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ON THE SOLVABILITY OF A DARBOUX TYPE NON-CHARACTERISTIC SPATIAL PROBLEM FOR THE WAVE EQUATION

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Abstract. The question of the correct formulation of a Darboux type non-characteristic spatial problem for the wave equation is investigated. The correct solvability of the problem is proved in the Sobolev space for surfaces of the temporal type on which Darboux type boundary conditions are given.

In the space of variables x_1, x_2, t let us consider the wave equation

$$\square u \equiv \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} = F, \tag{1}$$

where F and u are known and desired real functions, respectively.

We denote by $D : k_1 t < x_2 < k_2 t$, $0 < t < t_0$, $-1 < k_i = const < 1$, $i = 1, 2$, $k_1 < k_2$, a domain lying in the half-space $t > 0$ and bounded by the plane surfaces $S_i : k_i t - x_2 = 0$, $0 \le t \le t_0$, $i = 1, 2$, of the temporal type and by the plane $t = t_0$.

We shall consider a Darboux type problem formulated as follows: In the domain D find a solution $u(x_1, x_2, t)$ of equation (1) by the boundary conditions

$$u\big|_{S_i} = f_i, \quad i = 1, 2, \tag{2}$$

where f_i, $i = 1, 2$, are the known real functions on S_i and $(f_1 - f_2)|_{S_1 \cap S_2} = 0$.

It should be noted that in [1–5] Darboux type problems are studied for the cases where at least one of the surfaces S_1 and S_2 is the characteristic surface of equation (1) passing through the Ox_1-axis. Other multi-dimensional analogues of the Darboux problem are treated in [6–8].

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As distinct from the cases considered in [1–5], the fact that none of the surfaces S_1 and S_2 is characteristic results in the nonavailability of an integral representation for regular solutions of problem (1), (2). The latter circumstance somehow complicates the investigation of this problem. Below we shall prove the existence and uniqueness theorems for regular as well as for strong solutions of problem (1), (2) belonging to the class W_2^1.

Let $C_*^\infty(\overline{D})$ denote a space of functions belonging to the class $C^\infty(\overline{D})$ and having bounded supports, i.e.,

$$C_*^\infty(\overline{D}) = \big\{u \in C^\infty(\overline{D}) : \operatorname{diam}\operatorname{supp} u < \infty\big\}.$$

The spaces $C_*^\infty(S_i)$, $i = 1, 2$, are defined similarly.

The well-known Sobolev spaces will be denoted by $W_2^1(D)$, $W_2^2(D)$, $W_2^1(S_i)$, $i = 1, 2$. Note that the space $C_*^\infty(\overline{D})$ is an everywhere dense subspace of the spaces $W_2^1(D)$ and W_2^2, while $C_*^\infty(S_i)$ is an everywhere dense subspace of the space $W_2^1(S_i)$, $i = 1, 2$.

Definition. Let $f_i \in W_2^1(S_i)$, $i = 1, 2$, $F \in L_2(D)$. A function $u \in W_2^1(D)$ is called a strong solution of problem (1), (2) belonging to the class W_2^1 if there exists a sequence $u_n \in C_*^\infty(\overline{D})$ such that $u_n \to u$ in the space $W_2^1(D)$, $\Box u_n \to F$ in the space $L_2(D)$, and $u_n|_{S_i} \to f_i$ in $W_2^1(S_i)$, $i = 1, 2$, i.e., for $n \to \infty$

$$\|u_n - u\|_{W_2^1(D)} \to 0, \quad \|\Box u_n - F\|_{L_2(D)} \to 0,$$
$$\|u_n|_{S_i} - f_i\|_{W_2^1(S_i)} \to 0, \quad i = 1, 2.$$

We have

Lemma 1. *If $-1 < k_1 < 0$, $0 < k_2 < 1$, then for any $u \in W_2^1(D)$ there holds the a priori estimate*

$$\|u\|_{W_2^1(D)} \le C\Big(\sum_{i=1}^{2} \|f_i\|_{W_2^1(S_i)} + \|F\|_{L_2(D)}\Big), \tag{3}$$

where $f_i = u|_{S_i}$, $i = 1, 2$, $F = \Box u$, and the positive constant C does not depend on u.

Proof. Since the space $C_*^\infty(D)$ $(C_*^\infty(S_i))$ is an everywhere dense subspace of the spaces $W_2^1(D)$ and $W_2^2(D)$ $(W_2^1(S_i))$, by virtue of the well-known theorems on embedding the space $W_2^2(D)$ into the space $W_2^1(D)$, and the space $W_2^2(D)$ into $W_2^1(S_i)$ it is sufficient for us to prove that the a priori estimate (3) holds for functions u of the class $C_*^\infty(\overline{D})$.

We introduce the notation

$$
\begin{aligned}
&D_\tau=\big\{(x,t)\in D: t<\tau\big\},\ \ D_{0\tau}=\partial D_\tau\cap\big\{t=\tau\big\},\ \ 0<\tau\le t_0,\\
&S_{i\tau}=\partial D_\tau\cap S_i,\ \ i=1,2,\ \ S_\tau=S_{1\tau}\cup S_{2\tau},\\
&\alpha_1=\cos(\widehat{n,x_1}),\ \ \alpha_2=\cos(\widehat{n,x_2}),\ \ \alpha_3=\cos(\widehat{n,t}).
\end{aligned}
$$

Here $n=(\alpha_1,\alpha_2,\alpha_3)$ denotes the external normal unit vector to ∂D_τ. One can easily verify that

$$
n\big|_{S_{1\tau}}=\Big(0,\frac{-1}{\sqrt{1+k_1^2}},\frac{k_1}{\sqrt{1+k_1^2}}\Big),\ \ n\big|_{S_{2\tau}}=\Big(0,\frac{1}{\sqrt{1+k_2^2}},\frac{-k_2}{\sqrt{1+k_2^2}}\Big),
$$
$$
n\big|_{D_{0\tau}}=(0,0,1).
$$

