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CRITERIA OF UNITARY EQUIVALENCE OF HERMITIAN OPERATORS WITH A DEGENERATE SPECTRUM

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ABSTRACT. Nonimprovable, in general, estimates of the number of necessary and sufficient conditions for two Hermitian operators to be unitarily equaivalent in a unitary space are obtained when the multiplicities of eigenvalues of operators can be more than 1. The explicit form of these conditions is given. In the Appendix the concept of conditionally functionally independent functions is given and the corresponding necessary and sufficient conditions are presented.

Let $\mathbf{P}$, $\mathbf{Q}$ be the operators from a unitary n-dimensional space $\mathbb{U}^n$ in $\mathbb{U}^n$, and P, Q be the matrices of these operators in some orthonormal basis. Description of a system of invariants of these matrices which enables one to find out whether the given operators are unitarily equivalent is the classical problem of the theory of invariants (see, e.g., [1, §2.2], [2] and the references cited therein). In the author's paper [3] it is shown that two matricess $P, Q \in M_n(\mathbb{C})$ are unitarily equivalent iff the following conditions are fulfilled:

$$\begin{gathered} \operatorname{tr}\left\{P_+^l P_- P_+^m P_-^2\right\} = \operatorname{tr}\left\{Q_+^l Q_- Q_+^m Q_-^2\right\}, \quad 0 \leq l \leq m \leq n-1, \\ \operatorname{tr}\left\{P_+^l\right\} = \operatorname{tr}\left\{Q_+^l\right\}, \quad 1 \leq l \leq n, \end{gathered} \tag{1}$$

where A_+ (A_-) denotes the Hermitian (skew-Hermitian) part of the matrix $A \in M_n(\mathbb{C})$:

$$A_\pm = (A \pm A^*)/2.$$

Formulas (1) contain $n(n+3)/2$ of complex (but only n^2+1 of real) conditions, and all these conditions are independent if no additional restrictions are imposed on the entries of the matrices $P_\pm$, $Q_\pm$. However, if such restrictions are imposed, in particular, if some eigenvalues of the operator $\mathbf{P}_+$ have multiplicity ≥ 2, then not all of conditions (1) are independent [3].

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There arises a problem of finding a minimal number of necessary and sufficient conditions of type (1) for the matrices $P, Q \in M_n(\mathbb{C})$ to be unitarily equivalent.

Let the operators $\mathbf{P}, \mathbf{Q} : \mathbb{U}^n \to \mathbb{U}^n$ be Hermitian and have eigenvalues whose multiplicities can be more than 1. In this paper we derive an estimate (not improvable in the general case) of the number of necessary and sufficient conditions for unitary equivalence of the Hermitian matrices $P, Q \in M_n(\mathbb{C})$ corresponding to these operators and state these conditions in explicit form. It is shown that the number of independent conditions $\mathfrak{n} \leq n$ and all cases are found when the equality $\mathfrak{n} = n$ is fulfilled.

Let n eigenvalues of the Hermitian operator $\mathbf{P} : \mathbb{U}^n \to \mathbb{U}^n$ form the multiset [4, §3.4]

$$\mathcal{P} = \{p_i^{r_i} | i = \overline{1, m}\}, \ \ \sum_{i=1}^{m} r_i = n, \ \ p_i \neq p_j \ \text{ for } \ i \neq j, \ \ r_i \in \mathbb{N}.$$

Let us denote

$$t_k(P) = \operatorname{tr} P^k = \sum_{i=1}^{m} r_i p_i^k, \quad k = 0, 1, 2, \dots, \tag{2}$$

and let

$$\begin{gathered} T_k(P) = [T_{ij}(P)]_{1 \leq i,j \leq k} = [t_{i+j-2}(P)] \in M_k(\mathbb{R}), \\ D_k(P) = \det T_k(P), \quad k \in \mathbb{N}, \end{gathered} \tag{3}$$

stand for Hankel matrices and their determinants. It is obvious that $D_1(P) = t_0(P) = n$.

