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SOME PARTITIONS CONSISTING OF JORDAN CURVES

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Abstract. Some decompositions of the three-dimensional sphere and three-dimensional ball into Jordan curves are considered. In particular, it is proved that for every strictly positive real number $p \leq 1$ there exists a partition of the unit three-dimensional Euclidean sphere into circles whose radii are equal to p.

Let S^2 be the unit two-dimensional sphere in the Euclidean space $\mathbb{R}^3$ and let S^3 be the unit three-dimensional sphere in the Euclidean space $\mathbb{R}^4$. The well-known Hopf fibration $\varphi : S^3 \to S^2$ can be expressed analytically as

$$\varphi(x_1, x_2, x_3, x_4) = (z_1, z_2, z_3),$$

where we have

$$\begin{aligned} z_1 &= 2(x_1x_3 + x_2x_4), \\ z_2 &= 2(x_2x_3 - x_1x_4), \\ z_3 &= x_1^2 + x_2^2 - x_3^2 - x_4^2. \end{aligned}$$

Moreover, if (z_1, z_2, z_3) is any point of the sphere S^2 and a point (x_1, x_2, x_3, x_4) belongs to the set $\varphi^{-1}((z_1, z_2, z_3))$, then $\varphi^{-1}((z_1, z_2, z_3))$ can be expressed as the set of all points $(y_1, y_2, y_3, y_4) \in S^3$ which satisfy the equalities

$$\begin{aligned} y_1 &= x_1 \cos\theta - x_2 \sin\theta, \\ y_2 &= x_1 \sin\theta + x_2 \cos\theta, \\ y_3 &= x_3 \cos\theta - x_4 \sin\theta, \\ y_4 &= x_3 \sin\theta + x_4 \cos\theta. \end{aligned} \qquad (0 \leq \theta \leq 2\pi)$$

From these equalities one can see that for any point $(z_1, z_2, z_3) \in S^2$ the preimage $\varphi^{-1}((z_1, z_2, z_3))$ is a circle of radius 1, lying on the plane of $\mathbb{R}^4$

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generated by the following two vectors:

$$(x_1, x_2, x_3, x_4), \qquad (-x_2, x_1, -x_4, x_3).$$

In other words, the Hopf fibration gives us a set-theoretic partition of the unit sphere S^3 into circles of radii 1 and it is well known that this partition has a number of interesting geometrical and topological properties.

Now, let p be a real number such that $0 < p < 1$. A natural question arises: does there exist a partition (in the set-theoretic sense) of the sphere S^3 into circles of radii p? It turns out that such a partition always exists. In order to prove this fact (and a more general result) we need the following auxiliary assertion.

Lemma 1. *Let c be the cardinality of the continuum and let $(L_j)_{j\in\mathcal{J}}$ be a family of circles in the space $\mathbb{R}^4$ such that $\mathrm{card}(\mathcal{J}) < c$. Let x be an arbitrary point of the sphere S^3 not belonging to the set $\bigcup_{j\in\mathcal{J}} L_j$. Then for every strictly positive real number $p < 1$ there exists a circle L satisfying the conditions:*

(1) *$L \subset S^3$,*
(2) *$x \in L$,*
(3) *the radius of L is equal to p,*
(4) *L does not intersect the set $\bigcup_{j\in\mathcal{J}} L_j$.*

Proof. For any index $j \in \mathcal{J}$ let P_j be the affine plane in the space $\mathbb{R}^4$ containing the circle L_j. Since we have the inequality $\mathrm{card}(\mathcal{J}) < c$, there exists a hyperplane Γ in the space $\mathbb{R}^4$ passing through the origin and such that

$$x \in \Gamma, \quad (\forall j \in \mathcal{J})(\dim(P_j \cap \Gamma) \le 1).$$

Let us consider the set $\Gamma \cap S^3$. Obviously, this set is the unit two-dimensional sphere in the three-dimensional space Γ. Moreover, we have

$$(\forall j \in \mathcal{J})(\mathrm{card}(\Gamma \cap S^3 \cap L_j) \le 2).$$

Let us put

$$Z = \Gamma \cap S^3 \cap \Big(\bigcup_{j\in\mathcal{J}} L_j\Big).$$

Then we have

$$\mathrm{card}(Z) < c, \qquad x \notin Z.$$

Now, it is clear that there exists a circle $L \subset \Gamma \cap S^3$ such that its radius is equal to p and

$$x \in L, \qquad L \cap Z = \varnothing$$

Obviously, the circle L is a required one. $\square$

With the help of Lemma 1 we can obtain the following theorem.

