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ON KRULL DIMENSION OF ORE EXTENSIONS

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ABSTRACT. The Krull dimension of rings of skew polynomials is studied. Earlier the problem of Krull dimension was investigated only for some particular cases, namely, for Weyl algebras [2], a ring of differential operators [7,8], as well as for rings of Laurent skew polynomials [9–10].

Let R be a ring with unity and let $R[x]$ be a ring of left polynomials (i.e., polynomials with coefficients from the left to the powers of x) over R. Suppose that α is an endomorphism of R and δ is an α-differentiation of R, i.e., $\delta(a+b)=\delta(a)+\delta(b)$ and $\delta(ab)=\delta(a)b+\alpha(a)\delta(b)$ for any $a,b\in R$. Let $R[x;\alpha;\delta]$ denote the ring of left skew polynomials over R [1] (the additive group of this ring coincides with the one of $R[x]$ and the multiplication in it is defined by means of operators in $R[x]$ and the following commutation formula:

$$x\cdot a=\alpha(a)x+\delta(a),\quad a\in R). \tag{1}$$

If δ is the zero mapping of R, we use the notation $R[x;\alpha]$ for $R[x;\alpha;\delta]$.

Denote by $\mathrm{K.dim}(A)$ the Krull dimension of a ring A in the sense of Gabriel and Rentschler (i.e., the deviation of the set of left ideals of A) [2].

Theorem 1. *Let R be a ring with unity, let α be its automorphism, and let δ be a nilpotent ($\delta^d=0$) α-differentiation of R. Suppose that $\delta^{-i}(1_R)\neq\varnothing$ for $i=1,2,\dots,d-1$. Then*

$$\mathrm{K.dim}(R[x;\alpha;\delta])=\mathrm{K.dim}(R[x;\alpha])=\mathrm{K.dim}(R[x]).$$

Theorem 1 is a trivial consequence of the following propositions:

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Proposition 1. *Let R be a ring with unity, let α be its automorphism, and let δ be a nilpotent ($\delta^d = 0$) α-differentiation of R. If $\delta^{-i}(1_R) \neq \varnothing$ for $i = 1, 2, \dots, d-1$, then*

$$\mathrm{K.\,dim}(R[x;\alpha]) \leq \mathrm{K.\,dim}(R[x;\alpha;\delta]).$$

Proposition 2. *Let α be an injective endomorphism of a ring R with unity satisfying*

$$\alpha(\mathfrak{a}) < R\alpha(\mathfrak{m}) \Rightarrow \mathfrak{a} < \mathfrak{m}$$

for any left ideals $\mathfrak{a}$ and $\mathfrak{m}$ of R, where

$$R\alpha(\mathfrak{m}) = \Big\{ \sum_{p=1}^{n} \lambda_p \alpha(m_p);\ \ m_p \in \mathfrak{m};\ \ \lambda_p \in R \Big\}.$$

Then

$$\mathrm{K.\,dim}(R[x]) \leq \mathrm{K.\,dim}(R[x;\alpha]).$$

Proposition 3. *Let α be an automorphism of a ring R with unity. Then*

$$\mathrm{K.\,dim}(R[x;\alpha;\delta]) \leq \mathrm{K.\,dim}(R[x]).$$

The proofs of these propositions as well as of the other ones given in this paper are based on

Lemma 1 [2]. *Let E and F be partially ordered sets. If there exists a strictly isotonic mapping $\Phi : E \to F$, then* $\operatorname{dev} E \leq \operatorname{dev} F$.

To prove Proposition 1, we shall also need

Lemma 2. *Let α be an automorphism of a ring R with unity, and let δ be an α-differentiation of R. Then the condition*

$$f_1c_1 + f_2c_2 + \dots + f_nc_n = c, \tag{2}$$

where $f_1, f_2, \dots, f_n \in R[x;\alpha;\delta]$ and $c_1, c_2, \dots, c_n, c \in R$, implies

$$c = b_1c_1 + b_2c_2 + \dots + b_nc_n$$

with $b_1, b_2, \dots, b_n \in R$.