Since all eigenvalues of the operator $\mathbf{P} : \mathbb{U}^n \to \mathbb{U}^n$ are real, the rank of matrix $T_k(P)$ for sufficiently large k is equal to the signature of this matrix and to the number m of various eigenvalues [5, Ch. 16, §9]:

$$\operatorname{rank} T_k(P) = m, \quad k \geq m. \tag{4}$$

Lemma. *The following formulas hold for the determinants* (3)*:*

$$D_k(P) = \\ = \sum_{1 \leq i_1 < \cdots < i_k \leq m} \Big\{ r_{i_1} \cdots r_{i_k} \prod_{1 \leq j < l \leq k} \big(p_{i_j} - p_{i_l}\big)^2 \Big\} > 0, \ \ 1 \leq k \leq m. \tag{5}$$

Proof. By (2) we have

$$D_k(P) = \det AB = \det \Big[\sum_{j=1}^{m} a_{ij}(P) b_{jl}(P) \Big], \tag{6}$$

where $A \in M_{k,m}$, $B \in M_{m,k}$ stand for

$$A = [a_{ij}(P)] = [r_j p_j^{i-1}], \;\; B = [b_{jl}(P)] = [p_j^{i-1}], \;\; 1 \le j \le m, \;\; 1 \le i,l \le k.$$

(As usual, $M_{k,l} = M_{k,l}(\mathbb{R})$ denotes the set of real matrices of dimension $k \times l$, $k,l \in \mathbb{N}$, and $M_k = M_{k,k}$.)

Applying the Cauchy–Binet formula to (6), we obtain (5). □

Corollary. *The power series (Newtonian sums) $t_k(P)$ for $k \ge 2m$ are rational sums of the variables $t_0(P), \dots, t_{2m-1}(P)$.*

Proof. It follows from (4) that

$$D\begin{pmatrix} 1, & \cdots, & m, & m+1 \\ 1, & \cdots, & m, & m+k \end{pmatrix} = 0, \quad k \in \mathbb{N},$$

for all minors surrounding the minor $D_m(P) > 0$, which implies

$$t_{2m+k-1}(P) D_m(P) =$$
$$= \sum_{i,j=1}^{m} t_{m+k+j-1}(P) t_{m+i-1}(P) A_m \binom{j}{i}(P), \quad k \in \mathbb{N}, \tag{7}$$

where $A_m\binom{j}{i}(P)$ is the algebraic complement of the element $t_{j+i-2}(P)$ in $D_m(P)$ depending evidently only on $t_k(P)$, $k = \overline{0, 2m-2}$. By the induction with respect to k we obtain the desired result from (8). □

Theorem 1. *The mapping*

$$\{t_k(P) | k = \overline{0, 2m-1}\} \mapsto \{p_i^{r_i} | i = \overline{1,m}\} \tag{8}$$

is bijective.

Proof. The injectivity of mapping (8) follows from (2). Let us prove the surjectivity. It is obvious that the Hermitian operator $\mathbf{P} : \mathbb{U}^n \to \mathbb{U}^n$ having $m \le n$ various eigenvalues $p_1, \dots, p_m$ satisfies the operator identity

$$\mathbf{P}^m - \sum_{k=1}^{m} c_k \mathbf{P}^{m-k} = \mathbf{0} \tag{9}$$

with $c_k = (-1)^k \sum_{1 \le i_1 < \cdots < i_k \le m} p_{i_1} p_{i_2} \cdots p_{i_k}$, $k = \overline{1,m}$.

By multiplying (9) by $\mathbf{P}^0, \dots, \mathbf{P}^{m-1}$ and calculating traces we obtain the following system of equations for coefficients c_k:

$$\sum_{k=1}^{m} c_k t_{m-k+i}(P) = t_{m+i}(P), \quad i = \overline{0, m-1}, \tag{10}$$

whose determinant is $(-1)^{m(m-1)/2} D_m(P) \neq 0$. After calculating the unknowns c_k, $k=\overline{1,m}$, from (10) we find $p_1, p_2, \dots, p_m$ ($p_i \neq p_j$ for $i \neq j$) as roots of the polynomial $z^m - \sum_{k=1}^m c_k z^{m-k} = 0$.

