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ON A DARBOUX TYPE MULTIDIMENSIONAL PROBLEM FOR SECOND-ORDER HYPERBOLIC SYSTEMS

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Abstract. The correct formulation of a Darboux type multidimensional problem for second-order hyperbolic systems is investigated. The correct formulation of such a problem in the Sobolev space is proved for temporal type surfaces on which the boundary conditions of a Darboux type problem are given.

In the space of variables $x_1, \dots, x_n, t$ we consider the system of linear differential equations of second-order

$$Lu \equiv u_{tt} - \sum_{i,j=1}^{n} A_{ij} u_{x_i x_j} + \sum_{i=1}^{n} B_i u_{x_i} + Cu = F, \quad n > 2, \tag{1}$$

where A_{ij}, B_i, C are given real $m \times m$ matrices, F the given and u the unknown m-dimensional vector, $m > 1$.

It will be assumed below that the matrices A_{ij} are symmetrical and constant, and the inequality

$$\sum_{i,j=1}^{n} A_{ij} \eta_i \eta_j \geq c_0 \sum_{i=1}^{n} |\eta_i|^2, \quad c_0 = \text{const} > 0, \tag{2}$$

holds for any m-dimensional real vectors η_i, $i = 1, \dots, n$.

Condition (2) readily implies that system (1) is hyperbolic.

Let G be a dihedral angle in the space R^{n+1} of variables $x_1, \dots, x_n, t$ with a temporal type noncharacteristic boundary ∂G, i.e.,

$$\Big(E\alpha_{n+1}^2 - \sum_{i,j=1}^{n} A_{ij} \alpha_i \alpha_j \Big) \eta\eta \leq 0, \quad \eta \in R^m, \tag{3}$$

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where the vector $(\alpha_1, \dots, \alpha_n, \alpha_{n+1})$ is the unit normal vector to ∂G at points differing from points of the edges of G, and E is the $m \times m$ unit matrix.

Let Γ be a hyperplane parallel to the edges γ of the angle G and severing from this angle a subdomain $D \subset G$ whose sections by two-dimensional planes perpendicular to the edge γ are triangular.

It is further assumed that in system (1) elements of the matrices B_i, C are bounded measurable functions in the domain D, and the right-hand part of this system $F \in L_2(D)$.

Consider a Darboux type problem formulated as follows: In the domain D find a solution $u(x_1, \dots, x_n, t)$ of system (1) by the boundary condition

$$u\big|_{\partial D \cap \partial G} = f, \tag{4}$$

where f is the known real vector.

Note that for the case where at least one face of the angle G is characteristic, problems of this type for a second-order hyberbolic equation are studied in [1–4]. In [5] a Darboux type problem for a wave equation is studied on the assumption that both faces of G are of temporal type. Some multidimensional analogues of the Darboux problems are treated in [6–8]. Also note [9] where a Goursat type multidimensional problem for hyperbolic systems (1) is considered in the conical domain. In this paper, problem (1), (4) is investigated in the Sobolev space $W_2^1(D)$.

For simplicity let $D: k_1 t < x_n < k_2 t$, $0 < t < t_0$; $k_i = \text{const}$, $i = 1, 2$, $k_1 < k_2$, be the domain lying in the half-space $t > 0$ and bounded by the plane hypersurfaces $S_i : k_i t - x_n = 0$, $0 \le t \le t_0$, $i = 1, 2$, of temporal type and by the hyperplane $t = t_0$. It is obvious that for such a domain D condition (3), along with the property of the surfaces S_1 and S_2 being noncharacteristic, is equivalent to the inequalities

$$k_i^2 < \min(\mu_1, \dots, \mu_m), \quad i = 1, 2, \tag{5}$$

where μ_j, $j = 1, \dots, m$, are the eigenvalues of the symmetrical matrix A_{nn} which by virtue of (2) is positively definite. Then the boundary condition (4) takes the form

$$u\big|_{S_i} = f_i, \quad i = 1, 2, \tag{6}$$

where f_i, $i = 1, 2$, are the known real vector-functions on S_i and $(f_1 - f_2)|_{S_1 \cap S_2} = 0$.

Denote by $C_*^\infty(\overline{D})$ the space of functions belonging to the class $C^\infty(\overline{D})$ and having bounded supports, i.e.,

$$C_*^\infty(\overline{D}) = \big\{u \in C^\infty(\overline{D}) : \ \operatorname{diam} \operatorname{supp} u < \infty\big\}.$$

The spaces $C_*^\infty(S_i)$, $i = 1, 2$, are defined in a similar manner.

Denote by $W_2^1(D)$, $W_2^2(D)$, $W_2^1(S_i)$, $i=1,2$, the known Sobolev spaces. Note that the space $C_*^\infty(\overline{D})$ is a dense everywhere subspace of the spaces $W_2^1(D)$ and $W_2^2(D)$, while $C_*^\infty(S_i)$ is a dense everywhere subspace of the space $W_2^1(S_i)$, $i=1,2$.

