} \quad x, y \in R.$$

Multiply the above identity by n and use $p[x,y] = 0$ to get $nf(x,y) = \pm[x,y]$. Hence by Lemma 4 R is commutative. $\square$

Lemma 5. *Let R be a ring with 1 satisfying (c_1) or (c_2) or (c_3) or (c_4) such that either $Q(s,t)$ or $Q(s)$ holds. Then $C(R)$ is nil.*

Proof. Let $x = e_{22}$ and $y = e_{21} + e_{22}$ in (c_1) and (c_2). Then by Lemma 3, x and y fail to satisfy the polynomial identities (c_3) and (c_4) for any prime p. Similarly, $x = e_{11}$ and $y = e_{21}$ fail to satisfy (c_3) and (c_4). Thus $C(R)$ is a nil ideal. $\square$

Proof of Theorem 1. According to Lemma 5, $C(R)$ is nil. Now let R satisfy (c_1). Then by contradiction we assume that there exists a non-commutative ring with 1 satisfying (c_1). Another step is to pass to the subdirectly irreducible case, and with $Q(s,t)$ this reduction can be obtained as in [9]. Therefore, without loss of generality, we assume that there is a ring R such that

(α) R is a non-commutative ring with 1, satisfies (c_1), and R is subdirectly irreducible with heart $H = C(R)$ with $H^2 = (0)$.

Using the condition $Q(s)$, we must slightly modify the arguments in [8] because this condition is not preserved under homomorphism. Then sssuming that we have a non-commutative ring A with unity satisfying (c_1) and that $Q(s)$ holds, we have $a, b \in A$ with $s[a,b] \neq 0$. By Zorn's lemma we get an ideal M which is maximal with respect to the exclusion of $s[a,b]$. Then the ring $\overline{A} = A/M$ is not commutative, satisfies (c_1), and is subdirectly irreducible with the heart containing $\overline{s}[\overline{a},\overline{b}]$. Hence the ring $R = \overline{A}$ may not inherit $Q(s)$. This shows that s does not annihilate all commutators of R. To summarize these paragraphs, if $Q(s,t)$ holds, then we have a ring R satisfying (α). If $Q(s)$ holds, then, in addition to (α), we have

(β) s does not annihilate all commutators of R.

Now we define a mapping $F: R \to R$ for fixed $y, w \in R$ by

$$F(x) = \pm[x, (y+w)^t - y^t - w^t] \quad \text{for all} \quad x \in R. \tag{1}$$

Replace x by $x+1$ in (1) to get

$$F(x+1) = \pm[x, (y+w)^t - y^t - w^t] = F(x).$$

Multiplying (1) by x^n on the right, we get

$$\begin{aligned} F(x)x^n &= \pm[x, (y+w)^t - y^t - w^t]x^n = \\ &= \pm[x, (y+w)^t]x^n \mp [x, y^t]x^n \mp [x, w^t]x^n = \\ &= x^r[x^s, y+w] - \{\pm[x, y^t]x^n\} - \{\pm[x, w^t]x^n\} = \\ &= x^r[x^s, y] + x^r[x^s, w] - x^r[x^s, y] - x^r[x^s, w] = 0. \end{aligned}$$

Using Lemma 2, we obtain $F(x) = 0$ for all x, y in R. Hence

$$\begin{aligned} &\pm [x, (y+w)^t - y^t - w^t]x^n = 0, \\ &\pm [x, (y+w)^t] \mp [x, y^t] \mp [x, w^t] = 0 \end{aligned}$$

