GEORGIAN MATHEMATICAL JOURNAL: Vol. 3, No. 3, 1996, 239-262

TWO-DIMENSIONAL STEADY-STATE OSCILLATION PROBLEMS OF ANISOTROPIC ELASTICITY

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Abstract. The paper deals with the two-dimensional exterior boundary value problems of the steady-state oscillation theory for anisotropic elastic bodies. By means of the limiting absorption principle the fundamental matrix of the oscillation equations is constructed and the generalized radiation conditions of Sommerfeld–Kupradze type are established. Uniqueness theorems of the basic and mixed type boundary value problems are proved.

1. Introduction

In the paper we treat the uniqueness theorems of basic and mixed type exterior boundary value problems (BVPs) for equations of two-dimensional steady-state elastic oscillations of anisotropic bodies. In this case questions regarding the correctness of BVPs have not been investigated so far. Here any analogy with the isotropic case is completely violated, since the geometry of the characteristic surface becomes highly complicated and the fundamental matrix cannot be written explicitly in terms of elementary functions. This in turn creates another difficulty in obtaining asymptotic estimates.

As is well known, even for the metaharmonic equation

$$\Delta v(x) + k^2 v(x) = 0, \quad k^2 > 0, \quad x \in \mathbb{R}^2, \tag{1.1}$$

the decay condition at infinity $v(x) = O(|x|^{-1/2})$ as $|x| \to +\infty$ is not sufficient to establish the uniqueness of solutions. The strengthening of the condition by increasing its decay order results in such a solution being a zero function and therefore the nonhomogeneous BVPs become unsolvable. To

1991 *Mathematics Subject Classification.* 35E05, 73C15.

Key words and phrases. Oscillation theory, theory of elasticity, uniqueness theorems, anisotropic medium.

1072-947X/96/0500-0239$09.50/0

obtain the uniqueness, some additional restrictions have to be imposed on solutions at infinity. They were found by A. Sommerfeld [1] for Eq. (1.1):

$$v(x) = o(1), \quad \frac{\partial v(x)}{\partial |x|} \pm ik\, v(x) = o(|x|^{-1/2}). \tag{1.2}$$

Sommerfeld's principle was applied to derive rigorous proofs of the uniqueness and existence theorems of solutions of the exterior BVPs for metaharmonic and n-metaharmonic equations [2–8].

The exterior BVPs of steady-state oscillations for isotropic elastic bodies were studied by V. Kupradze [9,10]. The uniqueness theorems for the equation

$$\mu\Delta u(x) + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\, u(x) + \rho\omega^2 u(x) = 0, \quad \omega > 0, \quad x \in \mathbb{R}^2, \tag{1.3}$$

were proved in the class of vectors $u = (u_1, u_2)^\top$ representable as the sum of two metaharmonic vectors u^1 and u^2 satisfying the radiation conditions (1.2) with the parameters $k_1^2 = \rho\omega^2(\lambda+2\mu)^{-1}$ and $k_2^2 = \rho\omega^2\mu^{-1}$.

Classes of functions and vectors, in which the exterior BVPs for Eqs. (1.1) and (1.3) have unique solutions, can be selected by other currently available methods such as Ignatovsky's principle of limiting absorption, Tikhonov's principle of limiting amplitude, and Mandelstam's energy principle [11–14].

Quite a number of authors have dealt with similar questions for general scalar elliptic and hypoelliptic equations (with constant and variable coefficients). Applying the limiting absorption principle, D. Eidus [15] studied a second-order elliptic equation whose principal part coincides with the Laplacian at infinity. B. Vainberg [16, 17] studied general higher-order hypoelliptic operators of two variables and obtained a wide set of conditions at infinity ensuring the uniqueness of solutions for equations over the whole plane. The results were further extended to scalar equations with many independent variables [18–22]. Coercive BVPs for elliptic equations in infinite domains with a compact boundary and the limiting absorption and limiting amplitude principles were studied in [23, 24] by means of the functional methods. Some particular results concerning the uniqueness of solutions to Dirichlet and Neumann type exterior BVPs of anisotropic elasticity are obtained in [25, 26].

In the present paper, the limiting absorption principle is used to construct the fundamental matrix of oscillation equations in the most general anisotropic case and to establish generalized radiation conditions of the Sommerfeld–Kupradze type. Special functional classes are introduced, in which the basic and mixed exterior BVPs of steady-state oscillations have unique solutions for an arbitrary value of the oscillation parameter.

2. Preliminaries

Denote by $\mathbb{R}^2$ the two-dimensional Euclidean space. Let $\Omega^+ \subset \mathbb{R}^2$ be a compact domain with a smooth boundary $\Omega^+ = S$ and $\Omega^- = \mathbb{R}^2 \backslash \overline{\Omega}^+$; $\overline{\Omega}^+ = \Omega^+ \cup S$. Nonhomogeneous equations of steady-state oscillations of the elasticity theory for anisotropic bodies can be written in the matrix form [11]

$$C(D,\omega)u(x) \equiv C(D)u(x) + \omega^2 u(x) = F(x), \quad x \in \omega, \tag{2.1}$$

where $u = (u_1,\ u_2)^\top$ is a complex displacement vector, $\omega > 0$ is an oscillation parameter (frequency), F describes external (bulk) forces, Ω stands for either $\Omega^\pm$ or $\mathbb{R}^2$,

$$\begin{aligned} C(\xi) &= \| C_{kp}(\xi) \|_{2\times 2} = \| c_{kjpq}\xi_j\xi_q \|_{2\times 2}, \\ C_{11}(\xi) &\equiv L(\xi) = A_{11}\xi_1^2 + 2A_{13}\xi_1\xi_2 + A_{33}\xi_2^2, \\ C_{12}(\xi) &= C_{21}(\xi) \equiv M(\xi) = A_{13}\xi_1^2 + (A_{12}+A_{33})\xi_1\xi_2 + A_{23}\xi_2^2, \\ C_{22}(\xi) &\equiv N(\xi) = A_{33}\xi_1^2 + 2A_{23}\xi_1\xi_2 + A_{22}\xi_2^2, \end{aligned} \tag{2.2}$$

$D = \nabla = (D_1, D_2)$, $D_j = \partial/\partial x_j$, $A_{11} = c_{1111}$, $A_{12} = A_{21} = c_{1122}$, $A_{13} = A_{31} = c_{1112}$, $A_{22} = c_{2222}$, $A_{23} = A_{32} = c_{2122}$ and $A_{33} = c_{1212}$ are the elastic constants, which from the physical standpoint satisfy the usual symmetry assumption $c_{kjpq} = c_{pqkj} = c_{jkpq}$; here and in what follows the superscript $\top$ denotes transposition, while summation over repeated indices is meant from 1 to 2 (unless stated otherwise). Without loss of generality, the density of the elastic medium under consideration can be assumed to be equal to unity.

The components of the strain tensor $e_{kj} = e_{jk} = 2^{-1}(D_k u_j + D_j u_k)$ and the stress tensor τ_{kj} are connected by means of the Hooke's law

$$\tau_{kj} = c_{kjpq}e_{pq}, \quad k,j = 1,2. \tag{2.3}$$

Let $\tau = (\tau_{11}, \tau_{22}, \tau_{12})^\top$ and $e = (e_{11}, e_{22}, 2e_{12})^\top$. Then Eq. (2.3) reduces to $\tau = Ae$ with $A = \|A_{kj}\|_{3\times 3}$.

The potential energy density for the real strain components is calculated by the formula

$$2W = \tau_{kj}e_{kj} = Ae \cdot e \tag{2.4}$$

which from the mechanical viewpoint is assumed to be a positive-definite quadratic form in the variables $e_{kj} = e_{jk}$. This implies that A is a positive-definite matrix, i.e., $A_{11} > 0,\ A_{11}A_{22} - A_{12}^2 > 0,\ \det A > 0$. Obviously, form (2.4) is positive definite for the complex variables $e_{kj} = e_{jk}$ as well (for two complex vectors u and v, the scalar product $u \cdot v$ is defined as $u_k\bar{v}_k$).

The above results give rise to [27]

$$\forall \xi \in \mathbb{R}^2 \ : \det C(\xi) \geq \delta_0 |\xi|^4, \quad L(\xi) \geq \delta_0 |\xi|^2, \quad N(\xi) \geq \delta_0 |\xi|^2,$$
$$C(\xi)\zeta \cdot \zeta = C_{kj}(\xi)\zeta_k \bar{\zeta}_j \geq \delta_0 |\xi|^2 |\zeta|^2,$$

where δ_0 is some positive number depending on the elastic moduli and $\zeta = (\zeta_1, \zeta_2)^\top$ is an arbitrary complex vector.

The operator

$$T(D,n) = \|T_{kp}(D,n)\|_{2\times 2}, \quad T_{kp}(D,n) = c_{kjpq} n_j D_q, \tag{2.5}$$

is called the stress operator and the vector Tu, with components $\tau_{kj} n_j = [T(D,n)u]_k = c_{kjpq} n_j D_q u_p$, is called the stress vector acting on a part of the line with the unit normal $n = (n_1, n_2)^\top$. From (2.2) and (2.5) we obtain

$$T_{11}(\xi,\eta) = T_{11}(\eta,\xi) = \tfrac{1}{2}[\eta_1 D_1 L(\xi) + \eta_2 D_2 L(\xi)],$$
$$T_{21}(\xi,\eta) + T_{12}(\xi,\eta) = \eta_1 D_1 M(\xi) + \eta_2 D_2 M(\xi),$$
$$T_{22}(\xi,\eta) = T_{22}(\eta,\xi) = \tfrac{1}{2}[\eta_1 D_1 N(\xi) + \eta_2 D_2 N(\xi)].$$

3. Fundamental Matrices

Let us consider the matrix differential equation in the space of slowly increasing generalized functions (tempered distributions) [28]

$$[C(D) + \omega^2 I]\Gamma(x,\omega) = \delta(x) I, \tag{3.1}$$

where $I = \|\delta_{kj}\|_{2\times 2}$ is the unit matrix and $\delta(\cdot)$ is Dirac's distribution.