Definition. Let $f_i \in W_2^1(S_i)$, $i=1,2$, $F \in L_2(D)$. A vector-functon $u \in W_2^1(D)$ will be called a strong solution of problem (1), (6) from the class W_2^1 if there exists a sequence $u_n \in C_*^\infty(\overline{D})$ such that $u_n \to u$ in the space $W_2^1(D)$, $Lu_n \to F$ in the space $L_2(D)$, and $u_n|_{S_i} \to f_i$ in $W_2^1(S_i)$, $i=1,2$, i.e., for $n\to\infty$

$$\|u_n-u\|_{W_2^1(D)} \to 0, \quad \|Lu_n-F\|_{L_2(D)} \to 0,$$
$$\|u_n|_{S_i}-f_i\|_{W_2^1(S_i)} \to 0, \quad i=1,2.$$

We have

Lemma 1. *If inequalities* (5) *are fulfilled, $k_1<0$ and $k_2>0$, then for any $u \in W_2^2(D)$ the a priori estimate*

$$\|u\|_{W_2^1(D)} \le c\Big(\sum_{i=1}^2 \|f_i\|_{W_2^1(S_i)} + \|F\|_{L_2(D)}\Big) \tag{7}$$

holds, where $f_i = u|_{S_i}$, $i=1,2$, $F=Lu$, and the positive constant c does not depend on u.

Proof. Since the space $C_*^\infty(\overline{D})$ ($C_*^\infty(S_i)$) is a dense everywhere subspace of the spaces $W_2^1(D)$ and $W_2^2(D)$ ($W_2^1(S_i)$), by virtue of the familiar theorems of embedding the space $W_2^2(D)$ into the space $W_2^1(D)$ and the space $W_2^2(D)$ into $W_2^1(S_i)$ it is sufficient to show that the a priori estimate (7) holds for functions u of the class $C_*^\infty(\overline{D})$.

We introduce the notation

$$D_\tau = \big\{(x,t)\in D:\ t<\tau\big\}, \quad D_{0\tau} = \partial D_\tau \cap \{t=\tau\}, \quad 0<\tau\le t_0,$$
$$S_{i\tau} = \partial D_\tau \cap S_i, \quad i=1,2, \quad S_\tau = S_{1\tau}\cup S_{2\tau},$$
$$\alpha_i = \cos(\widehat{\nu,x_i}), \quad i=1,\dots,n, \quad \alpha_{n+1} = \cos(\widehat{\nu,t}).$$

Here $\nu=(\alpha_1,\dots,\alpha_n,\alpha_{n+1})$ is the unit normal vector to ∂D_τ and, as one can readily see,

$$\nu\big|_{S_{1\tau}} = \bigg(0,\dots,0,\frac{-1}{\sqrt{1+k_1^2}},\frac{k_1}{\sqrt{1+k_1^2}}\bigg),$$
$$\nu\big|_{S_{2\tau}} = \bigg(0,\dots,0,\frac{1}{\sqrt{1+k_2^2}},\frac{-k_2}{\sqrt{1+k_2^2}}\bigg), \quad \nu\big|_{D_{0\tau}} = (0,\dots,0,1).$$

Therefore for $k_1<0$, $k_2>0$ we find by virtue of inequality (3)

$$\alpha_{n+1}\big|_{S_\tau}<0,\ \ \alpha_{n+1}^{-1}\Big(E\alpha_{n+1}^2-\sum_{i,j=1}^n A_{ij}\alpha_i\alpha_j\Big)\eta\eta\Big|_{S_\tau}\geq 0,\ \ \eta\in R^m. \quad (8)$$

After performing a scalar multiplication of both sides of system (1) by the vector $2u_t$, where $u\in C_*^\infty(\overline{D})$, $F=Lu$, and integrating the resulting expression with respect to D_τ, we obtain

$$2\int_{D_\tau}\Big(F-\sum_{i=1}^n B_i u_{x_i}-Cu\Big)u_t dxdt=$$
$$=\int_{D_\tau}\Big(\frac{\partial(u_tu_t)}{\partial t}+2\sum_{i,j=1}^n A_{ij}u_{x_j}u_{tx_i}\Big)dxdt-2\int_{S_\tau}\sum_{i,j=1}^n A_{ij}u_tu_{x_j}\alpha_i ds=$$
$$=\int_{D_{0\tau}}\Big(u_tu_t+\sum_{i,j=1}^n A_{ij}u_{x_i}u_{x_j}\Big)dx+$$
$$+\int_{S_\tau}\alpha_{n+1}^{-1}\Big[\sum_{i,j=1}^n A_{ij}(\alpha_{n+1}u_{x_i}-\alpha_iu_t)(\alpha_{n+1}u_{x_j}-\alpha_ju_t)+$$
$$+\Big(E\alpha_{n+1}^2-\sum_{i,j=1}^n A_{ij}\alpha_i\alpha_j\Big)u_tu_t\Big]ds. \quad (9)$$