By substituting the found values of p_i in the first m equations of system (2) and recalling that the determinant of this system (with respect to unknowns r_i) is

$$\det \left[p_i^{k-1}\right]_1^m = \prod_{1\le k<j\le m} (p_j - p_k) \neq 0,$$

we obtain $r_1, r_2, \dots, r_m$. □

Note that to calculate the Jacobian of mapping (8) it is sufficient to know the determinant of the Vandermonde 2-multiple matrix [6]

$$J = \frac{\partial(t_0, t_1, \dots, t_{2m-1})}{\partial(p_1, r_1, \dots, p_m, r_m)} = \Big(\prod_{i=1}^m r_i\Big) \det(p_1, \dots, p_m; 2) =$$
$$= m! \Big(\prod_{i=1}^m r_i\Big) \prod_{1\le k<j\le m} (p_j - p_k)^{2^2} \neq 0.$$

Since all $r_i \geq 1$, we can define the diagonal matrix $R = \operatorname{diag}\{r_1^{-1}, \dots, r_m^{-1}\}$. Following (2) and (3), we use the notation

$$t_k(R) = \operatorname{tr} R^k, \quad D_k(R) = \det T_k(R), \quad k \in \mathbb{N} \ \ (t_0(R) = m). \tag{11}$$

Theorem 2. *The mapping*

$$\left\{ \operatorname{tr} P^k | k = \overline{0, 2m-1}\right\} \mapsto \left\{ \operatorname{tr} R^k | k \in \mathbb{N}\right\}$$

is injective and all $\operatorname{tr} R^k$, $k \in \mathbb{N}$, *are the rational functions of the arguments* $t_0(P), \dots, t_{2m-1}(P)$.

Proof. For the sake of brevity we denote $t_k = \operatorname{tr} P^k$, $T_k = T_k(P)$, $\widetilde{t}_k = \operatorname{tr} R^k$. From the first m equations of (2) we find

$$r_i = \frac{\det R^{(i)}}{\det[p_i^{k-1}]_1^m}\,, \quad i = \overline{1,m}, \tag{12}$$

where $R^{(i)} = [R_{kl}^{(i)}] \in M_m$ is the matrix whose kth row has the form

$$\left[p_1^{k-1}, \dots, p_{i-1}^{k-1}, t_{k-1}, p_{i+1}^{k-1}, \dots, p_m^{k-1}\right], \quad k = \overline{1,m}.$$

Due to (12) we obtain

$$r_i^2 = \frac{\det[r_l R_{kl}^{(i)}] \det[R_{kl}^{(i)}]}{\det[r_j p_j^{k-1}] \det[p_j^{k-1}]}, \quad i = \overline{1,m}.$$

If we cancel r_i in both sides and multiply the determinants in the right-hand side according to the "row by row" rule, we obtain

$$r_i = \frac{\det[t_{k+j-2} + t_{k-1}t_{j-1} - r_i p_i^{k+j-2}]_1^m}{\det[t_{k+j-2}]_1^m}, \quad i = \overline{1,m}. \tag{13}$$

To simplify (13) let us introduce one-column matrices $\mathbf{0} = [0] \in M_{m,1}$, $\mathbf{t} = [t_{k-1}] \in M_{m,1}$, $\mathbf{p}_i \in [p_i^{k-1}] \in M_{m,1}$ $(i = \overline{1,m})$. Use the following matrix identity for $(m+2)\times(m+2)$ block-matrices $(i = \overline{1,m})$:

$$\begin{bmatrix} T_m & \mathbf{t} & -r_i\mathbf{p}_i \\ -\mathbf{t}' & +1 & 0 \\ -\mathbf{p}_i' & 0 & +1 \end{bmatrix} \begin{bmatrix} E_m & \mathbf{0} & \mathbf{0} \\ \mathbf{t}' & +1 & 0 \\ \mathbf{p}_i' & 0 & +1 \end{bmatrix} = \begin{bmatrix} T_m + \mathbf{t}\mathbf{t}' - r_i\mathbf{p}_i\mathbf{p}_i' & \mathbf{t} & -r_i\mathbf{p}_i \\ \mathbf{0}' & +1 & 0 \\ \mathbf{0}' & 0 & +1 \end{bmatrix}$$