Therefore for $-1<k_1<0$, $0<k_2<1$ we have

$$
\alpha_3\big|_{S_{i\tau}}<0,\ \ i=1,2,\quad \alpha_3^{-1}(\alpha_3^2-\alpha_1^2-\alpha_2^2)\big|_{S_i}>0,\ \ i=1,2. \tag{4}
$$

On multiplying both parts of equation (1) by $2u_t$, where $u\in C_*^\infty(\overline{D})$, $F=\Box u$, and integrating the obtained expression with respect to D_τ, we obtain due to (4)

$$
\begin{aligned}
&2\int\limits_{D_\tau}Fu_t\,dx\,dt=\int\limits_{D_\tau}\Big(\frac{\partial u_t^2}{\partial t}+2u_{x_1}u_{tx_1}+2u_{x_2}u_{tx_2}\Big)dx\,dt-\\
&-2\int\limits_{S_\tau}(u_{x_1}u_t\alpha_1+u_{x_2}u_t\alpha_2)ds=\int\limits_{D_{0\tau}}(u_t^2+u_{x_1}^2+u_{x_2}^2)dx+\\
&+\int\limits_{S_\tau}\big[(u_t^2+u_{x_1}^2+u_{x_2}^2)\alpha_3-2(u_{x_1}u_t\alpha_1+u_{x_2}u_t\alpha_2)\big]ds=\\
&=\int\limits_{D_{0\tau}}(u_t^2+u_{x_1}^2+u_{x_2}^2)dx+\int\limits_{S_\tau}\alpha_3^{-1}\big[(\alpha_3u_{x_1}-\alpha_1u_t)^2+(\alpha_3u_{x_2}-\alpha_2u_t)^2+\\
&+(\alpha_3^2-\alpha_1^2-\alpha_2^2)u_t^2\big]ds\ge\int\limits_{D_{0\tau}}(u_t^2+u_{x_1}^2+u_{x_2}^2)dx+\\
&+\int\limits_{S_\tau}\alpha_3^{-1}\big[(\alpha_3u_{x_1}-\alpha_1u_t)^2+(\alpha_3u_{x_2}-\alpha_2u_t)^2\big]ds.
\end{aligned} \tag{5}
$$

Assuming

$$w(\tau)=\int\limits_{D_{0\tau}}(u_t^2+u_{x_1}^2+u_{x_2}^2)dx,\quad \widetilde{u}_i=\alpha_3u_{x_i}-\alpha_iu_t,\ \ i=1,2,$$

$$C_1=\max\left(\frac{\sqrt{1+k_1^2}}{|k_1|},\ \frac{\sqrt{1+k_2^2}}{|k_2|}\right),$$

we find by (5) that

$$\begin{aligned}
w(\tau)&\le C_1\int\limits_{S_\tau}(\widetilde{u}_1^2+\widetilde{u}_2^2)ds+\int\limits_{D_\tau}(F^2+u_t^2)dx\,dt\le\\
&\le C_1\int\limits_{S_\tau}(\widetilde{u}_1^2+\widetilde{u}_2^2)ds+\int\limits_0^\tau d\xi\int\limits_{D_{0\xi}}u_t^2dx+\int\limits_{D_\tau}F^2dx\,dt\le\\
&\le C_1\int\limits_{S_\tau}(\widetilde{u}_1^2+\widetilde{u}_2^2)ds+\int\limits_0^\tau w(\xi)d\xi+\int\limits_{D_\tau}F^2dx\,dt. \qquad (6)
\end{aligned}$$

Let (x,τ_x) be the point at which the surface $S_1\cup S_2$ intersects with the straight line parallel to the t-axis and passing through the point $(x,0)$. We have

$$u(x,\tau)=u(x,\tau_x)+\int\limits_{\tau_x}^{\tau}u_t(x,t)dt,$$

which implies

$$\begin{aligned}
\int\limits_{D_{0\tau}}u^2(x,\tau)dx&\le 2\int\limits_{D_{0\tau}}u^2(x,\tau_x)dx+2|\tau-\tau_x|\int\limits_{D_{0\tau}}dx\int\limits_{\tau_x}^{\tau}u_t^2(x,t)dt=\\
&=2\int\limits_{S_\tau}\alpha_3^{-1}u^2ds+2|\tau-\tau_x|\int\limits_{D_\tau}u_t^2dx\,dt\le\\
&\le C_2\Big(\int\limits_{S_\tau}u^2ds+\int\limits_{D_\tau}u_t^2dx\,dt\Big), \qquad (7)
\end{aligned}$$

where $C_2=2\max(C_1,t_0)$.

On introducing the notation

$$w_0(\tau)=\int\limits_{D_{0\tau}}(u^2+u_t^2+u_{x_1}^2+u_{x_2}^2)dx$$

and combining inequalities (6) and (7), we obtain

$$w_0(\tau) \le C_2\Big[\int\limits_{S_\tau}(u^2+\widetilde{u}_1^2+\widetilde{u}_2^2)ds+\int\limits_0^\tau w_0(\xi)d\xi+\int\limits_{D_\tau}F^2dx\,dt\Big],$$

which by virtue of Gronwall's lemma implies

$$w_0(\tau) \le C_3\Big[\int\limits_{S_\tau}(u^2+\widetilde{u}_1^2+\widetilde{u}_2^2)ds+\int\limits_{D_\tau}F^2dx\,dt\Big], \tag{8}$$

where $C_3 = const > 0$.