and

$$\pm[x, (y+w)^t] = \pm[x, y^t] \pm [x, w^t] = 0. \tag{2}$$

Substituting $w = 1$ in (2), we get

$$\pm\left(t[x,y] + \sum_{k=2}^{t-1} \binom{t}{k} [x, y^k]\right) = 0. \tag{3}$$

By (3) we can write

$$t[x,y] + \sum_{k=2}^{t-1} \binom{t}{k} [x, y^k] = 0. \tag{3'}$$

Replacing y in turn by $z_1y, z_2y, \ldots, z_{t-1}y$ in $(3')$, we get

$$\begin{array}{l}
z_1t[x,y] + \binom{t}{2}z_1^2[x,y^2] + \binom{t}{3}z_1^3[x,y^3] + \cdots + \binom{t}{t-1}z_1^{t-1}[x,y^{t-1}] = 0,\\
z_2t[x,y] + \binom{t}{2}z_2^2[x,y^2] + \binom{t}{3}z_2^3[x,y^3] + \cdots + \binom{t}{t-1}z_2^{t-1}[x,y^{t-1}] = 0,\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
z_{t-1}t[x,y] + \binom{t}{2}z_{t-1}^2[x,y^2] + \binom{t}{3}z_{t-1}^3[x,y^3] + \cdots + \binom{t}{t-1}z_{t-1}^{t-1}[x,y^{t-1}] = 0.
\end{array}$$

The above identities can be written in the matrix form:

$$A_{(t-1)\times(t-1)}W_{(t-1)\times 1} =$$

$$= \begin{pmatrix} z_1 & z_1^2 & z_1^3 & \cdots & z_1^{t-1} \\ z_2 & z_2^2 & z_2^3 & \cdots & z_2^{t-1} \\ z_3 & z_3^2 & z_3^3 & \cdots & z_3^{t-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_{t-1} & z_{t-1}^2 & z_{t-1}^3 & \cdots & z_{t-1}^{t-1} \end{pmatrix} \begin{pmatrix} t[x,y] \\ \binom{t}{2}[x,y^2] \\ \binom{t}{3}[x,y^3] \\ \vdots \\ \binom{t}{t-1}[x,y^{t-1}] \end{pmatrix} = 0.$$

Multiplying the above by $\operatorname{adj}(A)$, we have $\det(A)$. In particular, $\det(A)t[x,y] = 0$ for all $x, y \in R$.

Let z_i^j be the (i,j)th entry in the matrix A. Since factoring z_i out of the ith row of A gives a Vandermonde matrix, $\langle R, +\rangle$ is torsion free. Therefore $[x,y] = 0$ for all $x, y \in R$, and R is commutative. (In short, if $\langle R, +\rangle$ is torsion free, then the standard Vandermonde determinant argument shows that the homogeneous components must vanish on R (see [9]). So $[x,y] = 0$, since R is torsion free. Thus R is commutative.)

As in [9], the subdirect irreducibility gives a unique prime p such that R has elements of additive order p, from which it follows that $pH = (0)$ (as a special case p annihilates all commutators).

Let p not divide t. Then (3) yields a polynomial identity of the type in Proposition 1 and hence R is commutative. Thus p must divide t. Therefore p cannot divide s, which is obvious if $Q(s,t)$ holds, and is also true if $Q(s)$ holds, since s does not annihilate all commutators as p does. Hence in either of the cases $Q(s,t)$ and $Q(s)$ we get

$$p \quad \text{divides} \quad t \quad \text{but does not divide} \quad s. \tag{4}$$

By interchanging the roles of x and y in (2) we obtain

$$\pm[y,(x+w)^t] = \pm[y,x^t] \pm [y,w^t]. \tag{5}$$

Replace x by $x+1$ in (c_1) to get the identities

$$(x+1)^r[(x+1)^s, y] = \pm[x,y^t](x+1)^n$$

and

$$\sum_{j=0}^{r}\sum_{k=1}^{s}\binom{r}{j}\binom{s}{k}x^j[x^k,y]=\pm\sum_{j=1}^{n}\binom{n}{j}[x,y^t]x^s. \tag{6}$$