Proof. Suppose that k is the maximum of degrees of polynomials $f_1, f_2, \dots, f_n$. We can write these polynomials as

$$\begin{aligned}
&f_1 = a_{1k}x^k + a_{1,k-1}x^{k-1} + \dots + a_{10},\\
&f_2 = a_{2k}x^k + a_{2,k-1}x^{k-1} + \dots + a_{20},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
&f_n = a_{nk}x^k + a_{n,k-1}x^{k-1} + \dots + a_{n0},\\
&a_{ip} \in R,\ \ i = 1, 2, \dots, n,\ \ p = 0, 1, \dots, k.
\end{aligned}$$

Here some of the a_{ip}'s may be equal to zero.

Calculate the left side of (2) (using a^α and a^δ instead of $\alpha(a)$ and $\delta(a)$).

$$
\begin{aligned}
f_1c_1 + f_2c_2 + \cdots + f_nc_n &= \sum_{i=1}^n a_{ik}c_i^{\alpha^k}x^k + \\
&+\sum_{i=1}^n \Big(a_{ik}(c_i^{\delta\alpha^{k-1}} + c_i^{\alpha\delta\alpha^{k-2}} + \cdots + c_i^{\alpha^{k-1}\delta}) + a_{i,k-1}c_i^{\alpha^{k-1}}\Big)x^{k-1} + \cdots + \\
&+\sum_{i=1}^n \Big(a_{ik}(c_i^{\delta^d\alpha^{k-d}} + c_i^{\delta^{d-1}\alpha\delta\alpha^{k-d-1}} + \cdots + c_i^{\delta^{d-1}\alpha^{k-d}\delta} + c_i^{\delta^{d-2}\alpha\delta^2\alpha^{k-d-1}} + \\
&+ \cdots + c_i^{\alpha^{k-d}\delta^d}) + a_{i,k-1}(c_i^{\delta^{d-1}\alpha^{k-d}} + \cdots + c_i^{\alpha^{k-d}\delta^{d-1}}) + \cdots + \\
&+a_{id}c_i^{\alpha^d}\Big)x^d + \cdots + \sum_{i=1}^n (a_{ik}c_i^{\delta^k} + a_{i,k-1}c_i^{\delta^{k-1}} + \cdots + a_{i0}c_i).
\end{aligned}
$$

Taking into consideration (2), we have

$$
\begin{aligned}
&\sum_{i=1}^n a_{ik}c_i^{\alpha^k} = 0,\\
&\sum_{i=1}^n \Big(a_{ik}(c_i^{\delta\alpha^{k-1}} + c_i^{\alpha\delta\alpha^{k-2}} + \cdots + c_i^{\alpha^{k-1}\delta} + a_{i,k-1}c_i^{\alpha^{k-1}}\Big) = 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\sum_{i=1}^n \Big(a_{ik}(c_i^{\delta^{k-1}\alpha} + c_i^{\delta^{k-2}\alpha\delta} + \cdots + c_i^{\alpha\delta^{k-1}}) + \\
&\qquad\qquad + a_{i,k-1}(c_i^{\delta^{k-2}\alpha} + \cdots + c_i^{\alpha\delta^{k-2}}) + \cdots + a_{i1}c_i^{\alpha}\Big) = 0,\\
&\sum_{i=1}^n (a_{ik}c_i^{\delta^k} + a_{i,k-1}c_i^{\delta^{k-2}} + \cdots + a_{i0}c_i) = c.
\end{aligned}
$$

Using these equalities and calculating $\big(\sum_{i=1}^n a_{ik}c_i^{\alpha^k}\big)^{(\alpha^{-1}\delta)^k}$, we obtain

$$
\begin{aligned}
0 &= \Big(\sum_{i=1}^n a_{ik}^{\alpha^{-1}} c_i^{\alpha^{k-1}}\Big)^{\delta(\alpha^{-1}\delta)^{k-1}} = \Big(\sum_{i=1}^n (a_{ik}^{\alpha^{-1}\delta} c_i^{\alpha^{k-1}} + a_{ik}c_i^{\alpha^{k-1}\delta})\Big)^{(\alpha^{-1}\delta)^{k-1}} = \\
&= \Big(\sum_{i=1}^n (a_{ik}^{\alpha^{-1}\delta} c_i^{\alpha^{k-1}} - a_{ik}(c_i^{\delta\alpha^{k-1}} + \cdots + c_i^{\alpha^{k-2}\delta\alpha}) - a_{i,k-1}c_i^{\alpha^{k-1}})\Big)^{(\alpha^{-1}\delta)^{k-1}} = \\
&= \Big(\sum_{i=1}^n (a_{ik}^{\alpha^{-1}\delta\alpha^{-1}\delta} c_i^{\alpha^{k-2}} + a_{ik}^{\alpha^{-1}\delta} c_i^{\alpha^{k-2}\delta} - a_{ik}^{\alpha^{-1}\delta}(c_i^{\delta\alpha^{k-2}} + \cdots + c_i^{\alpha^{k-2}\delta}) -
\end{aligned}
$$