One can easily verify that the operator $\alpha_3\frac{\partial}{\partial x_i}-\alpha_i\frac{\partial}{\partial t}$ is the internal differential operator on the surface S_τ. Therefore the following inequality holds by virtue of (2):

$$\int\limits_{S_\tau}(u^2+\widetilde{u}_1^2+\widetilde{u}_2^2)ds \le C_4\sum_{i=1}^2\|f_i\|^2_{W_2^1(S_{i\tau})}, \quad C_4 = const > 0. \tag{9}$$

From (8) and (9) we obtain

$$w_0(\tau) \le C_5\Big(\sum_{i=1}^2\|f_i\|^2_{W_2^1(S_{i\tau})}+\|F\|^2_{L_2(D_\tau)}\Big), \ \ C_5 = const > 0. \tag{10}$$

The integration of both parts of inequality (10) with respect to τ gives us estimate (3). $\square$

In the sequel it will be assumed that $-1 < k_1 < 0$, $0 < k_2 < 1$, i.e., that inequalities (4) are fulfilled.

Lemma 2. *The dependence domain for the point $P_0(x_1^0, x_2^0, t^0) \in D$ of the solution $u(x_1, x_2, t)$ of problem* (1), (2) *belonging to the class $C^2(\overline{D})$ or $W_2^2(D)$ is contained within the characteristic cone of the past $\partial K_{P_0} : t = t^0 - \sqrt{(x_1-x_1^0)^2+(x_2-x_2^0)^2}$ with the vertex at the point P_0.*

Proof. We set

$$\Omega_{P_0} = D \cap K_{P_0}, \quad S_{iP_0} = S_i \cap \partial\Omega_{P_0}, \quad i = 1,2,$$

where $K_{P_0} : t < t^0 - \sqrt{(x_1-x_1^0)^2+(x_2-x_2^0)^2}$ is the interiority of the characteristic cone ∂K_{P_0}.

To prove the lemma it is sufficient to show that if

$$f_i\big|_{S_{iP_0}} \equiv u\big|_{S_{iP_0}} = 0, \ \ i = 1,2, \quad F\big|_{\Omega_{P_0}} \equiv \Box u\big|_{\Omega_{P_0}} = 0, \tag{11}$$

then $u|_{\Omega_{P_0}} = 0$.

Let us first consider the case with $u \in C^2(\overline{D})$. We denote by S_{3P_0} the remainder part of the boundary of the domain Ω_{P_0}, i.e., $S_{3P_0} = \partial\Omega_{P_0} \backslash (S_{1P_0} \cup S_{2P_0})$. Since by our construction the surface S_{3P_0} is part of the characteristic cone ∂K_{P_0} of equation (1), we have

$$\alpha_3\big|_{S_{3P_0}} = const > 0, \quad (\alpha_3^2 - \alpha_1^2 - \alpha_2^2)\big|_{S_{3P_0}} = 0, \tag{12}$$

where $n = (\alpha_1, \alpha_2, \alpha_3)$ is the external normal unit vector to $\partial\Omega_{P_0}$.

On multiplying both parts of equation (1) by $2u_t$ and integrating the obtained expression with respect to the domain Ω_{P_0}, we obtain due to (4), (11), (12) and the arguments used in deriving (5) the following inequality:

$$0 = 2\int\limits_{\Omega_{P_0}} F u_t dx\, dt = \int\limits_{\partial\Omega_{P_0}} \big[(u_t^2 + u_{x_1}^2 + u_{x_2}^2)\alpha_3 - 2(u_{x_1}u_t\alpha_1 + u_{x_2}u_t\alpha_2)\big] ds =$$

$$= \int\limits_{\partial\Omega_{P_0}} \alpha_3^{-1}\big[(\alpha_3 u_{x_1} - \alpha_1 u_t)^2 + (\alpha_3 u_{x_2} - \alpha_2 u_t)^2 + (\alpha_3^2 - \alpha_1^2 - \alpha_2^2)u_t^2\big] ds \geq$$

$$\geq \int\limits_{S_{3P_0}} \alpha_3^{-1}\big[(\alpha_3 u_{x_1} - \alpha_1 u_t)^2 + (\alpha_3 u_{x_2} - \alpha_2 u_t)^2\big] ds. \tag{13}$$

To obtain (13) we used the fact that the operator $\alpha_3 \frac{\partial}{\partial x_i} - \alpha_i \frac{\partial}{\partial t}$ is the internal differential operator on the surface $\partial\Omega_{P_0}$ and, in particular, on $S_{1P_0} \cup S_{2P_0}$. By virtue of (11) we have the equalities

$$\Big(\alpha_3 \frac{\partial u}{\partial x_i} - \alpha_i \frac{\partial u}{\partial t}\Big)\Big|_{S_{1P_0} \cup S_{2P_0}} = 0, \quad i = 1, 2.$$

Since $\alpha_3 > 0$ on S_{3P_0}, (13) implies

$$(\alpha_3 u_{x_i} - \alpha_i u_t)\big|_{S_{3P_0}} = 0, \quad i = 1, 2. \tag{14}$$

Taking into account that $u \in C^2(\overline{D})$ and the internal differential operators $\alpha_3 \frac{\partial}{\partial x_i} - \alpha_i \frac{\partial}{\partial t}$, $i = 1, 2$, are linearly independent on the two-dimensional connected surface S_{3P_0}, we immediately find by (14) that

$$u\big|_{S_{3P_0}} \equiv const. \tag{15}$$

But by (11)

$$u\big|_{S_{3P_0} \cap \left(S_{1P_0} \cup S_{2P_0}\right)} = 0,$$

which on account of (15) gives us

$$u\big|_{S_{3P_0}} \equiv 0. \tag{16}$$

From (16) it follows in particular that $u(P_0) = 0$.