Now we prove that $H\subseteq Z(R)$. Let $x\in H$ and $y\in R$. Thus using $H^2=(0)$, from (6) we have

$$s[x,y]=\pm[x,y^t]\quad\text{for all}\quad x\in H,\ \ y\in R. \tag{7}$$

Substitute $w=y$ into (5) to get

$$\pm[y,xy^{t-1}+yxy^{t-2}+\cdots+y^{t-1}x]=0$$

and so

$$\pm[x,y^t]=0\quad\text{for all}\quad x\in H,\ \ y\in R. \tag{8}$$

From (7), (8) and the fact that p does not divide s we have $[x,y]=0$ for $x\in H$, $y\in R$. Hence

$$H\subseteq Z(R). \tag{9}$$

Thus all commutators are central. By Lemma 5 we get

$$[x,y^t]=ty^{t-1}[x,y]=0\quad\text{for all}\quad x,y\in R. \tag{10}$$

This condition shows that t divisible by p annihilates all commutators. Hence (6) can be rewritten as

$$s[x,y]+\sum_{k=2}^{s}\binom{s}{k}[x^k,y]=\pm\sum_{j=1}^{r}\sum_{k=1}^{s}\binom{r}{j}\binom{s}{k}x^j[x^k,y]$$

which is the form of Proposition 1. Hence R is commutative. $\square$

Proof of Theorem 2. Let R satisfy (c_2). As above, it is easy to observe that the reduction to a subdirectly irreducible ring R satisfying (α) with $Q(s,t)$ holds and so do both conditions (α) and (β) with $Q(s)$. Replacing $x+1$ by x and $y+1$ by y in c_2 gives the following identities:

$$s[x,y]=\pm G(x,y),$$
$$t[x,y]=\pm H(x,y),$$

where G and H satisfy the conditions of Lemma 4. As in the proof of Theorem 1, we have a unique prime p such that $pH=(0)$. Thus by Proposition 1, p must divide both t and s, which is impossible if $Q(s,t)$ holds. Otherwise, since s is divisible by p, it must annihilate all commutators. Thus (β) gives a contradiction if $Q(s)$ holds. Hence R is commutative. $\square$

Since Theorems 3 and 4 can be proved in the same way, we omit their proofs.

A recent commutativity study [10] deals with rings satisfying related conditions of the form

$$[xy - p(yx), x] = 0$$

or

$$[xy - p(yx), y] = 0.$$

This becomes possible by interchanging the roles of x and y as

$$[yx - p(xy), y] = 0$$

or

$$[yx - p(xy), x] = 0.$$

The following result is proved in [10].

Theorem 5. *Let R be a ring with 1 such that for each $x, y \in R$, there exist $p(t)$, $q(t) \in t^2 Z[t]$ for which $[xy - p(yx), x] = 0$ and $[xy - q(yx), y] = 0$. Then R is commutative.*

Now we generalize Theorem 5.

Theorem 6. *Let R be a ring with unity 1 such that for each $x, y \in R$ there exists $p(x) \in x^2 Z[x]$ for which either $[yx - p(yx), x] = 0$ or $[yx - p(yx), y] = 0$. Then R is commutative.*

Proof. Suppose that $p(x) = a_2x^2 + a_3x^3 + a_4x^4 + \cdots + a_nx^n$, where $a_1, a_2, ..., a_n$ are integers. By hypothesis, we can write for all $x, y \in R$

$$\begin{aligned} [yx, x] &= [p(yx), x], \\ [x, y]x &= [p(yx), x]. \end{aligned} \tag{1'}$$

Putting $x + 1$ for x in $(1')$ gives

$$\begin{aligned}
[x, y](x+1) &= [p(y(x+1)), x], \\
[x, y]x + [x, y] &= \big[a_2(y(x+1))^2 + a_3(y(x+1))^3 + \cdots + a_n(y(x+1))^n, x\big], \\
[x, y]x + [x, y] &= \big[a_2y(x+1)y + a_3y(x+1)y(x+1)y + \cdots + \\
&\quad + a_ny(x+1)y(x+1)\cdots y(x+1)y, x\big](x+1) = \\
&= \big[a_2(yxy + y^2) + a_3(yxyxy + yxy^2 + y^2xy + y^3) + \cdots + \\
&\quad + a_n(yxyx\cdots y + (x+1), x\big](x+1), \\
[x, y]x + [x, y] &= \big[a_2yxy + a_3yxyxy + \cdots + a_nyxyxyx\cdots yx, x\big] + \\
&\quad + \big[a_2y^2 + a_3y^3 + \cdots + a_ny^n, x\big] + H(x, y),
\end{aligned}$$