(here $E_m \in M_m$ is the identity matrix and $'$ denotes transposition), which implies

$$\det\left[T_m + \mathbf{t}\mathbf{t}' - r_i\mathbf{p}_i\mathbf{p}_i'\right] =$$
$$= \det\begin{bmatrix} T_m & \mathbf{t} & -r_i\mathbf{p}_i \\ -\mathbf{t}' & +1 & 0 \\ -\mathbf{p}_i' & 0 & +1 \end{bmatrix} = \det\begin{bmatrix} T_m & \mathbf{0} & -r_i\mathbf{p}_i \\ \mathbf{0}' & 1+t_0 & -r_i \\ -\mathbf{p}_i' & +1 & +1 \end{bmatrix}.$$

The latter matrix is obtained from the previous one by adding the first and the $(m+1)$th rows and then subtracting the first colum from the $(m+1)$th one. Expanding the result with respect to the elements of the $(m+1)$th row we get for each $i = \overline{1,m}$ that

$$\det\left[T_m + \mathbf{t}\mathbf{t}' - r_i\mathbf{p}_i\mathbf{p}_i'\right] = (1+t_0+r_i)D_m + (1+t_0)\det\begin{bmatrix} T_m & -r_i\mathbf{p}_i \\ -\mathbf{p}_i' & 0 \end{bmatrix}.$$

Substitution into (13) and simplification give

$$r_i \det\begin{bmatrix} T_m & \mathbf{p}_i \\ -\mathbf{p}_i' & 0 \end{bmatrix} = D_m, \quad i = \overline{1,m}.$$

Hence we obtain

$$r_i^{-1} = \sum_{j=0}^{2m-2} p_i^j c_j, \tag{14}$$

where the coefficients c_j, $j = \overline{0, 2m-2}$, are expressed rationally through $t_0, \ldots, t_{2m-1}$ as follows:

$$c_j = D_m^{-1} \sum_{l=0}^{j} A_m \binom{j-l+1}{l+1}(P), \quad j = \overline{0, 2m-2}. \tag{15}$$

By virtue of (2), (7), (11), and (15) we find from (14) that

$$\widetilde{t}_{k-1}=\sum_{i=1}^{m} r_i^{-k+1}=\sum_{j=0}^{2k(m-1)} t_j \sum_{j_1+\cdots+j_k=j} c_{j_1}\cdots c_{j_k}=f_k(t_0,\dots,t_{2m-1}),$$

where $f_k(t_0,\dots,t_{2m-1})$ is a rational function of its arguments, $k\in\mathbb{N}$. $\square$

Let the primary specification (see [7]) of multiset $\mathcal{P}$ be also the multiset

$$\{r_i|\, i=\overline{1,m}\}=\{q_i^{s_i}|\, i=\overline{1,l}\},\quad q_i\neq q_j \text{ for } i\neq j,\quad q_i,s_i\in\mathbb{N}, \tag{16}$$

where

$$\sum_{i=1}^{l} q_i s_i=n,\quad \sum_{i=1}^{l} s_i=m. \tag{17}$$

Denote by p_{ij}, $j=\overline{1,s_i}$, the eigenvalues of operator $\mathbf{P}:\mathbb{U}^n\to\mathbb{U}^n$ each having the multiplicity equal to q_i, $i=\overline{1,l}$. Without loss of generality, multiset $\mathcal{P}$ will be assumed to be ordered so that

$$p_{ij}<p_{i,j+1},\quad j=\overline{1,s_i},\quad q_i<q_{i+1},\quad i=\overline{1,l-1}. \tag{18}$$

In these notation we have

$$t_k(P)=\sum_{i=1}^{l}\Big(q_i\sum_{j=1}^{s_i} p_{ij}^k\Big),\quad t_k(R)=\sum_{i=1}^{l} s_i q_i^{-k},\quad k=0,1,2,\dots.$$

Following the lemma, the determinants in (11) satisfy the conditions $D_k(R)>0$ for $k\le l$ and $D_k(R)=0$ for $k\ge l+1$. Hence on account of Theorem 2 we obtain $m-l$ conditions satisfied by values $t_0(P),\dots,t_{2m-1}(P)$. Thus the set $\{t_k(P)|\, k=\overline{0,2m-1}\}$ contains at most $m+l-1=\mathfrak{n}(P)$ independent elements.