If now we take an arbitrary point $Q\in\Omega_{P_0}$, then (11) implies that the above equalities will hold if the point P_0 is replaced by the point Q. Therefore, on repeating our previous reasoning for the domain Ω_Q, we find that $u(Q)=0$. Thus for the case with $u\in C^2(\overline{D})$ we obtain $u|_{\Omega_{P_0}}=0$.

Let now $u\in W_2^2(D)$ and equalities (11) be fulfilled. One can easily verify that inequality (13), where the point P_0 is replaced by the point Q, also holds for any point $Q\in\Omega_{P_0}$, i.e.,

$$\int\limits_{S_{3Q}}\alpha_3^{-1}\big[(\alpha_3u_{x_1}-\alpha_1u_t)^2+(\alpha_3u_{x_2}-\alpha_2u_t)^2\big]ds\le 0.$$

Hence by virtue of the fact that $\alpha_3|_{S_{3Q}}=const>0$ we obtain

$$\int\limits_{S_{3Q}}\big[(\alpha_3u_{x_1}-\alpha_1u_t)^2+(\alpha_3u_{x_2}-\alpha_2u_t)^2\big]ds=0. \tag{17}$$

Let Γ_Q denote a piecewise-smooth curve which is the boundary of the two-dimensional connected surface S_{3Q}. It is obvious that

$$\Gamma_Q=S_{3Q}\cap\big(S_{1Q}\cup S_{2Q}\big). \tag{18}$$

Recalling the fact that on the surface S_{3Q} the internal differential operators $\alpha_3\frac{\partial}{\partial x_i}-\alpha_i\frac{\partial}{\partial t}$, $i=1,2$, are independent for any $v\in W_2^1(S_{3Q})$ it is not difficult to obtain the estimate

$$\int\limits_{S_{3Q}}v^2ds\le C\Big(\int\limits_{\Gamma_Q}v^2ds+$$
$$+\int\limits_{S_{3Q}}\big[(\alpha_3v_{x_1}-\alpha_1v_t)^2+(\alpha_3v_{x_2}-\alpha_2v_t)^2\big]ds\Big), \tag{19}$$

where $C=const>0$ does not depend on v and the trace $v|_{\Gamma_Q}\in L_2(\Gamma_Q)$ is correctly defined by virtue of the respective embedding theorem.

Since $u\in W_2^2(D)$, the traces $u|_{S_{3Q}}\in W_2^1(S_{3Q})$, $u|_{\Gamma_Q}\in L_2(\Gamma_Q)$ are correctly defined by virtue of the embedding theorems. Therefore by (11) and (18) we have

$$u\big|_{\Gamma_Q}=0. \tag{20}$$

From (17), (19) and (20) we obtain

$$\int\limits_{S_{3Q}} u^2 ds \le C\Big(\int\limits_{\Gamma_Q} u^2 ds +$$

$$+\int\limits_{S_{3Q}} \big[(\alpha_3 u_{x_1}-\alpha_1 u_t)^2+(\alpha_3 u_{x_2}-\alpha_2 u_t)^2\big]ds\Big)=0$$

which immediately implies

$$\int\limits_{S_{3Q}} u^2 ds = 0, \quad u\big|_{S_{3Q}}=0, \quad \forall Q\in\Omega_{P_0}. \tag{21}$$

Since $u\in W_2^2(D)$, we conclude due to (21) and Fubini's theorem that

$$u\big|_{\Omega_{P_0}}=0. \quad \square$$

Remark 1. Lemma 2 implies that the wave process described by problem (1), (2) propagates at finite velocity. Therefore if $u\in C^\infty(\overline{D})$ is a solution of problem (1), (2) for $f_i\in C_*^\infty(S_i)$, $i=1,2$, $F\in C_*^\infty(\overline{D})$, then we have $u\in C_*^\infty(\overline{D})$.

For our further discussion we shall need

Lemma 3. *Let G be a bounded subdomain a of D having a piecewise-smooth boundary and bounded from above by the plane $t=t_0$ and from the sides by the planes S_1, S_2 and the piecewise-smooth surfaces S_3, S_4 of the temporal type on which the following inequalities are fulfilled:*

$$\alpha_3\big|_{S_3}<0, \quad \alpha_3\big|_{S_4}<0, \tag{22}$$

where $n=(\alpha_1,\alpha_2,\alpha_3)$ is the unit normal vector to ∂G and $S_3\cap S_4=\varnothing$. Let $K_{P_0}^+ : t>t^0+\sqrt{(x_1-x_1^0)^2+(x_2-x_2^0)^2}$ be the domain bounded by the characteristic cone of the future with the vertex at the point $P_0(x_1^0,x_2^0,t^0)$. Let $u_0\in C^\infty(\overline{G})$ and $g_i=u_0|_{\partial G\cap S_i}$, $i=1,2$, $F_0=\square u_0$, $X=\operatorname{supp} g_1\cup$ $\operatorname{supp} g_2\cup\operatorname{supp} F_0$, $Y=\bigcup\limits_{P_0\in X} K_{P_0}^+$.

We denote by S_3^ε, S_4^ε ε-neighborhoods of the surfaces S_3, S_4, where ε is a fixed sufficiently small positive number. Then if

$$u_0\big|_{S_3\cup S_4}=0, \tag{23}$$

$$Y\cap\big(S_3^3\cup S_4^3\big)=\varnothing, \tag{24}$$

then the function

$$u(P)=\begin{cases} u_0(P), & P\in G,\\ 0, & P\in D\backslash G\end{cases}$$

is a solution of problem (1), (2) *of the class* $C_*^\infty(\overline{D})$ *for*

$$f_i(P)=\begin{cases} g_i(P), & P\in\partial G\cap S_i,\\ 0, & P\in S_i\backslash(\partial G\cap S_i),\end{cases}\quad i=1,2,$$

$$F(P)=\begin{cases} F_0(P), & P\in G,\\ 0, & P\in D\backslash G.\end{cases}$$

Proof. To prove this lemma it is sufficient to show that the function $u_0\in C^\infty(\overline{G})$ vanishes on the set $G\cap(S_3^\varepsilon\cup S_4^\varepsilon)$.