$$-a_{ik}(c_i^{\delta\alpha^{k-2}\delta}+\cdots+c_i^{\alpha^{k-2}\delta^2})-a_{i,k-1}^{\alpha^{-1}\delta}c_i^{\alpha^{k-2}}-a_{i,k-1}c_i^{\alpha^{k-2}\delta})\Big)^{(\alpha^{-1}\delta)^{k-2}}=$$

$$=\Big(\sum_{i=1}^{n}\big(a_{ik}^{(\alpha^{-1}\delta)^2}c_i^{\alpha^{k-2}}-a_{ik}^{\alpha^{-1}\delta}(c_i^{\delta\alpha^{k-2}}+\cdots+c_i^{\alpha^{k-3}\delta\alpha})+a_{ik}(c_i^{\delta^2\alpha^{k-2}}+$$

$$+c_i^{\delta\alpha\delta\alpha^{k-3}}+\cdots+c_i^{\alpha^{k-3}\delta^2\alpha})+a_{i,k-1}(c_i^{\delta\alpha^{k-2}}+\cdots+c_i^{\alpha^{k-3}\delta\alpha})-$$

$$-a_{i,k-1}^{\alpha^{-1}\delta}c_i^{\alpha^{k-2}}+a_{i,k-2}c_i^{\alpha^{k-2}}\big)\Big)^{(\alpha^{-1}\delta)^{k-2}}=\cdots=\Big(\sum_{i=1}^{n}\big(a_{ik}^{(\alpha^{-1}\delta)^{k-1}}c_i^{\alpha}-$$

$$-a_{ik}^{(\alpha^{-1}\delta)^{k-2}}c_i^{\delta\alpha}+\cdots+(-1)^{k-1}a_{ik}c_i^{\delta^{k-2}\alpha}-a_{i,k-1}^{(\alpha^{-1}\delta)^{k-2}}c_i^{\alpha}+$$

$$+a_{i,k-1}^{(\alpha^{-1}\delta)^{k-3}}c_i^{\delta\alpha}+\cdots+(-1)^{k-1}a_{i,k-1}c_i^{\delta^{k-3}\alpha}+\cdots+(-1)^{k-1}a_{i1}c_i^{\alpha}\Big)^{\alpha^{-1}\delta}=$$

$$=\sum_{i=1}^{n}\big(a_{ik}^{(\alpha^{-1}\delta)^{k}}c_i-a_{i,k-1}^{(\alpha^{-1}\delta)^{k-1}}c_i+\cdots+$$

$$+(-1)^{k-1}a_{i1}^{\alpha^{-1}\delta}c_i+(-1)^{k-1}(a_{ik}c_i^{\delta^k}+a_{i,k-1}c_i^{\delta^{k-1}}+\cdots+a_{i1}c_i^{\delta})\big),$$

and therefore

$$c=\sum_{i=1}^{n}\Big((-1)^k\big(a_{ik}^{(\alpha^{-1}\delta)^k}-a_{i,k-1}^{(\alpha^{-1}\delta)^{k-1}}+\cdots+(-1)^{k-1}a_{i1}^{\alpha^{-1}\delta}\big)+a_{i0}\Big)c_i. \qquad \square$$

Proof of Proposition 1. Taking into account that the Krull dimension of the ring A is equal to $\operatorname{dev}(\operatorname{Id}A)$, where $\operatorname{Id}A$ is the set of left ideals of A, by Lemma 1 it suffices to construct a strictly isotonic mapping F from the set of left ideals of $R[x;\alpha]$ into the set of left ideals of $R[x;\alpha;\delta]$.