Remark. In terms of the partition theory formulas (17) imply that multiset (16) is the partitioning of the number n, which is a dimension of the space $\mathbb{U}^n$, and the rank of this partitioning is m. By the notation of [4] we have

$$\big(q_1^{s_1},\dots,q_l^{s_l}\big)\vdash n,\quad (s_1,\dots,s_l)\vdash m.$$

Following the Ramsay theorem (see [7]), $\mathfrak{n}(P)$ is the greatest number each of whose partitioning into l parts

$$(\mathfrak{n}_1,\dots,\mathfrak{n}_l)\vdash \mathfrak{n}(P)$$

contains at least one part having the property

$$\mathfrak{n}_i\le s_i,\quad 1\le i\le l.$$

Proposition. *For* $\mathfrak{n}(P)$ *we have the estimate*

$$\mathfrak{n}(P) = l + \sum_{i=1}^{l} s_i - 1 \le n,$$

the equality being fulfilled if either all eigenvalues of the operator $\mathbf{P} : \mathbb{U}^n \to \mathbb{U}^n$ *are simple:*

$$l = 1; \quad q_1 = 1, \quad s_1 = n; \quad (1^n) \vdash n,$$

or if one eigenvalue has multiplicity 2 *while the rest of the eigenvalues are simple:*

$$l = 2; \quad q_1 = 1, \quad s_1 = n-2; \quad q_2 = 2, \quad s_2 = 1; \quad (1^{n-2}, 2) \vdash n.$$

Proof. Formulas (17) imply

$$n = \sum_{i=1}^{l}(q_i - 1)(s_i - 1) + \sum_{i=1}^{l} s_i + \sum_{i=1}^{l}(q_i - 1) =$$
$$= \sum_{i=1}^{l}(q_i - 1)(s_i - 1) + \sum_{i=1}^{l} s_i + \sum_{i=1}^{l}(q_i - i) + l(l-1)/2.$$

Hence on account of the inequalities $s_i \ge 1$, $q_i \ge i$ we have the estimate

$$n \ge \sum_{i=1}^{l} s_i + l(l-1)/2,$$

in which the equality holds if $q_i = i$, $i = \overline{1,l}$, and $s_2 = \cdots = s_l = 1$. Note that $l(l-1)/2 \ge l - 1$ with equality for $l = 1, 2$. □

Example. Let the multiset of eigenvalues of the operator $\mathbf{P}$ have the form:

(a) $\{p_{1j}, p_i^{q_i} \,|\, j = \overline{1, s_1}, i = \overline{2, l}\}$, i.e., the operator $\mathbf{P}$ has s_1 simple eigenvalues; then (17) implies $l(l+1)/2 \le n - s_1 + 1$ and

$$\mathfrak{n}(P) = s_1 + 2l - 2, \quad s_1 \le \mathfrak{n}(P) \le \sqrt{9 + 8(n - s_1)} + s_1 - 3;$$

(b) $\{p_1, p_2^{n-1}\}$; then $l = 2$; $q_1 = 1$, $s_1 = 1$, $q_2 = n-1$, $s_2 = 1$; $\mathfrak{n}(P) = 3$;

(c) $\{p_1^n\}$; then $l = 1$; $s_1 = 1$, $q_1 = n$, $\mathfrak{n}(P) = 1$.

Let us construct an ordered set of invariants of an operator $\mathbf{P}$ (of a matrix P) of the form

$$I(P) = \Big\{ t_k(P) | \; k = \overline{1, \mathfrak{n}(P)}, \; \mathfrak{n}(P) = l + \sum_{i=1}^{l} s_i - 1 \Big\}. \tag{19}$$

Theorem 3. *For the Hermitian operator* $\mathbf{P} : \mathbb{U}^n \to \mathbb{U}^n$ *the set of invariants* (19) *is complete and all elements of this set are functionally independent as functions of the independent variables* (18).