where each homogeneous component of G has the sum of coefficients which is equal to zero. Thus H has no terms linear in y, and each term of H has a degree greater than 1 in x. Hence

$$[x,y]x + [x,y] = [p(yx),x] + [p(y),x] + H(x,y). \tag{2'}$$

Using (1′) and (2′), we obtain

$$[x,y] = [x,p(y)] + G(x,y).$$

Thus R is commutative by Kezlan's theorem [11] or Lemma 4. Similarly, with the help of Lemma 4, R is commutative if R satisfies $[yx - p(xy), y] = 0$. □

Similarly to the proof of Theorem 6, we can reprove the next theorem using Lemma 4.

Theorem 7 ([12]). *Let R be a ring with unity 1 satisfying*

$$[xy - p(yx), y] = 0 \quad \textit{for all} \quad x, y \in R,$$

where $p(x) \in x^2 Z[x]$. Then R is commutative.

Now let P be a ring property. If P is inherited by every subring and every homomorphic image, then P is called an h-property. More weakly, if P is inherited by every finitely generated subring and every natural homomorphic image modulo the annihilator of a central element, then P is called an H-property.

A ring property P such that a ring R has the property P if and only if all its finitely generated subrings have P is called an F-property.

Proposition 2 ([13, Proposition 1]). *Let P be an H-property, and P' be an F-property. If every ring R with unity 1 having the property P has the property P', then every s-unital ring having P has P'.*

Finally, Theorems 1–4, 6, and 7 are automatically generalized from a unital ring to s-unital ones due to Proposition 2. Indeed, we have

Theorem 8. *Let R be a left (resp., right) s-unital ring satisfying (c_1) (resp., (c_2)). Then R is commutative.*

Theorem 9. *Let R be a left s-unital (resp., right) ring satisfying (c_3) (resp., (c_4)). Then R is commutative.*

References

1. H. A. S. Abujabal and V. Perić, Commutativity of s-unital rings through a Streb result. *Radovi Mat.* **7**(1991), 73–92.

2. H. Komatsu, A commutativity theorem for rings. *Math. J. Okayama Univ.* **26**(1984), 109–111.

3. M. A. Quadri and M. A. Khan, A commutativity theorem for associative rings. *Math. Japonica* **33**(1988), 275–279.

4. H. G. Moore, Generalized n-like rings and commutativity. *Canad. Math. Bull.* **23**(1980), 449–452.

6. N. Jacobson, Structure of rings. *Amer. Math. Soc., Colloq. Publ.,* 1964.

7. W. K. Nicholson and A. Yaqub, Commutativity theorems for rings and groups. *Canad. Math. Bull.* **22**(1979), 419–423.

8. T. P. Kezlan, A note on commutativity of semi-prime PI rings. *Math. Japonica* **27**(1982), 267–268.

9. T. P. Kezlan, On identities which are equivalent with commutativity. *Math. Japonica* **29**(1984), 135–139.

10. H. E. Bell, M. A. Quadri, and M. Ashraf, Commutativity of rings with some commutator constraints. *Radovi Mat.* **5**(1989), 223–230.

11. T. P. Kezlan, On commutativity theorems for PI ring with unity. *Tamkang J. Math.* **24**(1993), 29–36.

12. H. E. Bell, M. A. Quadri, and M. A. Khan, Two commutativity theorems for rings. *Radovi Mat.* **3**(1987), 255–260.

13. Y. Hirano, Y. Kobayashi, and H. Tominaga, Some polynomial identities and commutativity of s-unital rings. *Math. J. Okayama Univ.* **24**(1982), 7–13.

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