Our aim is to construct the fundamental matrix $\Gamma(x,\omega)$ with entries maximally decreasing at infinity.

To this end, applying the generalized Fourier transform to (3.1), we get

$$[\omega^2 I - C(\xi)]\hat{\Gamma}(\xi,\omega) = I, \tag{3.2}$$

with $\hat{\Gamma}(\xi,\omega) = \mathcal{F}_{x\to\xi}[\Gamma(x,\omega)]$.

For summable functions the Fourier operators $\mathcal{F}_{x\to\xi}$ and $\mathcal{F}^{-1}_{\xi\to x}$ are defined as follows:

$$\mathcal{F}_{x\to\xi}[f] = \int_{\mathbb{R}^2} f(x)\exp(ix\xi)dx, \ \ \mathcal{F}^{-1}_{\xi\to x}[g] = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} g(\xi)\exp(-ix\xi)d\xi.$$

We introduce the notation

$$\Phi(\xi,\omega) = \det[\omega^2 I - C(\xi)] = L(\xi)N(\xi) - M^2(\xi) - \omega^2[L(\xi) + N(\xi)] + \omega^4. \tag{3.3}$$

Passing over to the polar coordinates

$$\xi_1 = \rho\cos\varphi, \quad \xi_2 = \rho\sin\varphi, \quad \rho = |\xi| \geq 0, \quad 0 \leq \varphi \leq 2\pi, \tag{3.4}$$

we easily get

$$\Phi(\xi,\omega)=\Phi(\eta)\rho^4-\omega^2[L(\eta)+N(\eta)]\rho^2+\omega^4=\Phi(\eta)(\rho^2-\rho_1^2)(\rho^2-\rho_2^2)=$$
$$=\Phi(\eta)[\rho^2-\omega^2k_1^2(\varphi)][\rho^2-\omega^2k_2^2(\varphi)], \tag{3.5}$$

where $\eta=\xi|\xi|^{-1}=(\cos\varphi,\sin\varphi)$,

$$\Phi(\eta)=\det C(\eta)=L(\eta)N(\eta)-M^2(\eta)>0, \quad \rho_j=\omega k_j(\varphi), \quad j=1,2, \tag{3.6}$$

and ωk_j are positive solutions of the equation $\Phi(\xi,\omega)=0$ with respect to ρ

$$k_j^2(\varphi)=\frac{1}{2\Phi(\eta)}\{L(\eta)+N(\eta)+(-1)^j\sqrt{[L(\eta)-N(\eta)]^2+4M^2(\eta)}\}>0. \tag{3.7}$$

Obviously, the function $\Phi(\xi,\omega)$ vanishes on the lines S_j defined by the equations $\rho=\omega k_j(\varphi)$, $j=1,2$. From now on it will be assumed that

I^0. $\nabla\Phi(\xi,\omega)\neq 0, \quad \xi\in S_j, \quad j=1,2;$ (3.8)

II^0. The curvatures of the lines S_1 and S_2 differ from zero for all $\varphi\in[0,2\pi]$.

It follows from the above conditions that the real zeros of the polynomial $\Phi(\xi,\omega)$ form two closed, non-intersecting, convex lines. For arbitrary $x\in\mathbb{R}^2\setminus\{0\}$ there exist exactly two points on each S_j, where the external unit normal vector is parallel to the vector x. At one of these points, called $\xi^j\in S_j$, the vectors $n(\xi^j)$ and x are of the same direction, while at the other one, called $\tilde{\xi}^j$, they are of the opposite directions. The evenness of the function $\Phi(\xi,\omega)$ in ξ implies that $\tilde{\xi}^j=-\xi^j$. It is clear also that if the polar angle φ_j corresponds to the point ξ^j, then $\pi+\varphi_j$ corresponds to the point $\tilde{\xi}^j$.

For arbitrary $\omega>0$ there exist positive constants δ_1 and δ_2 such that

$$0<\delta_1\le k_1(\varphi)<k_2(\varphi)\le\delta_2<\infty, \quad 0\le\varphi\le 2\pi. \tag{3.9}$$

From (3.6) and the evenness of the functions L, M, and N we get $k_j(\varphi)=k_j(\varphi+\pi)$, i.e., $k_j(\varphi)$ is a π-periodic function.

As the reciprocal matrix $[\omega^2I-C(\xi)]^{-1}$ does not exist for all $\xi\in\mathbb{R}^2$, we have to regularize the distribution $\hat{\Gamma}$ defined by (3.2). We will do this by means of the limiting absorption principle.

Consider the pseudo-oscillation equation

$$C(D,\tau_\varepsilon)\Gamma(x,\tau_\varepsilon)\equiv[C(D)+\tau_\varepsilon^2I]\Gamma(x,\tau_\varepsilon)=\delta(x)I, \tag{3.10}$$

where $\tau_\varepsilon=\omega+i\varepsilon$, $\varepsilon\in(-\varepsilon_1,\varepsilon_1)$, $\varepsilon_1>0$, $\varepsilon\neq 0$; $\Gamma(x,\tau_\varepsilon)$ is a fundamental matrix with entries from the space of tempered distributions. Therefore (3.10) is equivalent to

$$[\tau_\varepsilon^2I-C(\xi)]\hat{\Gamma}(\xi,\tau_\varepsilon)=I. \tag{3.11}$$

Note that

$$\Phi(\xi,\tau_\varepsilon)=\det[\tau_\varepsilon^2 I-C(\xi)]=\Phi(\eta)[\rho^2-\tau_\varepsilon^2 k_1^2(\varphi)][\rho^2-\tau_\varepsilon^2 k_2^2(\varphi)] \quad (3.12)$$

and by (3.7) the reciprocal matrix $[\tau_\varepsilon^2 I - C(\xi)]^{-1}$ exists for all $\xi \in \mathbb{R}^2$ and $\varepsilon \neq 0$. From (3.11) we get

$$\hat{\Gamma}(\xi,\tau_\varepsilon) = [\tau_\varepsilon^2 I - C(\xi)]^{-1} = [\Phi(\xi,\tau_\varepsilon)]^{-1} C^*(\xi,\tau_\varepsilon) \in L^2(\mathbb{R}^2), \quad (3.13)$$

with

$$C^*(\xi,\tau_\varepsilon) = \begin{Vmatrix} \tau_\varepsilon^2 - N(\xi), & M(\xi) \\ M(\xi), & \tau_\varepsilon^2 - L(\xi) \end{Vmatrix}. \quad (3.14)$$

Thus

$$\Gamma(x,\tau_\varepsilon) = \mathcal{F}^{-1}_{\xi\to x}[\hat{\Gamma}(\xi,\tau_\varepsilon)] = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} \Phi^{-1}(\xi,\tau_\varepsilon) C^*(\xi,\tau_\varepsilon) e^{-ix\xi} d\xi. \quad (3.15)$$

Lemma 1. *The entries of the matrix $\Gamma(x,\tau_\varepsilon)$ belong to the class $C^\infty(\mathbb{R}^2\backslash\{0\})$ and, together with all of their derivatives, decrease more rapidly than any negative power of $|x|$ as $|x| \to +\infty$.*

Proof. The correctness of the lemma follows from the relations

$$|x|^{2m} D_x^\alpha \Gamma_{kj}(x,\tau_\varepsilon) = \mathcal{F}^{-1}_{\xi\to x}[(-i\xi)^\alpha(-\Delta)^m \hat{\Gamma}_{kj}(\xi,\tau_\varepsilon)],$$

$$|(-i\xi)^\alpha(-\Delta)^m \hat{\Gamma}_{kj}(\xi,\tau_\varepsilon)| < c(\varepsilon)(1+|\xi|)^{-m-2+|\alpha|}, \quad c(\varepsilon) > 0,$$

where $\Delta = D_1^2 + D_2^2$ is the Laplace operator, $\alpha = (\alpha_1,\alpha_2)$ is an arbitrary multi-index, and $|\alpha| = \alpha_1 + \alpha_2$. $\square$

Formally, Eqs. (3.1) and (3.10) coincide when $\varepsilon = 0$. Therefore we have to investigate the limit of (3.15) as $\varepsilon \to 0$.