Let $P_0\in G\cap(S_3^\varepsilon\cup S_4^\varepsilon)$ be an arbitrary point of this set. We shall show that $u_0(P_0)=0$.

Let us use the notation of Lemma 2:

$$\Omega_{P_0}=G\cap K_{P_0},\quad S_{iP_0}=S_i\cap\partial\Omega_{P_0},\ \ i=1,2,3,4,\quad S_{5P_0}=\partial K_{P_0}\cap\partial\Omega_{P_0}.$$

It is obvious that $\partial\Omega_{P_0}=\overset{5}{\underset{i=1}{\cup}}S_{iP_0}$.

By the assumptions of Lemma 3 we have

$$\alpha_3\big|_{S_{iP_0}}<0,\ \ i=1,2,3,4;\ \alpha_3^{-1}(\alpha_3^2-\alpha_1^2-\alpha_2^2)\big|_{S_{iP_0}}>0,\ \ i=1,2,3,4; \tag{25}$$

$$\alpha_3\big|_{S_{5P_0}}>0,\quad (\alpha_3^2-\alpha_1^2-\alpha_2^2)\big|_{S_{5P_0}}=0, \tag{26}$$

where $n=(\alpha_1,\alpha_2,\alpha_3)$ is the unit normal vector to $\partial\Omega_{P_0}$.

Due to (23), (24) and $P_0\in G\cap(S_3^\varepsilon\cup S_4^\varepsilon)$ we have

$$u_0\big|_{S_{iP_0}}=0,\ \ i=1,2,2,4;\quad \Box u_0\big|_{\Omega_{P_0}}=F_0\big|_{\Omega_{P_0}}=0. \tag{27}$$

On multiplying both sides of the equation $\Box u_0=F_0$ by $2u_{0t}$ and on integrating the obtained expression by Ω_{P_0}, we find by virtue of (25)–(27) and the arguments used in deriving inequalities (5) and (13) that

$$0=2\int\limits_{\Omega_{P_0}}F_0\,\frac{\partial u_0}{\partial t}dx\,dt=\int\limits_{\partial\Omega_{P_0}}\alpha_3^1\Big[\Big(\alpha_3\,\frac{\partial u_0}{\partial x_1}-\alpha_1\,\frac{\partial u_0}{\partial t}\Big)^2+$$

$$+\Big(\alpha_3\,\frac{\partial u_0}{\partial x_2}-\alpha_2\,\frac{\partial u_0}{\partial t}\Big)^2+\big(\alpha_3^2-\alpha_1^2-\alpha_2^2\big)\Big(\frac{\partial u_0}{\partial t}\Big)^2\Big]ds\geq$$

$$\geq\int\limits_{S_{5P_0}}\alpha_3^{-1}\Big[\Big(\alpha_3\,\frac{\partial u_0}{\partial x_1}-\alpha_1\,\frac{\partial u_0}{\partial t}\Big)^2+\Big(\alpha_3\,\frac{\partial u_0}{\partial x_2}-\alpha_2\,\frac{\partial u_0}{\partial t}\Big)^2\Big]ds.$$

Since $\alpha_3|_{S_{5P_0}}>0$, the latter formula gives us

$$\Big(\alpha_3\,\frac{\partial u_0}{\partial x_i}-\alpha_i\,\frac{\partial u_0}{\partial t}\Big)\Big|_{S_{5P_0}}=0,\quad i=1,2.$$

The rest of the reasoning repeats the proof of Lemma 2. Therefore $u(P_0) = 0$. $\square$

Remark 2. One can easily verify that Lemma 3 remains valid if conditions (22) are not fulfilled on some set $\omega \subset S_3 \cup S_4$ of the zero two-dimensional measure, i.e., $\alpha_3|_\omega = 0$. In particular, if $\omega = \bigcup\limits_{i=1}^{m} \gamma_i$ is the union of the finite number of smooth curves $\gamma_i \subset S_3 \cup S_4$ and $\alpha_3|_\omega = 0$, $\alpha_3|_{(S_3 \cup S_4)\setminus\omega} < 0$, then Lemma 3 remains valid. We shall use this fact in the sequel when proving Theorem 1.

Remark 3. Also note that Lemmas 2 and 3 actually provide us with the technique for constructing the solution of problem (1), (2) to be described below when proving Theorem 1. This technique consists in reducing the initial problem (1), (2) to the mixed problem for a second-order hyperbolic equation in the finite cylinder.

It will be assumed below that in the boundary conditions (2) the functions f_1 and f_2 vanish on the straight line $\Gamma = S_1 \cap S_2$, i.e.,

$$f_i\big|_\Gamma = 0, \quad i = 1, 2. \tag{28}$$

Functions of the class $W_2^1(S_i)$ which satisfy equality (28) will be denoted by $\overset{\circ}{W}{}_2^1(S_i, \Gamma)$, i.e.,

$$\overset{\circ}{W}{}_2^1(S_i, \Gamma) = \big\{ f \in W_2^1(S_i) : f\big|_\Gamma = 0 \big\}, \ \ i = 1, 2.$$

We have

Theorem 1. *For any $f_i \in \overset{\circ}{W}{}_2^1(S_i, \Gamma)$, $i = 1, 2$, $F \in L_2(D)$ there exists a unique strong solution u of problem* (1), (2) *of the class W_2^1 for which estimate* (3) *holds.*

Proof. We denote by $S_i^0 : k_i t - x_2 = 0$, $0 \le t < +\infty$, $i = 1, 2$, the half-plane containing the support S_i from the boundary conditions (2), and by D_0 the dihedral angle between the half-planes S_1^0 and S_2^0. It is known that the function $f_i \in \overset{\circ}{W}{}_2^1(S_i, \Gamma)$ can be extended into the half-plane S_i^0 as a function $\widetilde{f}_i$ of the class $\overset{\circ}{W}{}_2^1(S_i^0)$, i.e., $(f_i - \widetilde{f}_i)|_{S_i} = 0$, $\widetilde{f}_i \in \overset{\circ}{W}{}_2^1(S_i^0)$, $i = 1, 2$. We assume that

$$\widetilde{F}(P) = \begin{cases} F(P), & P \in D, \\ 0, & P \in D_0 \setminus D. \end{cases}$$

Obviously, $\widetilde{F} \in L_2(D_0)$.