Proof. We introduce the notation

$$\phi_{ij} = r_{i,j+1} - r_{i1}, \quad j = \overline{1, s_i - 1}, \quad i = \overline{1, l},$$

where r_{ij} is the multiplicity of the eigenvalues p_{ij}, $j = \overline{1, s_i}$, $i = \overline{1, l}$. By virtue of the theorem on conditional functional independence (see the Appendix) it is sufficient to show that the functions $\{t_k(P) | \, k = \overline{1, \mathfrak{n}(P)}\}$ are conditionally functionally independent in the presence of constraints

$$\phi_{ij} = 0, \quad j = \overline{1, s_i - 1}, \ \ i = \overline{1, l}, \ \ t_0(P) = \sum_{i=1}^{l} q_i s_i = n.$$

Calculate the Jacobian

$$\widetilde{J} = \frac{\partial(t_0, t_1, \ldots, t_{\mathfrak{n}(P)}, \phi_{11} \ldots, \phi_{ls_l})}{\partial(p_{11}, r_{11} \ldots, p_{ls_l}, r_{ls_l})} \,.$$

Taking (3) into account and performing some simple calculations, we find

$$\widetilde{J} = \Big(\prod_{i=1}^{l} q_i^{s_i} \Big) \det \left[\widetilde{u}_1 | \ldots | \widetilde{u}_l \right], \tag{20}$$

where

$$\widetilde{u}_i = \left[u_{kj}^{(i)} \right]_{\substack{0 \le k \le \mathfrak{n}(P) \\ 1 \le j \le s_i}} = \Big[\sum_{j=1}^{s_i} p_{ij}^k, (p_{i1}^k)', \ldots, (p_{is_i}^k)' \Big]_{0 \le k \le \mathfrak{n}(P)}, \ \ i = \overline{1, l}.$$

Here $(p_{ij}^k)' = k p_{ij}^{k-1} = (\partial / \partial p_{ij}) p_{ij}^k$.

Applying induction with respect to l, let us show that the determinant in the right-hand side of (20) is not identically zero. Indeed, for $l = 1$ we have

$$\begin{aligned} \det[\widetilde{u}_1] &= \det \Big[\sum_{j=1}^{s_1} p_{1j}^k, (p_{11}^k)', \ldots, (p_{1s_1}^k)' \Big]_{0 \le k \le s_1} = \\ &= s_1 (s_1)! \prod_{1 \le l < j \le s_1} (p_{1j} - p_{1l}) \ne 0. \end{aligned}$$

Assume that $l \geq 2$ and $\det[\widetilde{u}_1|\cdots|\widetilde{u}_{l-1}] \neq 0$. After expanding $\det[\widetilde{u}_1|\cdots|\widetilde{u}_l]$ with respect to the last s_l+1 rows we obtain

$$\det\left[\widetilde{u}_1|\cdots|\widetilde{u}_l\right] = \det\left[\widetilde{u}_1|\cdots|\widetilde{u}_{l-1}\right] \times$$
$$\times \det\Big[\sum_{j=1}^{s_l} p_{lj}^{k-1}, (p_{l1}^{k-1})', \dots, (p_{ls_l}^{k-1})'\Big]_{\mathfrak{n}(P)-s_l\leq k\leq \mathfrak{n}(P)} + \cdots, \qquad (21)$$

where the points denote the terms of lower powers with respect to the variables $\{p_{lj}\,|\, j=\overline{1,s_l}\}$. Treating $\det[\sum_{j=1}^{s_l} p_{lj}^{k-1}, p_{l1}^{k-1'},\dots,p_{ls_l}^{k-1'}] = P(p_{l1})$ as the polynomial of the variable p_{l1}, we obtain