First of all, let us show by induction on k that for any $a,b\in R$ we can write $\delta^k(a\cdot b)$ as

$$\delta^k(ab)=a_k\delta^k(b)+a_{k-1}\delta^{k-1}(b)+\cdots+a_1\delta(b)+a_0b, \tag{3}$$

where $a_0,a_1,\dots,a_k\in R$ are the coefficients found from the representation

$$x^ka=a_kx^k+a_{k-1}x^{k-1}+\cdots+a_0.$$

Indeed, if $k=1$, then

$$\delta(ab)=\alpha(a)\delta(b)+\delta(a)b \ \text{ and } \ xa=\alpha(a)x+\delta(a).$$

Suppose the validity of (3) for some natural k. Then

$$\delta^{k+1}(ab)=\delta(\delta^k(ab))=\delta(a_k\delta^k(b)+a_{k-1}\delta^{k-1}(b)+\cdots+a_0b)=$$
$$=a'_{k+1}\delta^{k+1}(b)+a'_k\delta^k(b)+\cdots+a'_0b,$$

where $a'_{k+1}=\alpha(a_k)$, $a'_0=\delta(a_0)$, and $a'_n=\delta(a_n)+\alpha(a_{n-1})$ for $n=1,2,\dots,k$.

On the other hand,

$$x^{k+1}a = x(x^k a) = x(a_k x^k + a_{k-1}x^{k-1} + \dots + a_0) =$$
$$= \alpha(a_k)x^{k+1} + (\delta(a_k) + \alpha(a_{k-1}))x^k + \dots + (\delta(a_1) + \alpha(a_0))x + \delta(a_0) =$$
$$= a'_{k+1}x^{k+1} + a'_k x^k + \dots + a'_0.$$

The induction is thus completed.

Now calculate $x^d a$. Suppose that

$$x^d a = a_d x^d + a_{d-1}x^{d-1} + \dots + a_0, \quad a_0, a_1, \dots, a_d \in R.$$

We have

$$\delta^d(ab) = a_d\delta^d(b) + a_{d-1}\delta^{d-1}(b) + \dots + a_0 b, \quad b \in R,$$

where $a_0 = \delta^d(a)$. Since $\delta^d = 0$, we obtain

$$a_{d-1}\delta^{d-1}(b) + \dots + a_2\delta^2(b) + a_1\delta(b) = 0. \tag{4}$$

Since b is an arbitrary element of R and $\delta^{-1}(1_R) \neq \varnothing$, we can assume in (4) that $b \in \delta^{-1}(R)$. Taking into account that $\delta(0) = 0$ and $\delta(1_R) = \delta(1_R \cdot 1_R) = \delta(1_R) + \delta(1_R) = 0$, we have $a_1 = 0$. Further, assuming that $b \in \delta^{-2}(1_R)$, we obtain $a_2 = 0$, and so on. Finally, we have

$$a_0 = a_1 = \dots = a_{d-1} = 0.$$

Therefore $x^d a = a_d x^d$. Using the commutation formula (1), we easily obtain that $a_d = \alpha^d(a)$, and hence

$$x^d a = \alpha^d(a)x^d.$$

Moreover, if $p = md + q$, we can write

$$x^p a = x^q \alpha^{md}(a)x^{md}, \quad p, q, m \in \mathbb{N} \cup \{0\}. \tag{5}$$

Now we begin the construction of the mapping F.

Let I be an arbitrary left ideal of $R[x;\alpha]$. Let n be the minimum of degrees of nonzero polynomials from I. For any $k \geq n$ denote by $\mathfrak{a}_k$ the set of highest-degree coefficients of kth polynomials from I. Thus we obtain the sequence of left ideals of R:

$$\mathfrak{a}_n, \mathfrak{a}_{n+1}, \dots$$

satisfying

$$\alpha(\mathfrak{a}_{n+i}) \subseteq \mathfrak{a}_{n+i+1} \ \text{ for } \ i \in \mathbb{N} \cup \{0\}. \tag{6}$$

(This means that this sequence is α-nondecreasing.)