First we introduce the function

$$h \in C^\infty(\mathbb{R}^2), \quad h(\xi) = 1 \quad \text{for} \quad |\xi| < C_0, \quad h(\xi) = 0 \quad \text{for} \quad |\xi| > 2C_0, \quad (3.16)$$

with $C_0 > 2\delta_2\omega$ (see (3.9)) and represent $\Gamma(x,\tau_\varepsilon)$ as

$$\Gamma(x,\tau_\varepsilon) = \Gamma^1(x,\tau_\varepsilon) + \Gamma^2(x,\tau_\varepsilon),$$

where

$$\begin{aligned} \Gamma^1(x,\tau_\varepsilon) &= \mathcal{F}^{-1}_{\xi\to x}[(1-h(\xi))\hat{\Gamma}(\xi,\tau_\varepsilon)], \\ \Gamma^2(x,\tau_\varepsilon) &= \mathcal{F}^{-1}_{\xi\to x}[h(\xi)\hat{\Gamma}(\xi,\tau_\varepsilon)]. \end{aligned} \quad (3.17)$$

Lemma 2. *The entries of the matrix $\Gamma^1(x,\tau_\varepsilon)$ are uniformly continuous functions in τ_ε for $|x| \geq \delta_0 > 0$ and together with all of their derivatives decrease more rapidly than any negative power of $|x|$ as $|x| \to +\infty$.*

Proof. It is easy to see by (3.13) and (3.14) that

$$\hat{\Gamma}(\xi,\tau_\varepsilon)-\hat{\Gamma}(\xi,\tau)=(\tau_\varepsilon^2-\tau^2)[Q^1(\xi,\tau)-Q^2(\xi,\tau_\varepsilon,\tau)] \tag{3.18}$$

for $|\xi|>C_0>2\delta_2\omega$, where

$$\begin{aligned} Q^1(\xi,\tau)&=\Phi^{-1}(\xi,\tau)I=\Phi^{-1}(\xi,0)\{1+[\Phi(\xi,0)-\Phi(\xi,\tau)]\Phi^{-1}(\xi,\tau)\}I,\\ Q^2(\xi,\tau_\varepsilon,\tau)&=\Phi^{-1}(\xi,\tau_\varepsilon)\Phi^{-1}(\xi,\tau)\{\tau_\varepsilon^2+\tau^2-L(\xi)-N(\xi)\}C^*(\xi,\tau_\varepsilon). \end{aligned} \tag{3.19}$$

Lemma 2 now follows from (3.16), (3.17), and the relations $(1-h)Q^j\in L^1(\mathbb{R}^2)$, $\quad j=1,2$. $\square$

Lemma 3. *The estimates*

$$\begin{aligned} &|D^\alpha[\Gamma^1_{kj}(x,\tau_\varepsilon)-\Gamma^1_{kj}(x,\omega)]|\le\varepsilon C_1\nu_{|\alpha|}(x),\\ &|D^\alpha[\Gamma^1_{kj}(x,\tau_\varepsilon)-\Gamma^1_{kj}(x,0)]|\le C_1\nu_{|\alpha|}(x),\quad |\alpha|=0,1,2,3, \end{aligned}$$

hold for $0<|x|<\delta_0<1/2$ (i.e., in the vicinity of $x=0$); here C_1 is some positive constant independent of ε, $\varepsilon\in(-\varepsilon_1,\varepsilon_1)$ and

$$\nu_0(x)=\nu_1(x)=1,\quad \nu_2(x)=\ln|x|^{-1},\quad \nu_3(x)=|x|^{-1}. \tag{3.20}$$

Proof. Lemma 3 follows from (3.18), (3.19), and Lemma 2.17 of [29]. $\square$

Lemma 4. *The entries of the matrix $\Gamma^1(x,0)$ together with all of their derivatives decrease more rapidly than any negative power of $|x|$ as $|x|\to+\infty$ and in the vicinity of $x=0$ the estimates $\Gamma^1_{kj}(x,0)=O(\ln|x|)$, $D^\alpha\Gamma^1_{kj}(x,0)=O(|x|^{-|\alpha|})$ hold for an arbitrary multi-index α with $|\alpha|\ge1$.*

Proof. The first part of Lemma 4 can be proved similarly to Lemma 1. As to the second assertion, it follows from the representations

$$\Gamma^1_{kj}(x,0)=\mathcal{F}^{-1}_{\xi\to x}[\chi_1(\xi)(1-h(\xi))\hat{\Gamma}_{kj}(\xi,0)]+\mathcal{F}^{-1}_{\xi\to x}[\chi_2(\xi)\hat{\Gamma}_{kj}(\xi,0)], \tag{3.21}$$

$$\begin{aligned} D_p\Gamma^1_{kj}(x,0)&=\mathcal{F}^{-1}_{\xi\to x}[-i\xi_p(1-h(\xi))\hat{\Gamma}_{kj}(\xi,0)]=\\ &=\mathcal{F}^{-1}_{\xi\to x}[i\xi_p h(\xi)\hat{\Gamma}_{kj}(\xi,0)]+\mathcal{F}^{-1}_{\xi\to x}[-i\xi_p\hat{\Gamma}_{kj}(\xi,0)], \end{aligned} \tag{3.22}$$

where $\hat{\Gamma}(\xi,0)=[C(\xi)]^{-1}$, χ_1 and χ_2 are the characteristic functions of the domains $|\xi|\le2C_0$ and $|\xi|>2C_0$, respectively, with C_0 from (3.16).

In fact, the first summands of (3.21) and (3.22) are the analytic functions of x. The second summand of (3.21) is $O(\ln|x|)$ [29, Lemma 2.17], while the second summand of (3.22) is a homogeneous function of degree -1 [28, p. 30]. $\square$

The above lemmas readily imply

Lemma 5. *The limit*

$$\lim_{\varepsilon\to 0}\Gamma^1(x,\tau_\varepsilon)=\Gamma^1(x,\omega)=\frac{1}{4\pi^2}\int_{\mathbb{R}^2}[1-h(\xi)]\Phi^{-1}(\xi,\omega)C^*(\xi,\omega)e^{-ix\xi}d\xi$$

exists for arbitrary $x\in\mathbb{R}^2\setminus\{0\}$. *The entries of* $\Gamma^1(x,\omega)$ *together with all of their derivatives decrease more rapidly than any negative power of* $|x|$ *as* $|x|\to+\infty$ *and in the vicinity of the origin the relations*

$$\Gamma^1_{kj}(x,\omega)=O(\ln|x|),\quad D^\alpha\Gamma^1_{kj}(x,\omega)=O(|x|^{-|\alpha|}),\quad |\alpha|\geq 1$$

hold for an arbitrary multi-index α.

Let us now investigate the matrix $\Gamma^2(x,\tau_\varepsilon)$. It is easy to verify that the entries of the matrix

$$\Gamma^2(x,\tau_\varepsilon)=\frac{1}{4\pi^2}\int_{\mathbb{R}^2}h(\xi)\Phi^{-1}(\xi,\tau_\varepsilon)C^*(\xi,\tau_\varepsilon)e^{-ix\xi}d\xi \tag{3.23}$$

are analytic functions of x and together with all of their derivatives decrease more rapidly than any negative power of $|x|$ as $|x|\to+\infty$ for arbitrary $\varepsilon\neq 0$, since $\Phi(\xi,\tau_\varepsilon)\neq 0$, $\xi\in\mathbb{R}^2$.

In what follows we will show that there exist one-sided limits of (3.23) as $\varepsilon\to 0\pm$.

First let $\varepsilon\in(0,\varepsilon_1)$ with $0<\varepsilon_1<\frac{1}{3}\omega$. Then (see (3.9) and (3.16))

$$|\tau_\varepsilon k_j(\varphi)|<\tfrac{4}{3}\omega\delta_2<\tfrac{2}{3}C_0.$$

Next, let

$$x_1=r\cos\psi,\quad x_2=r\sin\psi,\quad r=|x|\geq 0,\quad \psi\in[0,2\pi]. \tag{3.24}$$

Then by (3.23)

$$\Gamma^2(x,\tau_\varepsilon)=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2C_0}h(\xi)\Phi^{-1}(\xi,\tau_\varepsilon)C^*(\xi,\tau_\varepsilon)e^{-i\rho r\cos(\varphi-\psi)}\rho d\rho d\varphi. \tag{3.25}$$

We introduce the complex ρ-plane $\rho=\rho'+i\rho''$. By (3.12) zeros of the function $\Phi(\xi,\tau_\varepsilon)$ with respect to ρ are $\pm\tau_\varepsilon k_j(\varphi)$ and obviously they belong to the first and third quarters of the complex ρ-plane. Therefore by (3.16) and the analyticity of the integrand in (3.25) in the vicinity of the points $\rho_1=\omega k_1(\varphi)$ and $\rho_2=\omega k_2(\varphi)$, we have the following representation

$$\Gamma^2(x,\tau_\varepsilon)=\frac{1}{4\pi^2}\int_0^{2\pi}\int_{l^-}h(\xi)\Phi^{-1}(\xi,\tau_\varepsilon)C^*(\xi,\tau_\varepsilon)e^{-i\rho r\cos(\varphi-\psi)}\rho d\rho d\varphi, \tag{3.26}$$

where $l^-=[0,\rho_1-\delta]\cup\Sigma^-_{1,\delta}\cup[\rho_1+\delta,\rho_2-\delta]\cup\Sigma^-_{2,\delta}\cup[\rho_2+\delta,C_0]$, $0<\delta<\min\{\rho_1,2^{-1}(\rho_2-\rho_1)\}$, and $\Sigma^-_{j,\delta}$ is the lower semicircle (centered at the point ρ_j, with radius δ) oriented counterclockwise.

Obviously, the limit of (3.26) exists when $\varepsilon \to 0+$. We introduce the notation

$$\Gamma^2(x,\omega,1)=\lim_{\varepsilon\to 0+}\Gamma^2(x,\tau_\varepsilon)=$$
$$=\frac{1}{4\pi^2}\int_0^{2\pi}\int_{l^-}h(\xi)\Phi^{-1}(\xi,\omega)C^*(\xi,\omega)e^{-i\rho r\cos(\varphi-\psi)}\rho d\rho d\varphi. \quad (3.27)$$

Similarly to the above, we get

$$\Gamma^2(x,\omega,2)=\lim_{\varepsilon\to 0-}\Gamma^2(x,\tau_\varepsilon)=$$
$$=\frac{1}{4\pi^2}\int_0^{2\pi}\int_{l^+}h(\xi)\Phi^{-1}(\xi,\omega)C^*(\xi,\omega)e^{-i\rho r\cos(\varphi-\psi)}\rho d\rho d\varphi, \quad (3.28)$$

where l^+ is the mirror image of l^- ($\Sigma^-_{j,\delta}$ is to be replaced by the clockwise oriented upper semicircle $\Sigma^+_{j,\delta}$).