$$\begin{aligned} P(p_{l1}) &= \det\left[p_{l1}^{k-1}, (p_{l1}^{k-1})', \dots, (p_{ls_l}^{k-1})'\right]_{\mathfrak{n}(P)-s_l\leq k\leq \mathfrak{n}(P)} + \cdots = \\ &= p_{l1}^{2\mathfrak{n}(P)-4} \det\left[(p_{l2}^{k-1})', \dots, (p_{ls_l}^{k-1})'\right]_{\mathfrak{n}(P)-s_l\leq k\leq \mathfrak{n}(P)-2} + \cdots = \\ &= p_{l1}^{2\mathfrak{n}(P)-4} \frac{(\mathfrak{n}(P)-3)!}{(\mathfrak{n}(P)-s_l-2)!} \prod_{j=2}^{s_l} p_{lj}^{\mathfrak{n}(P)-s_l-2} \prod_{2\leq t<j\leq s_l} (p_{lj}-p_{lt}) + \cdots . \end{aligned}$$

After substituting this result into (21) and taking into account the assumption of induction we find that $\det[\widetilde{u}_1|\cdots|\widetilde{u}_l]$ is the polynomial of p_{l1} of the power

$$2\mathfrak{n}(P) - 4 = 2\Big(l + \sum_{i=1}^{l} s_i - 1\Big) - 4 \geq 4l - 6 \geq 2$$

with a higher coefficient which is not identically zero. Therefore $\widetilde{J} \neq 0$. □

Corollary. *Two given Hermitian matrices $P, Q \in M_n(\mathbb{C})$ are unitarily equivalent iff $\mathfrak{n}(P)$ real equalities*

$$I(P) = I(Q) \qquad (22)$$

are fulfilled, where $I(P)$ is determined by (19).

Proof. The necessity of conditions (22) is obvious. The sufficiency follows from the fact that the mapping

$$I(P) \mapsto \{p_{ij}, q_i |\, j=\overline{1,s_i},\ i=\overline{1,l}\}$$

is injective by virtue of Theorem 3. □

Example.

(a) The Hermitian matrix $P \in M_n(\mathbb{C})$ is proportional with a coefficient $a \in \mathbb{R}$ to E_n iff $D_2(P) = t_0t_2 - t_1^2 = 0$, or, which is the same,

$$\operatorname{tr} P^2 = (\operatorname{tr} P)^2/n. \qquad (23)$$

It is obvious that $P = n^{-1}(\operatorname{tr} P)E_n$.

(b) The Hermitian matrix $P \in M_n(\mathbb{C})$ is equal to zero iff condition (23) is fulfilled and $\operatorname{tr} P = 0$.

Remark. If some additional restrictions are imposed on elements of the Hermitian matrix P, i.e., if not all of these elements are independent (e.g., for $\operatorname{tr} P = 0$), then the complete system of invariants of the operator $\mathbf{P} : \mathbb{U}^n \to \mathbb{U}^n$ contains less than $\mathfrak{n}(P)$ independent elements.

Appendix

Assume that $n, m \in \mathbb{N}$, we are given $n+m$ differentiable functions of $n+m$ variables $x_1, \dots, x_n, y_1 \dots, y_m$

$$f_i : \mathbb{R}^{n+m} \to \mathbb{R}, \quad i = \overline{1, n+m}, \tag{A.1}$$

such that the Jacobian J_1 differs from zero:

$$J_1 = \frac{\partial(f_1, \dots, f_{n+m})}{\partial(x_1, \dots, x_n, y_1, \dots, y_m)} \neq 0$$

and the variables $x_1, \dots, x_n, y_1, \dots, y_m$ satisfy m constraints

$$\phi_j(x_1, \dots, x_n, y_1, \dots, y_m) = 0, \quad j = \overline{1, m}, \tag{A.2}$$

where each of functions $\phi_j : \mathbb{R}^{n+m} \to \mathbb{R}$, $j = \overline{1, m}$, is differentiable with respect to $n+m$ arguments $x_1, \dots, x_n, y_1, \dots, y_m$ and

$$J_2 = \frac{\partial(\phi_1, \dots, \phi_m)}{\partial(y_1, \dots, y_m)} \neq 0.$$