If $C_0^\infty(D_0)$, $C_0^\infty(S_i^0)$, $i = 1, 2$, are the spaces of finite infinitely differentiable functions, then, as we know, these spaces are everywhere dense in the spaces $L_2(D_0)$, $W_2^1(S_i^0)$, $i=1,2$, respectively. Therefore there are sequences $F_n\in C_0^\infty(D_0)$ and $f_{in}\in C_0^\infty(S_i^0)$, $i=1,2$, such that

$$\lim_{n\to\infty}\|\widetilde{F}-F_n\|_{L_2(D_0)}=\lim_{n\to\infty}\|\widetilde{f}_i-f_{in}\|_{W_2^1(S_i^0)}=0,\quad i=1,2. \tag{29}$$

In the plane of the variables x_2,t we introduce the polar coordinates r, φ. The t-axis is assumed to be the polar axis, while the polar angle φ is counted from the t-axis and assumed to be positive in the clockwise direction. We denote by φ_i the value of the dihedral angle between the half-planes S_i^0 and $x_2=0$, $0\le t<+\infty$, $i=1,2$. Since the half-planes S_i^0 are of the temporal type ($-1<k_1<0$, $0<k_2<1$), we have $0<\varphi_i<\frac{\pi}{4}$, $i=1,2$.

In passing from the Cartesian cordinates x_1, x_2, t to the system of coordinates x_1, $\tau=\log r$, φ, the dihedral angle D_0 transforms to an infinite layer

$$H=\big\{-\infty<x_1<\infty,\ -\infty<\tau<\infty,\ -\varphi_1<\varphi<\varphi_2\big\} \tag{30}$$

while in terms of the previous notations for the functions u and F equation (1) takes the form

$$e^{-2\tau}L(\tau,\varphi,\partial)u=F, \tag{31}$$

where $\partial=(\frac{\partial}{\partial x_1},\frac{\partial}{\partial\tau},\frac{\partial}{\partial\varphi})$, and $L(\tau,\varphi,\partial)$ is a second-order hyperbolic type differential operator with respect to τ with infinitely differentiable coefficients which depend on τ and φ.

In the half-plane x_1,φ let us consider a convex domain Ω of the class C^∞ bounded by the straight line segments $\ell_1:\varphi=-\varphi_1$, $\ell_2:\varphi=\varphi_2$ and the curves $\gamma_1:x_1=g(\varphi)$, $-\varphi_1\le\varphi\le\varphi_2$, $\gamma_2:x_2=-g(\varphi)$, $-\varphi_1\le\varphi\le\varphi_2$. Here $g(\varphi)\in C^\infty(-\varphi_1,\varphi_2)\cap C[-\varphi_1,\varphi_2]$, $g(\varphi)>0$ for $-\varphi_1\le\varphi\le\varphi_2$, $g'(\varphi)>0$ for $-\varphi_1<\varphi<0$, $g'(0)=0$, $g'(\varphi)<0$ for $0<\varphi<\varphi_2$, $g''(\varphi)<0$ for $-\varphi_1<\varphi<\varphi_2$, and

$$\min\big(g(-\varphi_1),g(\varphi_2)\big)>1+t_0+d, \tag{32}$$

where

$$d=\max(d_1,d_2,d_3),\quad d_i=\sup_{(x_1,x_2,t)\in\operatorname{supp}f_i}|x_1|,\quad i=1,2,$$
$$d_3=\sup_{(x_1,x_2,t)\in\operatorname{supp}F}|x_1|.$$

We denote by $H_0\subset H$ a cylindrical domain $\Omega\times(-\infty,\infty)$ of the class C^∞ where $(-\infty,\infty)$ is the τ-axis, and by ∂H_0 its lateral surface $\partial\Omega\times(-\infty,\infty)$. When the inverse transformation $(x_1,\tau,\varphi)\to(x_1,x_2,t)$ takes place, the cylindrical domain H_0 will transform into the infinite domain $G_0\subset D_0$ bounded by the surfaces $\widetilde{S}_i=S_i^0\cap\partial G_0$, $i=1,2$, and also by the surfaces $\widetilde{S}_3$ and $\widetilde{S}_4$.

We shall show below that the surfaces $\widetilde{S}_3$ and $\widetilde{S}_4$ are of the temporal type and the following conditions are fulfilled on them:

$$\alpha_3\big|_{(\widetilde{S}_3\cup\widetilde{S}_4)\setminus\omega}<0,\quad \alpha_3\big|_{\omega}=0, \tag{33}$$

where ω is the union of two smooth curves ω_1 and ω_2 lying on $\widetilde{S}_3\cup\widetilde{S}_4$.