Consider all monomials of the form

$$\alpha^{(n+i)(d-1)}(a_{n+i})x^{(n+i)d}, \quad a_{n+i} \in \mathfrak{a}_{n+i}, \quad i \in \mathbb{N} \cup \{0\}, \tag{7}$$

and let J be the left ideal of $R[x;\alpha;\delta]$ generated by them. Define the mapping F by

$$F(I) = J,$$

and show that it is strictly isotonic.

Let us first study the structure of J. Taking into account (5) and the set of generators of J, we easily check that any polynomial from J with degree divisible by d is a monomial. Let

$$g = \alpha^{r(d-1)}(a_r)x^{dr}, \quad a_r \in \mathfrak{a}_r,$$

be any of them, and suppose that

$$g = \sum_{\lambda=1}^{m} f_\lambda g_\lambda, \quad f_\lambda \in R[x;\alpha;\delta],$$

where g_λ is of the form (7).

Take any $\lambda \in \{1, 2, \ldots, m\}$ and let the degree of g_λ be $d \cdot k$. By (5) we can assume: if $k > r$, then $f_\lambda = 0$; if $k = r$, then the degree of f_λ is less than d; if $k < r$, then f_λ contains only terms of the degree from the interval $[dr - dk; d(r+1) - dk)$ of natural numbers.

Suppose that

$$g_\lambda = \alpha^{k(d-1)}(a_k)x^{dk}, \quad k < r, \quad a_k \in \mathfrak{a}_k.$$

We can assume that

$$f_\lambda = c_1 x^{d(r+1)-1-dk} + \cdots + c_d x^{dr-dk}; \quad c_1, c_2, \ldots, c_d \in R.$$

Then using (5), we obtain

$$f_\lambda \cdot g_\lambda = (c_1 x^{d-1} + c_2 x^{d-2} + \cdots + c_d)\alpha^{dr-dk}(\alpha^{k(d-1)}(a_k))x^{dr}.$$

But

$$\alpha^{dr-dk}\big(\alpha^{k(d-1)}(a_k)\big) = \alpha^{rd-k}(a_k) = \alpha^{r(d-1)}\big(\alpha^{r-k}(a_k)\big).$$

By (6), $\alpha^{r-k}(a_k) \in \mathfrak{a}_r$. Therefore any product $f_\lambda \cdot g_\lambda$ such that the degree of g_λ is less than the degree of g can be replaced by the product $f'_\lambda \cdot g'_\lambda$, where the degree of f'_λ is less than d, the degree of g'_λ is equal to the degree of g, and $g'_\lambda \in J$. Hence by Lemma 2 the coefficient of g can be rewritten as

$$\sum_{\lambda=1}^{m} b_\lambda \alpha^{r(d-1)}(a_{r_\lambda}), \quad b_\lambda \in R.$$

Since $\mathfrak{a}_r$ is an ideal, we conclude that

$$a = \alpha^{r(d-1)}(a) \quad \text{with} \quad a \in \mathfrak{a}_r. \tag{8}$$

Now we can show that the mapping F constructed above is strictly isotonic.

Let I_1 be any left ideal of $R[x;\alpha]$ such that $I \subset I_1$, and let $F(I_1) = J_1$. We have to show that $J \subset J_1$. By the construction of F it is clear that $J \subseteq J_1$.

Since $I_1 \supset I$, there exists a polynomial

$$h = b_k x^k + b_{k-1}x^{k-1} + \cdots + b_0, \quad b_k, b_{k-1}, \ldots, b_0 \in R,$$

in I_1 such that it is not an element of I. More than that we can assume that b_k cannot be obtained as a left linear combination of the highest coefficients of polynomials of kth degree from I. Indeed, if it is not the case, we can take the appropriate difference $h - \sum_\mu c_\mu f_\mu$ which will be an element of $I_1 \backslash I$ whose degree will be less than that of h, and so on.