Formulas (3.5), (3.6), (3.7), (3.27), and (3.28) imply

$$\Gamma^2(x,\omega,m)=\frac{1}{4\pi^2}\int_0^{2\pi}\{\text{v.p.}\int_0^{2C_0}h(\xi)\Phi^{-1}(\xi,\omega)C^*(\xi,\omega)e^{-i\rho r\cos(\varphi-\psi)}\rho d\rho+$$
$$+(-1)^m\sum_{j=1}^{2}i\pi(-1)^j\frac{\Psi(\rho_j,\varphi;r,\psi)}{\rho_1-\rho_2}\}d\varphi,$$

where the interior integral is understood in the principal value sense and

$$\Psi(\rho,\varphi;r,\psi)=\frac{e^{-i\rho r\cos(\varphi-\psi)}}{\Phi(\eta)(\rho+\rho_1)(\rho+\rho_2)}C^*(\xi,\omega)=$$
$$=\frac{(\rho-\rho_1)(\rho-\rho_2)}{\Phi(\xi,\omega)}C^*(\xi,\omega)e^{-ix\xi}\rho.$$

Denoting

$$\tilde L(\varphi)=L(\eta),\quad \tilde M(\varphi)=M(\eta),\quad \tilde N(\varphi)=N(\eta), \quad (3.29)$$

it is easy to show that

$$\sum_{j=1}^{2}(-1)^j\frac{\Psi(\rho_j,\varphi;r,\psi)}{\rho_1-\rho_2}=-\sum_{j=1}^{2}\{[\Phi'_\rho(\xi,\omega)]^{-1}C_*(\xi,\omega)e^{-ix\xi}\rho\}_{\xi\in S_j}=$$
$$=-\sum_{j=1}^{2}\frac{(-1)^je^{-i\rho_j r\cos(\varphi-\psi)}}{2[(\tilde L-\tilde N)^2+4\tilde M^2]^{1/2}}\tilde C_j(\varphi), \quad (3.30)$$

where

$$\tilde C_j(\varphi)=\left\| \begin{matrix} 1-\tilde N(\varphi)k_j^2(\varphi), & \tilde M(\varphi)k_j^2(\varphi) \\ \tilde M(\varphi)k_j^2(\varphi), & 1-\tilde L(\varphi)k_j^2(\varphi) \end{matrix} \right\|. \quad (3.31)$$

With regard to these relations we have

$$\Gamma^2(x,\omega,m)=\frac{1}{4\pi^2}\int_0^{2\pi}\{\text{v.p.}\int_0^{2C_0}\frac{h(\xi)}{\Phi(\xi,\omega)}C^*(\xi,\omega)e^{-i\xi x}\rho d\rho\}d\varphi+$$
$$+\frac{i(-1)^{m+1}}{4\pi}\int_0^{2\pi}\sum_{j=1}^{2}\frac{(-1)^j e^{-i\omega k_j(\varphi)r\cos(\varphi-\psi)}}{2[(\tilde{L}-\tilde{N})^2+4\tilde{M}^2]^{1/2}}C_j^*(\varphi)d\varphi,\quad m=1,2. \quad (3.32)$$

The investigation of the asymptotic behavior of $\Gamma^2(x,\omega,m)$ as $|x|\to+\infty$ requires more detailed analytical and structural information about the curves S_j, $j=1,2$. Namely, note that if $x\in\mathbb{R}^2\backslash\{0\}$ and φ_j is the polar angle corresponding to the point $\xi^j\in S_j$, then $\tilde{\varphi}_j=\varphi_j+\pi$ corresponds to the point $\tilde{\xi}^j\in S_j$. The curvature $\varkappa$ of the curve S_j, defined by the equation $\rho=\omega k_j(\varphi)$, is calculated by the formula

$$\varkappa=[\rho^2+2(\rho')^2-\rho\rho''][\rho^2+(\rho')^2]^{-3/2}>0,\quad 0\le\varphi<2\pi. \quad (3.33)$$

It is obvious that φ_j and $\tilde{\varphi}_j$ are the stationary points of the function

$$g(\varphi)=r^{-1}(x_1\xi_1+x_2\xi_2)=\rho\cos(\varphi-\psi)=\omega k_j(\varphi)\cos(\varphi-\psi),\quad \xi\in S_j.$$

The relation

$$g''_{\varphi\varphi}(\varphi)=\omega\cos(\varphi-\psi)k_j^{-1}(\varphi)[k_j^2(\varphi)+(k_j'(\varphi))^2]\varkappa(\varphi)\neq 0$$

for $\varphi=\varphi_j$ or $\varphi=\tilde{\varphi}_j$, implies that φ and $\tilde{\varphi}_j$ are non-degenerate stationary points [30].

Further, for the exterior unit normal $n(\xi)$ to S_j we have $\cos(\varphi_j-\psi)=(n(\xi^j)\cdot\xi^j|\xi^j|^{-1})>0, \cos(\tilde{\varphi}_j-\psi)=-(n(\xi^j)\cdot\tilde{\xi}^j|\tilde{\xi}^j|^{-1})<0$, and consequently $g''_{\varphi\varphi}(\varphi_j)>0$, $g''_{\varphi\varphi}(\tilde{\varphi}_j)<0$. From (3.7) and the inequalities

$$(-1)^j\Phi'_\rho(\xi,\omega)>0,\quad \xi\in S_j,\quad j=1,2, \quad (3.34)$$

it follows that the exterior unit normal $n(\xi)$ to S_j can be represented as $n(\xi)=(-1)^j|\nabla\Phi(\xi,\omega)|^{-1}\nabla\Phi(\xi,\omega)$, $\xi\in S_j$. Note that

$$|\nabla\Phi(\xi,\omega)|=2\Phi(\eta)|\rho_1^2-\rho_2^2|[\rho_j^2+(\rho_j')^2]^{1/2},\quad \xi\in S_j. \quad (3.35)$$

Let us introduce the generalized v.p.-integral (see [30], Ch.3. §4)

$$\text{v.p.}\int_{\mathbb{R}^2}\frac{f(\xi)}{\Phi(\xi,\omega)}d\xi=\lim_{\varepsilon\to 0+}\int_{|\Phi(\xi,\omega)|>\varepsilon}\frac{f(\xi)}{\Phi(\xi,\omega)}d\xi.$$

This integral exists if $f\in C^\infty(\mathbb{R}^2)$ and $\operatorname{supp} f$ is compact (cf. [30]).

Applying the properties of singular integrals on smooth curves and inequality (3.8), we can rewrite the matrix $\Gamma^2(x,\omega,m)$ as

$$\Gamma^2(x,\omega,m)=\Psi^1(x)+\Psi^2_m(x), \quad (3.36)$$

where

$$\Psi^1(x) = \text{v.p.}\frac{1}{4\pi^2}\int_{\mathbb{R}^2} h(\xi)\Phi^{-1}(\xi,\omega)C^*(\xi,\omega)e^{-ix\xi}d\xi,$$

$$\Psi_m^2(x) = (-1)^{m+1}\frac{i}{4\pi}\int_0^{2\pi}\sum_{j=1}^{2}\frac{(-1)^j e^{-ir\omega k_j(\varphi)\cos(\varphi-\psi)}}{2[(\tilde{L}-\tilde{N})^2+4\tilde{M}^2]^{1/2}}\tilde{C}_j(\varphi)\,d\varphi \qquad (3.37)$$

$$= (-1)^{m+1}\frac{i}{4\pi}\int_0^{2\pi}\sum_{j=1}^{2}\{[\Phi'_\rho(\xi,\omega)]^{-1}C^*(\xi,\omega)e^{-ix\xi}\}_{S_j}\,\rho_j d\varphi.$$

Due to the relation $dS_j = [(\rho'_j)^2+\rho_j^2]^{1/2}\,d\varphi$ and (3.35)

$$\frac{\rho_j d\varphi}{[\Phi'_\rho(\xi,\omega)]_{S_j}} = \frac{\rho_j\;dS_j}{[(\rho'_j)2+\rho_j^2]^{1/2}[2\Phi(\eta)\rho_j(-1)^j(\rho_2^2-\rho_1^2)]} = \frac{(-1)^j\;dS_j}{\{|\nabla\Phi(\xi,\omega)|\}_{S_j}}.$$

Therefore

$$\Psi_m^2(x) = (-1)^{m+1}\frac{1}{4\pi}\sum_{j=1}^{2}(-1)^j\int_{S_j}\frac{C^*(\xi,\omega)e^{-ix\xi}}{|\nabla\Phi(\xi,\omega)|}\,dS_j. \qquad (3.38)$$

We can get an equivalent representation of Ψ_m^2 by means of special differential forms on S_j.

Note that if S_j is an $(n-1)$-dimensional level manifold defined by the equation $P(\xi_1,\cdots,\xi_n)=0$, with $P\in C^\infty$ and $\nabla P(\xi)\neq 0$, $\xi\in S_j$, then the Laray–Gelfand differential form Ω_P reads [30] as

$$\Omega_P = \sum_{k=1}^{n}(-1)^{k-1}|\nabla P|^{-2}\frac{\partial P}{\partial\xi_k}d\xi_1\wedge\cdots\wedge d\xi_{k-1}\wedge d\xi_{k+1}\wedge\cdots\hat{d}\xi_n. \qquad (3.39)$$

Let $\nu=(\nu_1,\cdots,\nu_n)$ be the positive normal vector defining the orientation of S_j. The measure on S_j, corresponding to the differential form (3.39) and chosen orientation, is given by the formula [35]

$$[\Omega_P] = \sum_{k=1}^{n}(-1)^{k-1}|\nabla P|^{-2}\frac{\partial P}{\partial\xi_k}(-1)^{k-1}\nu_k dS_j = |\nabla P|^{-2}\frac{\partial P}{\partial\nu}dS_j =$$
$$= |\nabla P|^{-2}(\varepsilon_j\nabla P|\nabla P|^{-1},\nabla P) = \varepsilon_j|\nabla P|^{-1}dS_j, \qquad (3.40)$$

where $\varepsilon_j=\operatorname{sgn}(\nu\cdot\nabla P)$, i.e., $\varepsilon_j=1$ if the positive normal ν and the vector ∇P have the same directions on S_j, and $\varepsilon_j=-1$ if they have the opposite directions.