As known, constraint equations (A.2) determine explicit functions

$$y_j = y_j(x_1, \dots, x_n), \quad j = \overline{1, m},$$

and functions (A.1) become composite functions of the variables $x_1, \dots, x_n$:

$$f_i = f_i\{(x_1, \dots, x_n, y_1(x_1, \dots, x_n), \dots, y_m(x_1, \dots, x_n)\}, \quad i = \overline{1, n+m}. \tag{A.3}$$

Definition. The functions in (A.1) will be called conditionally functionally dependent (independent) in the presence of constraints (A.2) provided that the corresponding composite functions (A.3) are functionally dependent (independent), i.e., if the Jacobian $\det[f_{i,k}]$ is equal to (different from) zero, where

$$f_{i,k} = \partial_{x_k} f_i + \sum_{l=1}^{m} \left(\partial_{y_l} f_i\right)\left(\partial_{x_k} y_l\right), \quad i, k = \overline{1, n}, \tag{A.4}$$

and $\partial_{x_k} y_l$ are uniquely determined by the system of equations

$$0 = \partial_{x_k}\phi_j + \sum_{l=1}^{m} \left(\partial_{y_l}\phi_j\right)\left(\partial_{x_k} y_l\right), \quad k = \overline{1,n}, \;\; j = \overline{1,m}. \qquad (A.5)$$

Theorem. *In the presence of constraints* $(A.2)$, *only those* n *functions from the set* $\{f_i | i = \overline{1, n+m}\}$ *will be conditionally functionally dependent for which the Jacobian*

$$J = \frac{\partial(f_1, \dots, f_n, \phi_1, \dots, \phi_m)}{\partial(x_1, \dots, x_n, y_1, \dots, y_m)} = \det \begin{bmatrix} \partial_{x_k} f_i & | & \partial_{y_l} f_i \\ -- & | & -- \\ \partial_{x_k}\phi_i & | & \partial_{y_l}\phi_j \end{bmatrix}$$

is equal to zero.

For $J \neq 0$ *the functions* $f_1, \dots f_n$ *are conditionally functionally independent.*

Proof. For the sake of brevity we denote the Jacobi matrices by

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \in M_{n+m},$$

$$\Phi = \left[\Phi_1 \,|\, \Phi_2\right] \in M_{m,n+m}, \quad Y = \left[\partial_{x_k} y_l\right] \in M_{n,m},$$

where

$$\begin{aligned} &F_{11} = \left[\partial_{x_k} f_i\right] \in M_n, \quad F_{12} = \left[\partial_{y_l} f_i\right] \in M_{n,m}, \quad i = \overline{1,n}, \\ &F_{21} = \left[\partial_{x_k} f_i\right] \in M_{m,n}, \quad F_{22} = \left[\partial_{y_l} f_i\right] \in M_m, \quad i = \overline{n+1, n+m}, \\ &\Phi_1 = \left[\partial_{x_k}\phi_j\right] \in M_{m,n}, \quad \Phi_2 = \left[\partial_{y_l}\phi_j\right] \in M_m, \quad j = \overline{1,m} \\ &\qquad\qquad\qquad (k = \overline{1,n}, \quad l = \overline{1,m}). \end{aligned}$$

In this notation formulas (A.4) and (A.5) have the form

$$[f_{i,k}] = F_{11} + F_{12}Y, \quad -\Phi_1 = \Phi_2 Y$$

and

$$J_1 = \det F \neq 0, \quad J_2 = \det \Phi_2 \neq 0.$$

Hence

$$\det\left[f_{i,k}\right] = \det\left[F_{11} - F_{12}\Phi_2^{-1}\Phi_1\right].$$

Applying now the known identity for a block matrix determinant (see, e.g., [5, Ch.2, §5]) we obtain

$$\det\left[f_{i,k}\right] = J_2^{-1} \det \begin{bmatrix} F_{11} & F_{12} \\ \Phi_1 & \Phi_2 \end{bmatrix} = J/J_2.$$

Thus the condition $\det[f_{i,k}] = 0$ is equivalent to the condition $J = 0$. $\square$

Corollary. *If* $\phi_j = f_{n+j}$, $j = \overline{1,m}$, *and* $J_1 \neq 0$ *then*

$$\det\left[f_{i,k}\right] = J_1/J_2 \neq 0.$$

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