Obviously, $\alpha^{k(d-1)}(b_k)x^{dk} \in J_1$. Let us show that this monomial does not belong to J. Suppose the contrary. Then by (8),

$$b_k x^k \in I$$

which contradicts the choice of h. $\square$

Before proving Proposition 2 note that for the first time an endomorphism of the type considered in this proposition has been studied by Lesieur [3]. He proved that the condition

$$\alpha(a) \subset R\alpha(\mathfrak{m}) \Rightarrow \mathfrak{a} \subset \mathfrak{m}$$

is equivalent to

$$\alpha(a) = \sum_{j=1}^{n} \lambda_j \alpha(b_j) \Rightarrow a = \sum_{j=1}^{n} \mu_j b_j, \quad a, \lambda_j, \mu_j, b_j \in R. \tag{9}$$

It was also shown by him that elements of the left ideal

$$\underbrace{\underbrace{R\alpha}(R\alpha(\cdots(R\alpha(\mathfrak{a}))\cdots))}_{n \text{ times}}$$

of R generally have the form

$$\sum_{i=1}^{d} \mu_i \alpha^n(a_i), \quad \mu_i \in R; \quad a_i \in \mathfrak{a}. \quad \square$$

Proof of Proposition 2. By Lemma 1 it suffices to construct a strictly isotonic mapping F from the set of left ideals of $R[x]$ into the set of left ideals of $R[x;\alpha]$.

Let I be any left ideal of $R[x]$. Let n be the minimum of the degrees of nonzero polynomials from I. Consider the nondecreasing sequence

$$\mathfrak{a}_n \subseteq \mathfrak{a}_{n+1} \subseteq \cdots, \quad n \in \mathbb{N} \cup \{0\}, \tag{10}$$

of left ideals of R, where $\mathfrak{a}_{n+i}$ $(i=0,1,2,\dots)$ is the set (in fact, the left ideal) of highest coefficients of all polynomials of degree $n+i$ from I. With the help of this sequence we can construct the α-nondecreasing sequence of left ideals of R:

$$\mathfrak{a}'_n \overset{\alpha}{\hookrightarrow} R\alpha(\mathfrak{a}'_{n+1}) \overset{\alpha}{\hookrightarrow} R\alpha(R\alpha(\mathfrak{a}'_{n+2})) \overset{\alpha}{\hookrightarrow} \cdots, \tag{11}$$

where $\mathfrak{a}'_{n+i} = \underbrace{\underbrace{R\alpha}(R\alpha(\cdots(R\alpha(\mathfrak{a}_{n+i}))\cdots))}_{n \text{ times}}$, $i=0,1,2,\dots$.

Consider all monomials of the form

$$a_{n+i}x^{n+i}, \quad a_{n+i} \in \underbrace{R\alpha(R\alpha(\cdots(R\alpha(\mathfrak{a}'_{n+i}))\cdots))}_{i \text{ times}}, \quad i=0,1,\dots, \tag{12}$$

and let J be the left ideal of $R[x;\alpha]$ generated by them.

Define the mapping F by

$$F(I)=J.$$

Let $I_1 \supset I$ be any left ideal, and let $J_1 = F(I_1)$. It follows from the construction of F that $J \subseteq J_1$.

As in proving Proposition 1, choose a polynomial

$$g = a_d x^d + \cdots + a_0, \quad a_0, \cdots, a_d \in R, \quad d \in \mathbb{N} \cup \{0\},$$

from I_1 which does not belong to I. Here we can also assume that the highest term of g cannot be obtained as a left linear combination of highest terms of the polynomials from g having the same degree. Obviously, $\alpha^d(a_d)x^d \in J_1$. Show that this monomial is not in J.

Suppose the contrary. Then the monomial $\alpha^d(a_d)x^d$ can be represented as the sum of the products of monomials of the form

$$c_{d-p}x^{d-p} \cdot a_p x^p, \quad c_{d-p} \in R, \quad a_p \in \underbrace{R\alpha(R\alpha(\cdots(R\alpha(\mathfrak{a}_p))\cdots))}_{p \text{ times}}, \tag{13}$$

where $p \le d$ and $a_p = \sum_{i=1}^k \mu_i \alpha^p(b_i)$, $b_i \in \mathfrak{a}_p$.

If we carry out the multiplication in (13), we obtain

$$c_{d-p}x^{d-p}a_p x^p = \sum_{i=1}^k \lambda_i \alpha^d(b_i)x^d,$$

where $\lambda_i = c_{d-p}\alpha^{d-p}(\mu_i)$, $\lambda_i \in R$. Moreover, by (11) we can assume that $b_i \in \mathfrak{a}_d$ $(i=1,2,\dots,k)$.