Due to (3.40)

$$(-1)^j\int_{S_j}\frac{f(\xi)}{|\nabla P(\xi)|}dS_j = (-1)^j\varepsilon_j\int_{S_j}f(\xi)\Omega_P.$$

Let us choose a positive normal on S_j by the equation $\nu = |\nabla P|^{-1}\nabla P$, i.e., $\varepsilon_j = 1$.

Making use of these results for $P(\xi) \equiv \Phi(\xi,\omega)$ $(n=2,\ \varepsilon_j=1,\ j=1,2)$, from (3.38) we have

$$\Psi^2_m(x) = (-1)^{m+1}\frac{i}{4\pi}\sum_{j=1}^{2}(-1)^j\int_{S_j} C^*(\xi,\omega)e^{-ix\xi}\Omega_{\Phi(\xi,\omega)}. \tag{3.41}$$

The asymptotic behavior of such integrals for large $|x|$ is studied by many authors (see [23, 30, 33, 34]). Due to Theorems 1, 2 and Remark 1 of [33] and Theorem 4.4 of [30] integrals (3.37) and (3.41) admit the following representation ($|x| = r \to +\infty$):

$$\Psi^1(x) = \frac{-i}{4\pi}\sum_{j=1}^{2}(-1)^j[b_j(x) - \tilde{b}_j(x)] + O(r^{-3/2}), \tag{3.42}$$

$$\Psi^2_m(x) = (-1)^{m+1}\frac{i}{4\pi}\sum_{j=1}^{2}(-1)^j\{b_j(x) + \tilde{b}_j(x)\} + O(r^{-3/2}), \tag{3.43}$$

where the points $\xi^j, \tilde{\xi}^j \in S_j$ correspond to the point x, $n(\xi)$ is the exterior unit normal at $\xi \in S_j$, b_j and $\tilde{b}_j$ denote the principal contributions corresponding to the non-degenerate stationary points ξ^j and $\tilde{\xi}^j$, respectively, in the asymptotic expansion (as $|x| \to +\infty$) of the integral

$$B_j(x) = \int_{S_j} C^*(\xi,\omega)e^{-ix\xi}\Omega_{\Phi(\xi,\omega)} = \int_{S_j}\frac{C^*(\xi,\omega)e^{-ix\xi}}{|\nabla\Phi(\xi,\omega)|}\,dS_j =$$
$$= \int_0^{2\pi}\frac{e^{-ir\omega k_j(\varphi)\cos(\varphi-\psi)}}{2[(\tilde{L}-\tilde{N})^2 + 4\tilde{M}^2]^{1/2}}\tilde{C}_j(\varphi)\,d\varphi$$

and read as follows:

$$b_j(x) = \overline{\tilde{b}_j(x)} = \left(\frac{2\pi}{r}\right)^{1/2}\frac{e^{-i\pi/4}}{[\varkappa(\xi^j)]^{1/2}}\,\frac{e^{-ix\xi^j}}{|\nabla\Phi(\xi^j,\omega)|}C^*(\xi^j,\omega); \tag{3.44}$$

here $\varkappa(\xi)$ is the curvature of S_j at the point ξ (see (4.9)).

Taking into consideration that $\tilde{\xi}^j = -\xi^j$, $\varkappa(\tilde{\xi}^j) = \varkappa(\xi^j)$, $|\nabla\Phi(\tilde{\xi}^j,\omega)| = |\nabla\Phi(\xi^j,\omega)|$, $C^*(\tilde{\xi}^j,\omega) = C^*(\xi^j,\omega)$ and denoting

$$Q_j(x) = \left(\frac{2\pi}{r}\right)^{1/2}\frac{C^*(\xi^j,\omega)}{[\varkappa(\xi^j)]^{1/2}|\nabla\Phi(\xi^j,\omega)|},$$

from (3.44) we get

$$b_j(x) = Q_j(x)e^{-i\pi/4}e^{-ix\xi^j}, \quad \tilde{b}_j(x) = Q_j(x)e^{i\pi/4}e^{ix\xi^j}.$$

Therefore (3.36), (3.42), and (3.43) imply

$$\Gamma^2(x,\omega,1)=\sum_{j=1}^{2}\frac{i}{2\pi}(-1)^j Q_j(x)e^{i\pi/4}e^{ix\xi^j}+O(|x|^{-3/2}), \tag{3.45}$$

$$\Gamma^2(x,\omega,2)=\sum_{j=1}^{2}\frac{i}{2\pi}(-1)^{j+1} Q_j(x)e^{-i\pi/4}e^{-ix\xi^j}+O(|x|^{-3/2}). \tag{3.46}$$

Thus we have proved

Lemma 6. *The entries of the matrix* $\Gamma^2(x,\omega,m)$ *defined by equalities* (3.27), (3.28) *are* C^∞*-smooth functions with respect to* x *on* $\mathbb{R}^2$. *For sufficiently large* $|x|$ *the asymptotic formulas* (3.45) *and* (3.46) *hold. The convergence in* (3.27) *and* (3.28) *is uniform with respect to the parameter* $x \in \mathbb{R}^2$.

Lemmas 2-6 imply

Theorem 7. *If conditions* I^0 *and* II^0 *are satisfied, then*
(i) *the limits*

$$\lim_{(-1)^{m+1}\varepsilon\to 0+}\Gamma(x,\tau_\varepsilon)=\Gamma(x,\omega,m), \quad \forall x\in\mathbb{R}^2\backslash\{0\},$$

exist, where $\tau_\varepsilon=\omega+i\varepsilon$, $\Gamma(x,\tau_\varepsilon)$ *is defined by* (3.15)*; the above limits exist uniformly with respect to the variable* x *for* $|x|\geq\delta>0$ *with some positive* δ*;*

(ii) $\Gamma(x,\omega,m)$ $(m=1,2)$ *are the fundamental matrices of the operator* $C(D,\omega)$ *and*

$$\begin{aligned}\Gamma(x,\omega,m) &= \mathcal{F}^{-1}_{\xi\to x}\{[1-h(\xi)][\omega^2 I - C(\xi)]^{-1}\} + \\ &+ \frac{1}{4\pi^2}\mathrm{v.p.}\int_{\mathbb{R}^2} h(\xi)[\omega^2 I - C(\xi)]^{-1}e^{-i\xi x}\,d\xi + \\ &+ (-1)^{m+1}\frac{i}{8\pi}\int_0^{2\pi}\sum_{j=1}^{2}\frac{(-1)^j e^{-i\omega r k_j(\varphi)\cos(\varphi-\psi)}}{[(\tilde L-\tilde N)^2+4\tilde M^2]^{1/2}}\tilde C_j(\varphi)\,d\varphi;\end{aligned} \tag{3.47}$$

$$\Gamma(x,\omega,m)=\Gamma(-x,\omega,m)=[\Gamma(x,\omega,m)]^\top$$

(*see* (3.7), (3.16), (3.29), (3.31), (3.32))*;*

(iii) *in the vicinity of the origin* $(0<|x|<1/2)$ *the inequalities*

$$|D^\alpha[\Gamma(x,\omega,m)-\Gamma(x)]| < c\nu_{|\alpha|}(x), \quad |\alpha|=0,1,2,3,$$

$$|D^\alpha[\Gamma(x,\tau_\varepsilon)-\Gamma(x,\omega,m)| < |\varepsilon| c\nu_{|\alpha|}(x), \quad (-1)^{m+1}\varepsilon>0,$$

hold, where $\nu_{|\alpha|}$ *is defined by* (3.20) *and* $\Gamma(x)$ *is the fundamental matrix of the statics operator* $C(D)$: $\Gamma(x)=\mathcal{F}^{-1}_{\xi\to x}[C^{-1}(-i\xi)]$ (*cf.* [31])*;*

iv) *for sufficienty large $|x|$ $(|x| \to +\infty)$*

$$\Gamma(x,\omega,1)=\sum_{j=1}^{2}|x|^{-1/2}R_{1j}(x)e^{ix\xi^j}+O(|x|^{-3/2}),$$
$$\Gamma(x,\omega,2)=\sum_{j=1}^{2}|x|^{-1/2}R_{2j}(x)e^{-ix\xi^j}+O(|x|^{-3/2}),$$

where the point $\xi^j\in S_j$ corresponds to x,

$$R_{1j}(x)=\overline{R_{2j}(x)}=(-1)^j\frac{i}{\sqrt{2\pi}}\frac{e^{i\pi/4}}{\sqrt{\varkappa(\xi^j)}|\nabla\Phi(\xi^j,\omega)|}C^*(\xi^j,\omega).$$

Corollary 8. *If $y\in\Omega_0$, where Ω_0 is a bounded subset of $\mathbb{R}^2$, then (for $|x|\to+\infty$)*

$$D_x^\alpha D_y^\beta\Gamma(x-y,\omega,1)=\sum_{j=1}^{2}|x|^{-1/2}R_{1j}(x)(i\xi^j)^\alpha(-i\xi^j)^\beta e^{i(x-y)\xi^j}+O(|x|^{-3/2}),$$
$$D_x^\alpha D_y^\beta\Gamma(x-y,\omega,2)=\sum_{j=1}^{2}|x|^{-1/2}R_{2j}(x)(-i\xi^j)^\alpha(i\xi^j)^\beta e^{i(y-x)\xi^j}+O(|x|^{-3/2}),$$

where $\xi^j\in S_j$ corresponds to x, and α and β are arbitrary multi-indices.