Indeed, one can easily verify that the surfaces $\widetilde{S}_1$ and $\widetilde{S}_2$ are the images of the cylindrical surfaces $S_1'=\ell_1\times(-\infty,\infty)\subset\partial H_0$ and $S_2'=\ell_2\times(-\infty,\infty)\subset\partial H_0$, while the surfaces $\widetilde{S}_3$ and $\widetilde{S}_4$ are the images of the surfaces $S_3^0=\gamma_1\times(-\infty,\infty)\subset\partial H_0$ and $S_4^0=\gamma_2\times(-\infty,\infty)\subset\partial H_0$ when the inverse transformation $(x_1,\tau,\varphi)\to(x_1,x_2,t)$ takes place. We divide the surface S_3^0 into two parts $S_3^0=S_{3+}^0\cup S_{3-}^0$, where

$$\begin{gathered}S_{3+}^0=\gamma_{1+}\times(-\infty,\infty),\quad S_{3-}^0=\gamma_{1-}\times(-\infty,\infty),\\ \gamma_{1+}:x_1=g(\varphi),\ \ 0<\varphi<\varphi_2,\quad \gamma_{1-}:x_1=g(\varphi),\ \ -\varphi_1<\varphi<0.\end{gathered}$$

It is easy to see that when the inverse transformation $(x_1,\tau,\varphi)\to(x_1,x_2,t)$ occurs, the image $\widetilde{S}_{3+}\subset\widetilde{S}_3$ of the surface S_{3+}^0 admits the parametric representation

$$\widetilde{S}_{3+}:x_1=g(\varphi),\ \ x_2=\sigma\sin\varphi,\ \ t=\sigma\cos\varphi;\ \ 0<\varphi<\varphi_2,\ \ 0<\sigma<+\infty.$$

Hence for the unit normal vector $n=(\alpha_1,\alpha_2,\alpha_3)$ to ∂G_0 on the segment $\widetilde{S}_{3+}$ we obtain the expression

$$n\big|_{\widetilde{S}_{3+}}=\left(\frac{\sigma}{\sqrt{\sigma^2+g'^2(\varphi)}},\frac{-g'(\varphi)\cos\varphi}{\sqrt{\sigma^2+g'^2(\varphi)}},\frac{g'(\varphi)\sin\varphi}{\sqrt{\sigma^2+g'^2(\varphi)}}\right). \tag{34}$$

Taking into account the structure of the domain Ω, by (34) we find that $\widetilde{S}_{3+}$ is a surface of the temporal type on which $\alpha_3|_{\widetilde{S}_{3+}}<0$. Similar statements are proved also for the other segments $\widetilde{S}_{3-}$, $\widetilde{S}_{4+}$ and $\widetilde{S}_{4-}$ of the surfaces $\widetilde{S}_3$ and $\widetilde{S}_4$. To prove finally that condition (33) is fulfilled, it remains for us only to note that component α_3 of the unit normal vector n vanishes on the curves

$$\omega_1=\partial\widetilde{S}_{3+}\cap\partial\widetilde{S}_{3-},\quad \omega_2=\partial\widetilde{S}_{4+}\cap\partial\widetilde{S}_{4-}$$

which are the images of the straight lines $\widetilde{\omega}_1:x_1=g(0)$, $\varphi=0$, $-\infty<\tau<\infty$, and $\widetilde{\omega}_2:x_1=-g(0)$, $\varphi=\pi$, $-\infty<\tau<\infty$, and the third component α_3 of the unit normal vector n is equal to zero.

On the boundary ∂G_0 of the domain G_0 we define a function $\nu_n(x_1,x_2,t)$ of the class C^∞ as follows:

$$\nu_n\big|_{\widetilde{S}_i}=f_{in},\ \ i=1,2,\quad \nu_n\big|_{\widetilde{S}_3}=\nu_n\big|_{\widetilde{S}_4}=0,\ \ n=1,2,\dots.$$

The fact that the function $\nu_n \in C_0^\infty(\partial G_0)$ is implied by the structure of G_0, by inequality (32), and also by the smoothness and positioning of the carriers of the functions $f_{in} \in C_0^\infty(S_i^0)$, $i = 1, 2$.

In passing to the variables x_1, τ, φ, the functions ν_n and F_n will transform into some functions for which we shall use the previous notation. It is obvious that

$$\nu_n \in C_0^\infty(\partial H_0), \quad F_n \in C_0^\infty(H_0). \tag{35}$$

By virtue of (35) there are numbers $h_{in} = const$, $h_{1n} < h_{2n}$ such that

$$\begin{aligned} &\operatorname{supp} \nu_n \subset \partial H_0 \cap \{h_{1n} < \tau < h_{2n}\}, \\ &\operatorname{supp} F_n \subset H_0 \cap \{h_{1n} < \tau < h_{2n}\} = H_n. \end{aligned} \tag{36}$$

Note that when the inverse transformation $J^{-1} : (x_1, \tau, \varphi) \to (x_1, x_2, t)$ takes place, the upper base $\partial H_n \cap \{\tau = h_{2n}\}$ of the finite cylinder H_n will transform into the surface lying higher than the plane $t = t_0$, i.e., $\inf t > t_0$ for $(x_1, x_2, t) \in \partial(J^{-1}(H_n)) \cap \{\log r = h_{2n}\}$.

Assume that

$$h_{n1}^0 = h_{1n} - 1, \;\; h_{2n}^0 = h_{2n} + 1, \;\; H_n^0 = H_0 \cap \{h_{1n}^0 < \tau < h_{2n}^0\}.$$

Denote by S_{0n} the lateral surface of the finite cylinder H_n^0, and by Ω_0 the lower base of H_n^0.