Thus we have

$$\alpha^d(a_d) = \sum_{j=1}^n \sum_{i=1}^k \lambda_{ij}\alpha^d(b_{ij}), \quad \lambda_{ij} \in R; \quad b_{ij} \in \mathfrak{a}_d.$$

This equality can be rewritten as

$$\alpha^d(a_d) = \sum_{l=1}^{q} \nu_l \alpha^d(b_l), \quad \nu_l \in R, \quad b_l \in \mathfrak{a}_d, \quad q \le nk.$$

Using here equality (9) d times, we get

$$a_d = \sum_{l=1}^{q} \beta_l b_l, \quad \beta_l \in R, \quad b_l \in \mathfrak{a}_d.$$

But this contradicts the choice of g.

Thus we have proved that F is strictly isotonic. □

Proof of Proposition 3. Let I be any left ideal of $R[x;\alpha;\delta]$. Consider all the monomials of the form

$$\alpha^{-n}(a_n)x^n, \quad a_n \in \mathfrak{a}_n, \tag{14}$$

where $\mathfrak{a}_n$ is the left ideal of all polynomials of degree n from I.

Let J be the left ideal of $R[x]$ generated by monomials (14). Define the mapping F from the set of left ideals of $R[x;\alpha;\delta]$ into the set of left ideals of $R[x]$ by

$$F(I) = J.$$

Since α is the automorphism of R and $\alpha^{-n}(\mathfrak{a}_n) \subseteq \alpha^{-(n+1)}(\mathfrak{a}_{n+1})$, $\forall n \in \mathbb{N}$, it can be proved quite analogously to the proofs of Propositions 1 and 2 that F is strictly isotonic. □

Remark 1. As an example let us show that the equality given in Proposition 3 can be strict.

Let $R = K[y]$, where K is a field of characteristic zero. Let α be the identical automorphism of R, and let δ be the partial differentiation by the variable y. Then δ is the α-differentiation of R, and thus we obtain the ring of skew polynomials $R[x;\alpha;\delta]$ which can be considered as the Weyl algebra over K. Its Krull dimension is equal to 1 [2]. On the other hand, $R[x] = K[y][x] = K[y;x]$. Therefore, the Krull dimension of $R[x]$ is 2.

Remark 2. Let R be any division ring and let α be its automorphism. Then the definition of the Krull dimension and Proposition 3 imply that

$$\mathrm{K.dim}(R[x;\alpha;\delta]) = 1.$$

Taking into consideration the already known results concerning the Krull dimension of polynomial rings, from Theorem 1 we obtain

Corollary 1. *Let R be a left Noetherian ring with the Krull dimension n. Let α be an automorphism of R and let δ be a nilpotent ($\delta^d = 0$) α-differentiation of R. Moreover, let $\delta^{-i}(1) \neq \varnothing$, $i = 1, 2, \dots, d-1$. Then*

$$\mathrm{K.dim}(R[x;\alpha;\delta]) = \mathrm{K.dim}(R[x;\alpha]) = n+1.$$

Let R be a ring and let α and δ satisfy the conditions of Theorem 1. Denote by $R[[x;\alpha;\Delta]]$ the ring of left skew formal power series over $R[s]$ for which

$$xa = a^{\delta_0}x + a^{\delta_1}x^2 + \cdots + a^{\delta_{d-1}}x^{d-1}, \quad a \in R,$$

where $\delta_0 = \alpha$, $\delta_1 = \delta\alpha, \dots, \delta_{d-1} = \delta^{d-1}\alpha$.

Proposition 4. $\mathrm{K.dim}(R[x;\alpha;\delta]) \leq \mathrm{K.dim}(R[[x;\alpha;\Delta]])$.