4. Radiation Conditions and the Formulation of the Basic Boundary Value Problems. Integral Representations of Solutions

Let us introduce the following classes (cf. [16-19]).

A function (vector, matrix) u belongs to the class $SK_m(\Omega)$, $m=1,2$, if it (each component of the vector, each element of the matrix) is C^1-smooth in the vicinity of infinity and for sufficiently large $|x|$ the relations

$$u(x)=\sum_{j=1}^{2}u^{(j)}(x),\quad u^{(j)}(x)=O(|x|^{-1/2}),$$
$$\frac{\partial u^{(j)}(x)}{\partial x_p}+i(-1)^m\xi_p^j u^{(j)}(x)=O(|x|^{-3/2}),\quad j,p=1,2, \tag{4.1}$$

hold, where the point $\xi^j\in S_j$ corresponds to the radius-vector x (see Sect.3); here $\Omega=\Omega^-$ or $\Omega=\mathbb{R}^2$; (no summation over j in (4.1)).

These conditions are generalized Sommerfeld–Kupradze type radiation conditions in anisotropic elasticity (cf. [1,2,3]).

It is easy to verify that the entries of the matrix $\Gamma(x,\omega,m)$ constructed in the previous section satisfy conditions (4.1) and consequently $\Gamma(\cdot,\omega,m) \in SK_m(\mathbb{R}^2\backslash\{0\})$.

Now we can formulate the basic BVPs for Eq. (2.1).

Problem for the whole plane (PL_m-Problem):
Find a regular vector $u \in C^2(\mathbb{R}^2)$ satisfying Eq. (2.1) in $\mathbb{R}^2$ and radiation conditions (4.1) at infinity, i.e., $u \in C^2(\mathbb{R}^2) \cap SK_m(\mathbb{R}^2)$.

Exterior boundary value problems $(I)^-_m - (IV)^-_m$:
Find a regular vector $u \in C^1(\overline{\Omega}^-) \cap C^2(\Omega^-)$ satisfying Eq. (2.1) in $\overline{\Omega}^-$, the radiation conditions at infinity ($u \in SK_m(\Omega^-)$) and one of the following boundary conditions on $\partial\Omega^-$:

Problem $(\mathrm{I})^-_m : [u]^- = f;$

Problem $(\mathrm{II})^-_m : [Tu]^- = f;$

Problem $(\mathrm{III})^-_m : [(n\cdot u)]^- = f_1,\ [(t\cdot Tu)]^- = f_2;$

Problem $(\mathrm{IV})^-_m : [(t\cdot u)]^- = f_1,\ [(n\cdot Tu)]^- = f_2;$

here $n(x)$ and $t(x)$ are the exterior unit normal vector and the unit tangent vector at $x \in \partial\Omega^-$, respectively; $[\cdot]^\pm$ denote one-sided (from $\Omega^\pm$) limits on $\partial\Omega^\pm$.

Similarly, one can formulate the interior BVPs for a bounded domain Ω^+ (obviously, in this case the classes SK_m are not involved in the formulation of the BVPs).

Let u, v be regular vectors in Ω^+ and $u, v \in C^2(\Omega^+) \cap C^1(\overline{\Omega}^+)$ with $\partial\Omega^+ \in C^{2+\gamma}$, $0 < \gamma \leq 1$. Applying the Green formula

$$\int\limits_{\Omega^+}[C(D,\omega)u\cdot v - u\cdot C(D,\omega)v]dx = \int\limits_{\partial\Omega^+}\{(Tu)^+(v)^+ - (u)^+(Tv)^+\}dS \tag{4.2}$$

and Theorem 7, by standard arguments (cf. [32]) we easily get the following integral representation of a regular vector in bounded domains:

$$\begin{aligned} u(x) = \int_{\partial\Omega^+}&\{[T(D_y,n(y))\Gamma(y-x,\omega,m)]^\top[u(y)]^+ - \\ &-\Gamma(x-y,\omega,m)[T(D_y,n(y))u(y)]^+\}\,dS + \\ &+\int_{\Omega^+}\Gamma(x-y,\omega,m)C(D,\omega)u(y)\,dy, \quad x\in\Omega^+. \end{aligned} \tag{4.3}$$

Lemma 9. *Let $u \in C^1(\overline{\Omega}^-) \cap C^2(\Omega^-) \cap SK_m(\Omega^-)$, $\operatorname{supp} C(D,\omega)u$ be a compact domain in $\mathbb{R}^2$, and $S = \partial\Omega^- \in C^{2+\gamma}$, $0<\gamma\leq 1$. Then*

$$u(x) = \int_{\Omega^-}\Gamma(x-y,\omega,m)C(D,\omega)u(y)\,dy +$$

$$+\int_S \{\Gamma(x-y,\omega,m)[T(D_y,n(y))u(y)]^- -$$
$$-[T(D_y,n(y))\Gamma(y-x,\omega,m)]^\top [u(y)]^-\}\,dS,\quad x\in\Omega^-. \tag{4.4}$$

Proof. Let us write the integral representation formula for the bounded domain $\Omega_R^- = \Omega^- \cap B_R$, where $B_R = \{x : |x| < R\}$ with sufficiently large R, $S \subset B_R$, $\operatorname{supp} C(D,\omega)u \subset B_R$, $\Sigma_R = \partial B_R$. Due to (4.3)

$$u(x) = \int_{\Omega_R^-} \Gamma(x-y,\omega,m)C(D,\omega)u(y)\,dy + \int_S \{\Gamma(x-y,\omega,m)[Tu(y)]^- -$$
$$-[T(D_y,n(y))\Gamma(y-x,\omega,m)]^\top [u(y)]^-\}\,dS - \int_{\Sigma_R} \{\Gamma(x-y,\omega,m)][Tu(y)] -$$
$$-[T(D_y,n(y))\Gamma(y-x,\omega,m)]^\top [u(y)]\}\,d\Sigma_R,\quad x\in\Omega_R^-. \tag{4.5}$$

For fixed x and sufficiently large R the last summand in (4.5) is the only term depending on R. Let us integrate Eq. (4.5) with respect to R from A to $2A$, divide the result by A, and pass to the limit as $A \to +\infty$:

$$u(x) = \int_{\Omega^-} \Gamma(x-y,\omega,m)C(D,\omega)u(y)\,dy + \int_S \{\Gamma(x-y,\omega,m)[Tu(y)]^- -$$
$$-[T(D_y,n(y))\Gamma(y-x,\omega,m)]^\top [u(y)]^-\}\,dS + \lim_{A\to\infty} \Psi(A), \tag{4.6}$$

where Ψ consists of terms like

$$\Psi_{kj}(A) = \frac{1}{A}\int_A^{2A}\int_{\Sigma_R} b_{kj}(\eta) g^k(R\eta) h^j(R\eta)\,dR\,d\Sigma_R,\quad \eta = y|y|^{-1}, \tag{4.7}$$

with the scalar functions g^k and h^j satisfying the radiation conditions

$$\begin{aligned} &g^k(R\eta) = O(R^{-1/2}),\ \tfrac{\partial}{\partial R} g^k(R\eta) + i(-1)^m \lambda_k(\eta) g^k(R\eta) = O(R^{-3/2}),\\ &h^j(R\eta) = O(R^{-1/2}),\ \tfrac{\partial}{\partial R} h^j(R\eta) + i(-1)^m \lambda_j(\eta) h^j(R\eta) = O(R^{-3/2}); \end{aligned} \tag{4.8}$$

here points $\xi^p \in S_p$ correspond to the vector $\eta \in \Sigma_1$ and

$$\lambda_p(\eta) = (\eta\cdot\xi^p) > 0,\quad p = 1,2. \tag{4.9}$$

From (4.7) we have (cf. [18, 32])

$$\Psi_{kj}(A) = \frac{1}{A}\int_A^{2A}\int_{\Sigma_R} \frac{b_{kj}(\eta)}{\lambda_k(\eta)+\lambda_j(\eta)} \{[\lambda_k(\eta) g^k(R\eta)] h^j(R\eta) +$$
$$+ g^k(R\eta)[\lambda_j(\eta) h^j(R\eta)]\}\,dR\,d\Sigma_R =$$
$$= \frac{1}{A}\int_A^{2A}\int_0^{2\pi} \frac{i(-1)^{m+1} b_{kj}(\eta)}{\lambda_k(\eta)+\lambda_j(\eta)} \{\tfrac{\partial}{\partial R} g^k(R\eta) h^j(R\eta) +$$
$$+ g^k(R\eta)\tfrac{\partial}{\partial R} h^j(R\eta) + O(R^{-2})\} R\,dR\,d\varphi =$$

$$= \frac{1}{A}\int_0^{2\pi} \frac{i(-1)^{m+1} b_{kj}(\eta)}{\lambda_k(\eta)+\lambda_j(\eta)} \left\{ \int_A^{2A} \tfrac{\partial}{\partial R}[g^k(R\eta)h^j(R\eta)]R\,dR \right\} d\varphi +$$

$$+O(A^{-1}) = \frac{1}{A}\int_0^{2\pi} \frac{i(-1)^{m+1} b_{kj}(\eta)}{\lambda_k(\eta)+\lambda_j(\eta)} \{[Rg^k(R\eta)h^j(R\eta)]_A^{2A} -$$

$$- \int_A^{2A} g^k(R\eta)h^j(R\eta)\,dR\}\,d\varphi + O(A^{-1}) = O(A^{-1}).$$

Therefore $\lim_{A\to+\infty}\Psi(A) = 0$ and representation (4.4) holds. □

Corollary 10.
(i) *Let the conditions of Lemma* 9 *be fulfilled. Then*

$$\lim_{R\to+\infty}\int_{\Sigma_R} \{[T(D_y,n(y))\Gamma(y-x,\omega,m)]^\top [u(y)] -$$
$$-\Gamma(x-y,\omega,m)[Tu(y)]\}\,d\Sigma_R = 0, \quad x\in\Omega^-; \tag{4.10}$$