Theorem 1. *Let I be a set of cardinality c and let $(p_i)_{i\in I}$ be a family of real numbers such that*

$$(\forall i \in I)(0 < p_i < 1).$$

Then there exists a partition $(L_i)_{i\in I}$ of the unit sphere S^3 satisfying the conditions:

(1) $(\forall i \in I)$ *(L_i is a circle),*

(2) $(\forall i \in I)$ *(the radius of L_i is equal to p_i).*

Proof. Let α be the least ordinal number of cardinality c. Without loss of generality we may assume that

$$I = \{\xi : \xi < \alpha\}.$$

Let $(x_\xi)_{\xi<\alpha}$ be an injective family of all points of the unit sphere S^3. Let us construct by the transfinite recursion a family $(L_\xi)_{\xi<\alpha}$ of circles satisfying the following relations:

(a) for any $\zeta < \xi < \alpha$ we have $L_\zeta \cap L_\xi = \varnothing$,

(b) for each $\xi < \alpha$ the radius of the circle L_ξ is equal to p_ξ,

(c) for each $\xi < \alpha$ the point x_ξ belongs to the set $\bigcup_{\zeta\le\xi} L_\zeta$.

Suppose that for an ordinal number $\xi < \alpha$ a partial family of circles $(L_\zeta)_{\zeta<\xi}$ has already been defined. Of course, we have

$$S^3 \setminus \bigcup_{\zeta<\xi} L_\zeta \neq \varnothing.$$

Let y be an arbitrary point from the set $S^3 \setminus \bigcup_{\zeta<\xi} L_\zeta$. Let us put

$$x = \begin{cases} x_\xi & \text{if } \ x_\xi \notin \bigcup_{\zeta<\xi} L_\zeta, \\ y & \text{if } \ x_\xi \in \bigcup_{\zeta<\xi} L_\zeta. \end{cases}$$

Then we can apply Lemma 1 to the point x and to the family of circles $(L_\zeta)_{\zeta<\xi}$. By this lemma there exists a circle $L \subset S^3$ such that its radius is equal to p_ξ and

$$x \in L, \qquad L \cap \Big(\bigcup_{\zeta<\xi} L_\zeta\Big) = \varnothing.$$

Let us put $L_\xi = L$. It is clear that in this way we shall be able to construct the required partition $(L_\xi)_{\xi<\alpha}$ of the unit sphere S^3. $\square$

Remark 1. If for every $i \in I$ we have $p_i = p$, then we obtain a partition of the sphere S^3 into pairwise congruent circles (all of radius p). It is interesting to investigate for which real numbers p there exists an effective partition of the sphere S^3 into circles having radii p ("effective" means here that the axiom of choice will not be used). The example of the Hopf fibration φ shows us that for $p = 1$ an effective partition of S^3 into circles with radii p

does exist. It is worth observing here that by the method described above we can create many (non-effective) partitions of the sphere S^3 into circles with radii 1. Thus we see that there exists a partition of S^3 into circles with radii 1, essentially different from the Hopf partition. Of course, these partitions do not have some good geometrical or topological properties as the Hopf partition. Notice also that there exists a curve L satisfying the following three relations:

(1) L is homeomorphic to S^1;
(2) L is contained in S^3;
(3) there does not exist a partition of S^3 into curves congruent to L.

In connection with the mentioned fact see [1], where an analogous proposition is established for the three-dimensional Euclidean space $\mathbb{R}^3$.

Moreover, in [1] the following problem is posed: give a characterization of all Jordan curves L lying in $\mathbb{R}^3$ such that there exists a partition of $\mathbb{R}^3$ into curves congruent to L. Clearly, a similar problem can be formulated for the three-dimensional sphere S^3.

These two problems remain open.

Remark 2. Let I be an arbitrary set of cardinality c and let $(p_i)_{i\in I}$ be an arbitrary family of strictly positive real numbers. By the method described above one can prove the existence of a partition $(L_i)_{i\in I}$ of the space $\mathbb{R}^3$ into circles such that

$$(\forall i \in I)(L_i \text{ has the radius } p_i).$$

In particular, we see that there exists a partition of the space $\mathbb{R}^3$ into pairwise congruent circles. A similar result is true for the unit closed three-dimensional ball $B_3 \subset \mathbb{R}^3$ (see [1]).