For the hyperbolic equation (31) with $F = F_n$ let us consider, in the cylinder H_n^0, the following mixed problem:

$$e^{-2\tau} L(\tau, \varphi, \partial) v = F_n, \tag{37}$$

$$v\Big|_{\Omega_0} = 0, \quad \frac{\partial v}{\partial \tau}\Big|_{\Omega_0} = 0, \tag{38}$$

$$v\Big|_{S_{0n}} = \nu_n. \tag{39}$$

By virtue of (35) and the results from [9], [10] the mixed problem (37)–(39) has a unique solution $v = v_n$ of the class $C^\infty(\overline{H}_n^0)$. Note that if

$$H_{n1} = H_0 \cap \{h_{1n} - 1 < \tau < h_{1n}\} \subset H_n^0,$$

then by virtue of (36) this solution is identically zero in the cylinder H_{n1}, i.e., $v_n|_{H_{n1}} = 0$. We assume $H_n^- = H_0 \cap \{-\infty < \tau < h_{2n}^0\}$ and

$$u_n(\theta) = \begin{cases} v_n(\theta), & \theta \in H_n^0, \\ 0, & \theta \in H_n^- \backslash H_n^0. \end{cases}$$

Since $v_n|_{H_{n1}} = 0$, the function u_n belongs to the class $C^\infty(H_n^-)$, vanishes for $\tau < h_{1n}$ and is a solution of the following mixed problem in the semi-infinite cylinder H_n^- with the Cauchy zero data for $\tau = -\infty$:

$$e^{-2\tau} L(\tau, \varphi, \partial) u_n = F_n,$$
$$u_n\big|_{\partial H_n^- \cap \partial H_0} = \nu_n.$$

Returning to the initial variables x_1, x_2, t when the inverse transformation $J^{-1} : (x_1, \tau, \varphi) \to (x_1, x_2, t)$ takes place and retaining the previous notation for the functions u_n and F_n, we find that:

(1) the function u_n in the domain $G_n = J^{-1}(H_n^-) \cap \{0 < t < t_0\}$ belongs to the class $C^\infty(\overline{G}_n)$ and satisfies the equation

$$\Box u_n = F_n;$$

by the construction of the domains Ω, H_0, and G_0, the domain G_n does not depend on the number n and therefore will be denoted below by G;

(2) the function u_n on the lateral part $\overset{4}{\underset{i=1}{\cup}} \widetilde{S}_i^0$ of the boundary ∂G satisfies the conditions

$$u_n\big|_{\widetilde{S}_3^0 \cup \widetilde{S}_4^0} = 0, \quad u_n\big|_{\widetilde{S}_i^0} = f_{in}, \quad i = 1, 2,$$

where, as one can easily verify, the surface $\widetilde{S}_i^0$ is a part of the surface S_i for $i = 1, 2$, and is a part of the surface $\widetilde{S}_i$, figuring in conditions (33), for $i = 3, 4$.

Thus by virtue of (32), (33), Lemma 3, and Remark 2, the function

$$u_n^0(P) = \begin{cases} u_n(P), & P \in G, \\ 0, & P \in D \backslash G \end{cases}$$

belongs to the class $C_*^\infty(\overline{D})$ and is the solution of problem (1), (2) for $f_i = f_{in}$, $i = 1, 2$, and $F = F_n$.

Due to inequality (3) we have

$$\|u_n^0 - u_m^0\|_{W_2^1(D)} \leq C\Big(\sum_{i=1}^{2} \|f_{in} - f_{im}\|_{W_2^1(S_i)} + \|F_n - F_m\|_{L_2(D)}\Big). \qquad (40)$$

From (29) and (40) we conclude that the sequence of functions u_n^0 is fundamental in the space $W_2^1(D)$. Therefore by virtue of the fact that the space $W_2^1(D)$ is complete there is a function $u \in W_2^1(D)$ such that $u_n^0 \to u$ in $W_2^1(D)$, $\Box u_n^0 \to F$ in the space $L_2(D)$, and $u_n^0|_{S_i} \to f_i$ in $W_2^1(S_i)$, $i = 1, 2$, for $n \to \infty$. Therefore the function u is the strong solution of problem (1), (2) of the class W_2^1. The uniqueness of the strong solution of problem (1), (2) belonging to the class W_2^1 follows from inequality (3). $\Box$

Let us now consider the case where equation (1) contains the lowest terms

$$L_0 u \equiv \Box u + a u_{x_1} + b u_{x_2} + c u_t + d u = F, \tag{41}$$

where the coefficients a, b, c, and d are the known bounded measurable functions in the domain D.

In the space $W_2^1(D)$ we introduce the equivalent norm which depends on the parameter γ

$$\|u\|^2_{D,1,\gamma} = \int\limits_D e^{-\gamma t}(u^2 + u_t^2 + u_{x_1}^2 + u_{x_2}^2)dx\,dt, \quad \gamma > 0.$$

Arguments similar to those used in [4] allow us to prove

Lemma 4. *For any $u \in W_2^2(D)$ the following a priori estimate holds:*

$$\|u\|_{D,1,\gamma} \le \frac{C}{\sqrt{\gamma}}\Big(\sum_{i=1}^{2} \|f_i\|_{S_i,1,\gamma} + \|F\|_{D,0,\gamma}\Big), \tag{42}$$

where $f_i = u|_{S_i}$, $F = \Box u$, and the positive constant C does not depend on u and the parameter γ.

By virtue of estimate (42) the lowest terms in equation (41) for the above-introduced equivalent norms of the spaces $L_2(D)$, $W_2^1(D)$, $W_2^1(S_i)$, $i = 1,2$, give arbitrarily small perturbations for a sufficiently large value of the parameter γ, which fact enables one to prove by Theorem 1 and the results from [4] that problem (41), (2) has a unique solution in the class W_2^1.

The following theorem is valid:

Theorem 2. *For any $f_i \in \overset{\circ}{W}{}_2^1(S_i, \Gamma)$, $i = 1,2$, $F \in L_2(D)$ there exists a unique strong solution u of problem* (41), (2) *in the class W_2^1, for which estimate* (3) *holds.*

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