(ii) *The homogeneous PL_m-problem has only the trivial solution.*

Corollary 11. *Let F be a given vector function on $\mathbb{R}^2$ with compact support, i.e.,* $\operatorname{diam}\operatorname{supp}F < \infty$ *and $F \in C^{0,\gamma}(\mathbb{R}^2)$, $0<\gamma\le 1$. Then the corresponding non-homogeneous PL_m-problem is uniquely solvable and the solution can be represented by the convolution*

$$u(x) = [\Gamma(\cdot,\omega,m) * F](x) = \int_{\mathbb{R}^2} \Gamma(x-y,\omega,m)F(y)\,dy.$$

5. Uniqueness Theorems for the Exterior BVPs

Lemma 12. *Let u be a solution of the homogeneous exterior BVP $(K)_m^-$, $K = I,\cdots,IV$ $(F=0,\ f=0)$. Then $u(x) = O(|x|^{-3/2})$, as $|x|\to+\infty$.*

Proof. For definitness, let $m=1$. Due to Theorem 7 and Corollary 8 we have the asymptotic representations at infinity

$$\Gamma(x-y,\omega,1) = \sum_{j=1}^{2} r^{-1/2} C^*(\xi^j,\omega)\psi_j(\xi^j,y)e^{ix\xi^j} + O(r^{-3/2}), \tag{5.1}$$

$$[T(D_y,n(y))\Gamma(y-x,\omega,1)]^\top = -[T(D_x,n(y))\Gamma(x-y,\omega,1)]^\top =$$
$$= \sum_{j=1}^{2} r^{-1/2} C^*(\xi^j,\omega)[T(i\xi^j,n(y))]^\top \psi_j(\xi^j,y)e^{ix\xi^j} + O(r^{-3/2}), \tag{5.2}$$

where ψ_j is a scalar function and C^* is defined by (3.14).

Applying formulas (5.1), (5.2) and integral representation (4.4) for any $u \in C^1(\overline{\Omega^-}) \cap C^2(\Omega^-) \cap SK_1(\Omega^-)$ with compact supp $C(D,\omega)u$, we get

$$u(x) = \sum_{j=1}^{2} r^{-1/2} C^*(\xi^j, \omega) A_j(\xi^j) e^{ix\xi^j} + O(r^{-3/2}), \tag{5.3}$$

where the vector $A_j = (A_{j1}, A_{j2})^\top$ is uniquely defined by the vector u and does not depend on $|x|$ (in what follows we do not need the explicit expressions of A_{jk} and ψ_j).

Equation (5.3) implies

$$Tu(x) = ir^{-1/2} \sum_{j=1}^{2} T(\xi^j, n(x)) C^*(\xi^j, \omega) A_j(\xi^j) e^{ix\xi^j} + O(r^{-3/2}). \tag{5.4}$$

We need the following property of C^*: the vectors

$$\begin{aligned} V^1(\xi,\omega) &= (C^*_{11}(\xi,\omega), C^*_{21}(\xi,\omega))^\top = (\omega^2 - N(\xi), M(\xi))^\top, \\ V^2(\xi,\omega) &= (C^*_{12}(\xi,\omega), C^*_{22}(\xi,\omega))^\top = (M(\xi), \omega^2 - L(\xi))^\top, \end{aligned} \tag{5.5}$$

are linearly dependent for all $\xi \in S_1 \cup S_2$, since $\det C^*(\xi,\omega) = \Phi(\xi,\omega) = 0$, $\xi \in S_1 \cup S_2$. Therefore $V^2(\xi,\omega) = a(\xi) V^1(\xi,\omega)$, $\xi \in S_1 \cup S_2$, with $a(\xi) = [\omega^2 - L(\xi)][M(\xi)]^{-1} = [M(\xi)][\omega^2 - N(\xi)]^{-1}$.

Due to (5.3), (5.4), and (5.5)

$$u(x) = \sum_{j=1}^{2} r^{-1/2} V^1(\xi^j, \omega) \nu_j(\xi^j) e^{ix\xi^j} + O(r^{-3/2}), \tag{5.6}$$

$$Tu(x) = \sum_{j=1}^{2} ir^{-1/2} T(\xi^j, n(x)) V^1(\xi^j, \omega) \nu_j(\xi^j) e^{ix\xi^j} + O(r^{-3/2}), \tag{5.7}$$

where $\nu_j(\xi^j) = A_{j1}(\xi^j) + a(\xi^j) A_{j2}(\xi^j)$, $\quad j = 1, 2$.

Let now u be a solution of the homogeneous exterior BVP (K_1^-) and write the Green formula (4.2) for u, $v = u$ and $\Omega^+ = \Omega_R^- = \Omega^- \cap B_R$; taking into account the homogeneity of the problem, we have

$$\int_{\Sigma_R} \{T(D_x,\eta)u(x) \cdot u(x) - u(x) \cdot T(D_x,\eta)u(x)\}\, d\Sigma_R = 0,\ \eta = \frac{x}{|x|}. \tag{5.8}$$

From the convexity of curves S_j it follows that, if $\xi^j \in S_j$ corresponds to η, then $(\xi^1 - \xi^2, \eta) \neq 0$; cosequently (see (4.9))

$$\lambda_1(\eta) \neq \lambda_2(\eta), \quad \eta \in \Sigma_1. \tag{5.9}$$

Condition (5.9) enables us to carry out calculations and arguments similar to those involved in the proof of Lemma 9. After integration of Eq. (5.8) from A to $2A$ with respect to R and division by A we get

$$\frac{1}{A}\int_A^{2A}\int_{\Sigma_R}\{[T(D_x,\eta)u(x)]_k\overline{[u_k(x)]}-[u_k(x)]\overline{[T(D_x,\eta)u(x)]_k}\}\,dR\,d\Sigma_R=0.$$

Hence by (5.6) and (5.7)

$$2i\,\mathrm{Im}\left\{\frac{1}{A}\int_A^{2A}\int_{\Sigma_R}\sum_{p,j=1}^{2}\frac{i}{R}(T(\xi^j,\eta)V^1(\xi,\omega)\cdot V^1(\xi^p,\omega))\times\right.$$
$$\left.\times\nu_j(\xi^j)e^{ix\xi^j}\overline{\nu_p(\xi^p)}e^{-ix\xi^p}\,dR\,d\Sigma_R+O(A^{-1})\right\}=0. \qquad (5.10)$$

In (5.10) we have summands of the form

$$\Psi_{kj}^*(A)=\frac{1}{A}\int_A^{2A}\int_{\Sigma_R}b_{kj}(\eta)g^k(R\eta)\overline{h^j(R\eta)}\,dR\,d\Sigma_R,$$

where g^k and h^j satisfy radiation conditions (4.8).

With the help of (5.9) we can prove (see the corresponding arguments and manipulations in the proof of Lemma 9) that for different indices ($k\neq j$)

$$\Psi_{kj}^*(A)=O(A^{-1}). \qquad (5.11)$$

Thus (5.10) and (5.11) imply

$$\frac{1}{A}\int_A^{2A}\int_0^{2\pi}\sum_{j=1}^{2}\{[T(\xi^j,\eta)V^1(\xi^j,\omega)\cdot V^1(\xi^j,\omega)]|\nu_j(\xi^j)|^2\}\,dR\,d\varphi=O(A^{-1}).$$

Passing to the limit as $A\to+\infty$, from the latter equation we have

$$\sum_{j=1}^{2}\int_0^{2\pi}G(\xi^j,\eta)|\nu_j(\xi^j)|^2\,d\varphi=0, \qquad (5.12)$$

where $\xi^j\in S_j$ corresponds to η, $G(\xi,\eta)=(T(\xi,\eta)V^1(\xi,\omega)\cdot V^1(\xi,\omega)$. In what follows we will prove that $G(\xi^j,\eta)$ is a positive function. To this end, note that $[M(\xi^j)]^2=[\omega^2-L(\xi^j)][\omega^2-N(\xi^j)]$ for $\xi^j\in S_j$ and rewrite $G(\xi^j,\eta)$ in the form (see (2.2))

$$G(\xi^j,\eta)=[\omega^2-N(\xi^j)]\{T_{11}(\xi^j,\eta)[\omega^2-N(\xi^j)]$$
$$+[T_{12}(\xi^j,\eta)+T_{21}(\xi^j,\eta)]M(\xi^j)+T_{22}(\xi^j,\eta)[\omega^2-L(\xi^j)]\}. \qquad (5.13)$$

Applying Eqs. (2.6) and

$$D_j\Phi(\xi,\omega)=-\{[\omega^2-N(\xi)]D_jL(\xi)+[\omega^2-L(\xi)]D_jN(\xi)+2M(\xi)D_jM(\xi)\}$$

in (5.13), we get $G(\xi^j,\eta) = -\frac{1}{2}[\omega^2 - N(\xi^j)](n(\xi^j)\cdot\nabla\Phi(\xi^j,\omega))$, since $n(\xi^j) = \eta$, where $n(\xi^j)$ is the outward unit normal at $\xi^j \in S_j$. From the latter equation by (3.34) we have

$$G(\xi^j,\eta) = (-1)^{j+1}2^{-1}[\omega^2 - N(\xi^j)]|\nabla\Phi(\xi^j,\omega)|, \quad \xi^j \in S_j. \tag{5.14}$$