For our further consideration we need the following

Lemma 2. *Let S^2 be the unit sphere in the Euclidean space $\mathbb{R}^3$ and let B_3 be the closed unit ball in the same space. Let $(L_j)_{j\in\mathcal{J}}$ be a family of curves in $\mathbb{R}^3$ satisfying the conditions:*

(1) $\mathrm{card}(\mathcal{J}) < c$,
(2) $(\forall j \in \mathcal{J})$ *(L_j is an arc of a circle).*

Then for any two distinct points $x \in S^2$ and $y \in S^2$ there exists a curve L such that

(a) *L is an arc of a circle,*
(b) *the set of end-points of L is $\{x, y\}$,*
(c) $L \subset B_3$,
(d) $L \cap S^2 = \{x, y\}$,
(e) $(\forall j \in \mathcal{J})$ $(L_j \cap L \subseteq \{x, y\})$.

Proof. The reasoning is quite similar to the arguments used in the proof of Lemma 1. For any index $j \in \mathcal{J}$ let P_j be the plane in the space $\mathbb{R}^3$ containing the arc L_j. Since we have the inequality

$$\mathrm{card}(\mathcal{J}) < c,$$

there exists a plane Γ in the space $\mathbb{R}^3$ such that

$$x \in \Gamma, \quad y \in \Gamma, \qquad (\forall j \in \mathcal{J})(\dim(P_j \cap \Gamma) \leq 1).$$

Hence we obtain

$$(\forall j \in \mathcal{J})(\mathrm{card}(L_j \cap \Gamma) \leq 2).$$

Let us put

$$Z = \Gamma \cap \Big(\bigcup_{j \in \mathcal{J}} L_j \Big).$$

Obviously, we have

$$\mathrm{card}(Z) < c.$$

Now, it is clear that in $\Gamma \cap B_3$ there exists an arc of the circle L such that x and y are end-points of L and

$$L \cap S^2 = \{x, y\},$$
$$L \cap Z \subseteq \{x, y\}.$$

This arc L is a required one. $\square$

Theorem 2. *Let S^2 be the unit two-dimensional sphere in the space $\mathbb{R}^3$ and let B_3 be the closed unit ball in the same space. Then there exists a family $(L_i)_{i \in I}$ of curves satisfying the following conditions:*

(1) $(\forall i \in I)$*(L_i is an arc of a circle),*

(2) $(\forall i \in I)$ $(L_i \subset B_3)$*,*

(3) $(\forall i \in I)$ *($L_i \cap S^2$ is the set of end-points of L_i),*

(4) $(\forall i \in I)(\forall i' \in I)(i \neq i' \Rightarrow L_i \cap L_{i'} \subset S^2)$,

(5) *for any two distinct points $x \in S^2$ and $y \in S^2$ there exists an arc L_i such that $\{x, y\}$ is the set of end-points of L_i.*

Proof. Let α be the least ordinal number of cardinality c. Let $(\{x_\xi, y_\xi\})_{\xi < \alpha}$ be an injective family of all two-point subsets of the sphere S^2. Let us construct the required family of arcs $(L_\xi)_{\xi < \alpha}$ by the method of transfinite recursion. Suppose that for an ordinal number $\xi < \alpha$ a partial family $(L_\zeta)_{\zeta < \xi}$ has already been defined. For any $\zeta < \xi$ let X_ζ be the set of end-points of the arc L_ζ. Consider the family $(X_\zeta)_{\zeta < \xi}$. Let η be the least ordinal number such that

$$\{x_\eta, y_\eta\} \notin (X_\zeta)_{\zeta < \xi}.$$

We can apply Lemma 2 to the points x_η, y_η and to the family of arcs $(L_\zeta)_{\zeta<\xi}$. By this lemma there exists an arc of the circle L_ξ satisfying relations analogous to relations (a)–(e) of the lemma. Hence, in this way we shall be able to construct the family of arcs $(L_\xi)_{\xi<\alpha}$. Now, if we put

$$I = \{\xi : \xi < \alpha\},$$

then it is easy to check that the family $(L_i)_{i\in I}$ satisfies conditions (1)–(5) of Theorem 2. $\square$

Remark 3. Slightly changing the proof of Theorem 2 presented above, we can find a family of arcs $(L_i)_{i\in I}$ satisfying conditions (1)–(5) of this theorem and also satisfying the following condition:

(6) for each point $x \in B_3$ there exists an arc L_i such that $x \in L_i$.

It is reasonable to notice here that we can also prove a result analogous to Theorem 2 for a family $(L_i)_{i\in I}$ consisting of polygonal lines. Moreover, we may even assume that any polygonal line L_i $(i \in I)$ consists at most of two segments.

A natural question arises: is it possible to prove Theorem 2 effectively (i.e., without using the axiom of choice)?

Finally, notice that the formulations of both Theorems 1 and 2 are, in fact, due to elementary geometry of the Euclidean space, but the proof of these theorems requires essentially non-elementary methods.

References

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