Proof. In order to construct a strictly isotonic mapping from the set of left ideals of $R[x;\alpha;\delta]$ into the set of left ideals of $R[[x;\alpha;\Delta]]$, it suffices to associate with any left ideal I of $R[x;\alpha;\delta]$ the left ideal J of $R[[x;\alpha;\Delta]]$ generated by the monomials of the form

$$\alpha^{d^r}(a_d)\cdot x^{dr}; \quad a_d \in R,$$

where $a_d x^d$ is the highest-degree term of the dth degree polynomial from I. The fact that this mapping is strictly isotonic, can be proved as above. $\square$

Consider now the ring $R[x_1,\dots,x_n;\alpha_1,\dots,\alpha_n;\delta_1,\dots,\delta_n]$ of left skew polynomials in n variables over R [6], where

$$\begin{gathered} a^{\alpha_i\alpha_j} = a^{\alpha_j\alpha_i}, \quad a^{\delta_i\delta_j} = a^{\delta_j\delta_i}, \quad i,j = 1,2,\dots,n; \\ a^{\alpha_i\delta_j} = a^{\delta_j\alpha_i}, \quad i \neq j; \\ x_i x_j = x_j x_i, \quad x_i a = \alpha_i(a)x_i + \delta_i(a), \quad a \in R. \end{gathered} \tag{15}$$

It is easy to show that if the endomorphisms α_i ($i = 1,2,\dots,n$) of R and the corresponding α_i-differentiations δ_i ($i = 1,2,\dots,n$) of R satisfy (15), then $R[x_1,\dots,x_n;\alpha_1,\dots,\alpha_n;\delta_1,\dots,\delta_n]$ can be represented as the ring of left skew polynomials in one variable $A_{n-1}[x_n;\overline{\alpha}_n;\overline{\delta}_n]$ over $A_{n-1} = R[x_1,\dots,x_{n-1};\alpha_1,\dots,\alpha_{n-1};\delta_1,\dots,\delta_{n-1}]$, where the mappings $\overline{\alpha}_n$ and $\overline{\delta}_n$ are defined as follows: if

$$f = \sum_{\nu} a_\nu x_1^{\nu_1}\cdots x_{n-1}^{\nu_{n-1}} \in A_{n-1}, \quad a_\nu \in R,$$

then

$$\overline{\alpha}_n(f) = \sum_{\nu} \alpha_n(a_\nu) x_1^{\nu_1}\cdots x_{n-1}^{\nu_{n-1}}$$

and

$$\overline{\delta}_n(f) = \sum_{\nu} \delta_n(a_\nu) x_1^{\nu_1}\cdots x_{n-1}^{\nu_{n-1}}$$

(the fact that $\overline{\delta}_n$ is an $\overline{\alpha}_n$-differentiation of A_{n-1} can be checked by direct calculation). If α_n is an automorphism of R, then $\overline{\alpha}_n$ is an automorphism of A_{n-1}. This enables us to generalize Proposition 3. $\square$

Proposition 5. *Let α_i $(i = 1, 2, \ldots, n)$ be automorphisms of R. Then*

$$\mathrm{K.\,dim}(R[x_1, \ldots, x_n; \alpha_1, \ldots, \alpha_n; \delta_1, \ldots, \delta_n]) \leq \mathrm{K.\,dim}(R[x_1, \ldots, x_n]).$$

Taking into account that if δ_n is a nilpotent α_n-differentiation of R, then $\overline{\delta}_n$ is a nilpotent $\overline{\alpha}_n$-differentiation of A_{n-1}m from Theorem 1 we obtain

Theorem 2. *Let α_i $(i = 1, 2, \ldots, n)$ be automorphisms of R, and let δ_i $(i = 1, 2, \ldots, n)$ be nilpotent $(\delta_i^{d_i} = 0)$ α_i-differentiations of R such that $\delta_i^{-k_i}(1) \neq \varnothing$ for $k_i = 1, 2, \ldots, d_i - 1$. Then*

$$\mathrm{K.\,dim}(R[x_1, \ldots, x_n; \alpha_1, \ldots, \alpha_n; \delta_1, \ldots, \delta_n]) =$$
$$\mathrm{K.\,dim}(R[x_1, \ldots, x_n; \alpha_1, \ldots, \alpha_n]) = \mathrm{K.\,dim}(R[x_1, \ldots, x_n]).$$

If, in addition, R is a left Noetherian ring with finite Krull dimension, then

$$\mathrm{K.\,dim}(R[x_1, \ldots, x_n; \alpha_1, \ldots, \alpha_n; \delta_1, \ldots, \delta_n]) = \mathrm{K.\,dim}\, R + n.$$

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