Obviously, for $\xi \in S_j$ $(j = 1,2)$:

$$\omega^2 - N(\xi) = \omega^2 - \omega^2 k_j(\varphi)N(\zeta) = \omega^2[1 - k_j(\varphi)N(\zeta)], \quad \zeta = |\xi|^{-1}\xi.$$

Let us fix some $\varphi \in [0,2\pi]$ and consider

$$[1 - k_1(\varphi)N(\zeta)][1 - k_2(\varphi)N(\zeta)] = 1 - [k_1(\varphi) + k_2(\varphi)]N(\zeta) + \\ +k_1(\varphi)k_2(\varphi)N^2(\zeta) = -M^2(\zeta)[L(\zeta)N(\zeta) - M^2(\zeta)]^{-1}.$$

Due to (3.6) we deduce

$$[1 - k_1(\varphi)N(\zeta)][1 - k_2(\varphi)N(\zeta)] < 0 \tag{5.15}$$

for all $\varphi \in [0,2\pi]$ except for solutions of the equation $M(\zeta) = M(\cos\varphi,\sin\varphi) = 0$. Now the inequalities $k_1(\varphi) < k_2(\varphi)$ and $N(\zeta) > 0$ together with (5.15) imply $1 - k_1(\varphi)N(\zeta) > 0$, $1 - k_2(\varphi)N(\zeta) < 0$, almost everywhere on $[0,2\pi]$, i.e., $(-1)^{j+1}[\omega^2 - N(\xi)] > 0$, $\xi \in S_j$, $j = 1,2$. Therefore by (5.14)

$$G(\xi^j,\eta) = 2^{-1}|\omega^2 - N(\xi^j)|\,|\nabla\Phi(\xi^j,\omega)| > 0, \quad \xi^j \in S_j. \tag{5.16}$$

Consequently, from (5.12) and (5.16) it follows that $\nu_j(\xi^j) = 0$, which together with (5.6) proves Lemma 13. $\square$

Corollary 13. *Let u be a solution of the homogeneous exterior BVP $(K)^-_m$ $(K = I,\cdots,IV)$, $m = 1$ or $m = 2$. Then*

$$u \in SK_1(\Omega^-) \cap SK_2(\Omega^-). \tag{5.17}$$

Theorem 14. *The exterior homogeneous BVP $(K)^-_m$ $(K = I,\cdots,IV)$ has only the trivial solution.*

Proof. First let us note that a solution of homogeneous equation (2.1) is an analytic vector-function of two real variables x_1 and x_2 in Ω^-, since $C(D,\omega)$ is an elliptic operator.

Let u be a solution of the homogeneous BVP $(K)^-_m$. Denote by Σ_R a circumference with center at the origin $O \in \Omega^+$ and radius $R > 0$ such that the corresponding circle B_R contains the curve $S = \partial\Omega^\pm$. It is obvious that the vector u admits an extension from $\Omega^-\backslash B_R = \mathbb{R}^2\backslash B_R$ onto the whole plane $\mathbb{R}^2$ preserving the C^∞-smoothness. Let v be one of such extensions: $v \in C^\infty(\mathbb{R}^2)$, $v|_{\Omega^-\backslash B_R} = u|_{\Omega^-\backslash B_R}$ and denote

$$F(x) = C(D,\omega)v(x), \quad x \in \mathbb{R}^2. \tag{5.18}$$

It is obvious that v is a solution of PL_m-Problem (5.18) with compact supp $F \subset B_R$ and according to Corollary 13 $v \in SK_1(\mathbb{R}^2) \cap SK_2(\mathbb{R}^2)$, since u satisfies condition (5.17).

Due to Corollary 11 we have two representations for v:

$$v(x) = [\Gamma(\cdot,\omega,1) * F](x), \quad v(x) = [\Gamma(\cdot,\omega,2) * F](x). \tag{5.19}$$

Equalities (5.19) imply $\hat{v}(\xi) = \hat{\Gamma}(\xi,\omega,1)\hat{F}(\xi) = \hat{\Gamma}(\xi,\omega,2)\hat{F}(\xi)$, i.e.,

$$[\hat{\Gamma}(\xi,\omega,1) - \hat{\Gamma}(\xi,\omega,2)]\hat{F}(\xi) = 0, \tag{5.20}$$

where $\hat{F}(\xi)$ is an analytic vector of two complex variables ξ_1, ξ_2 and satisfies specific conditions at infinity, since it is the Fourier transform of the vector F with compact support (see [28]).

Note that by (3.47)

$$\Gamma(x,\omega,1) - \Gamma(x,\omega,2) = \sum_{j=1}^{2} \int_{S_j} \mathcal{D}^j(\xi) e^{-ix\xi}\, dS_j,$$

where

$$\mathcal{D}^j(\xi) = \frac{i}{4\pi} \frac{(-1)^j (n(\xi)\cdot\xi)}{\omega^2[(L-N)^2 + 4M^2]^{1/2}} C^*(\xi,\omega), \;\; \xi \in S_j.$$

Theorem 7 implies that $\Gamma(\cdot,\omega,m)$, $m = 1,2$, are regular functionals on the space of rapidly decreasing functions $\mathcal{S}(\mathbb{R}^2)$ (see [28]) and we have (for arbitrary $g = (g_1,g_2)^\top \in \mathcal{S}(\mathbb{R}^2)$)

$$\begin{aligned}\langle[\Gamma(x,\omega,1) - \Gamma(x,\omega,2)], g(x)\rangle &= \int_{\mathbb{R}^2} [\Gamma(x,\omega,1) - \Gamma(x,\omega,2)]\overline{g(x)}\, dx = \\ = \sum_{j=1}^{2} \int_{S_j} \mathcal{D}^j(\xi)\overline{\hat{g}(\xi)}\, dS_j &= \langle (2\pi)^{-2}[\hat{\Gamma}(\xi,\omega,1) - \hat{\Gamma}(\xi,\omega,2)], \hat{g}(\xi)\rangle, \end{aligned} \tag{5.21}$$

and consequently $\hat{\Gamma}(\cdot,\omega,1) - \hat{\Gamma}(\cdot,\omega,2)$ is concentrated on $S_1 \cup S_2$, i.e.,

$$\operatorname{supp}\, [\hat{\Gamma}(\xi,\omega,1) - \hat{\Gamma}(\xi,\omega,2)] = S_1 \cup S_2.$$

Now (5.20) together with (5.21) yields

$$\langle [\hat{\Gamma}(\xi,\omega,1) - \hat{\Gamma}(\xi,\omega,2)]\hat{F}, \hat{g}\rangle = 4\pi^2 \sum_{j=1}^{2} \int_{S_j} \mathcal{D}^j_{pq}(\xi)\hat{F}_q(\xi)\overline{\hat{g}_p(\xi)}\, dS_j = 0,$$

i.e.,

$$C^*(\xi,\omega)\hat{F}(\xi) = 0, \;\; \xi \in S_j, \;\; j = 1,2. \tag{5.22}$$

From the analyticity of $\hat{F}(\xi)$, Eq. (5.22), and condition (3.8) it follows that (cf. [18])

$$C^*(\xi,\omega)\hat{F}(\xi)=\Phi(\xi,\omega)\hat{H}(\xi),\quad \xi\in\mathbb{R}^2, \tag{5.23}$$

where $\hat{H}(\xi)$ is the analytic vector of two complex variables ξ_1, ξ_2 and satisfies at infinity the same estimates as $\hat{F}$; therefore $\hat{H}(\xi)$ is also the Fourier transform of some vector $H(x)$ with compact support and, evidently, $H\in SK_m(\mathbb{R}^2)$. By (5.18)

$$\hat{F}(\xi)=C(-i\xi,\omega)\hat{v}(\xi), \tag{5.24}$$

where $C(-i\xi,\omega)=I\omega^2-C(\xi)$. Note that

$$C^*(\xi,\omega)C(-i\xi,\omega)=C(-i\xi,\omega)C^*(\xi,\omega)=\Phi(\xi,\omega)I. \tag{5.25}$$

Substituting (5.24) into (5.23), with regard to (5.25) we get

$$C(-i\xi,\omega)C^*(\xi,\omega)[\hat{v}(\xi)-\hat{H}(\xi)]=0$$

and by the inverse Fourier transform

$$C(D,\omega)[C^*(D)+\omega^2 I][v(x)-H(x)]=0,$$

where

$$C^*(D)=\begin{Vmatrix} N(D) & -M(D) \\ -M(D) & L(D) \end{Vmatrix}.$$

Since $[C^*(D)+\omega^2 I][v(x)-H(x)]$ belongs to the class $SK_m(\mathbb{R}^2)$, it vanishes by Corollary 10. Therefore

$$\begin{Vmatrix} N(D)+\omega^2 & M(D) \\ M(D) & L(D)+\omega^2 \end{Vmatrix}\begin{pmatrix} v_1-H_1 \\ H_2-v_2 \end{pmatrix}=0,\quad x\in\mathbb{R}^2.$$

Applying again Corollary 10, we have $v(x)-H(x)=0$, $x\in\mathbb{R}^2$. Thus the vector $v(x)$ has a compact support. But then the same is valid for the vector u and consequently by analyticity $u(x)=0$, $x\in\Omega^-$. $\square$

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(Received 10.12.1993)

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