Assuming that

$$w(\tau)=\int_{D_{0\tau}}\Big(u_tu_t+\sum_{i,j=1}^n A_{ij}u_{x_i}u_{x_j}\Big)dx,\ \ \widetilde{u}_i=\alpha_{n+1}u_{x_i}-\alpha_iu_t,\ \ i=1,\dots,n,$$

and using (8), from (9) we have

$$w(\tau)\leq c_1\Big[\int_{S_\tau}\sum_{i,j=1}^n A_{ij}\widetilde{u}_i\widetilde{u}_j ds+\int_0^\tau w(t)dt+$$
$$+\int_{D_\tau}uudxdt+\int_{D_\tau}FFdxdt\Big],\ \ c_1=\text{const}>0. \quad (10)$$

Let (x,τ_x) be the point at which the surface $S_1\cup S_2$ intersects the straight line parallel to the t-axis passing through the point $(x,0)$. We have

$$u(x,\tau)=u(x,\tau_x)+\int_{\tau_x}^\tau u_t(x,t)dt,$$

which implies

$$\int_{D_{0\tau}}u(x,\tau)u(x,\tau)dx\leq$$

$$\leq 2\int_{D_{0\tau}} u(x,\tau_x)u(x,\tau_x)dx + 2|\tau-\tau_x|\int_{D_{0\tau}} dx\int_{\tau_x}^{\tau} u_t(x,t)u_t(x,t)dt =$$
$$= 2\int_{S_\tau} |\alpha_3^{-1}|uuds + 2|\tau-\tau_x|\int_{D_\tau} u_tu_tdxdt \leq$$
$$\leq c_2\Big(\int_{S_\tau} uuds + \int_{D_\tau} u_tu_tdxdt\Big), \quad c_2 = \text{const} > 0. \tag{11}$$

By introducing the notation

$$w_0(\tau) = \int_{D_{0\tau}} \Big(uu + u_tu_t + \sum_{i,j=1}^{n} A_{ij}u_{x_i}u_{x_j}\Big)dx$$

and combining inequalities (10) and (11) we obtain

$$w_0(\tau) \leq c_3\Big[\int_{S_\tau} \Big(uu + \sum_{i,j=1}^{n} A_{ij}\widetilde{u}_i\widetilde{u}_j\Big)ds + \int_0^{\tau} w_0(t)dt + \int_{D_\tau} FFdxdt\Big],$$

which by the Gronwall lemma leads us to

$$w_0(\tau) \leq c_4\Big[\int_{S_\tau} \Big(uu + \sum_{i,j=1}^{n} A_{ij}\widetilde{u}_i\widetilde{u}_j\Big)ds + \int_{D_\tau} FFdxdt\Big], \tag{12}$$

where $c_i = \text{const} > 0$, $i = 3, 4$.

One can readily see that the operator $\alpha_{n+1}\frac{\partial}{\partial x_i} - \alpha_i\frac{\partial}{\partial t}$ is the internal differential operator on the surface S_τ. Therefore the inequality

$$\int_{S_\tau} \Big(uu + \sum_{i,j=1}^{n} A_{ij}\widetilde{u}_i\widetilde{u}_j\Big)ds \leq c_5\sum_{i=1}^{2} \|f_i\|^2_{W_2^1(S_{i\tau})}, \quad c_5 = \text{const} > 0, \tag{13}$$

is fulfilled by virtue of (6).

Using condition (2) we have

$$c_6\Big(uu + u_tu_t + \sum_{i=1}^{n} u_{x_i}u_{x_i}\Big) \leq uu + u_tu_t + \sum_{i,j=1}^{n} A_{ij}u_{x_i}u_{x_j}, \quad c_6 = \text{const} > 0. \tag{14}$$

By virtue of (13) and (14) inequality (12) implies

$$\int_{D_{0\tau}} \Big(uu + u_tu_t + \sum_{i=1}^{n} u_{x_i}u_{x_i}\Big)dx \leq$$
$$\leq c_7\Big(\sum_{i=1}^{2} \|f_i\|^2_{W_2^1(S_{i\tau})} + \|F\|^2_{L_2(D_\tau)}\Big), \quad c_7 = \text{const} > 0. \tag{15}$$

The integration of both sides of (15) with respect to τ gives estimate (7).

For simplicity, when considering below the dependence domain and the existence theorem for a solution of problem (1), (6), we shall limit our consideration to the case where system (1) does not contain the lowest terms, namely,

$$L_0 u \equiv u_{tt} - \sum_{i,j=1}^{n} A_{ij} u_{x_i x_j} = F. \tag{16}$$

Let $K : -\infty < t \leq |x| g(\frac{x}{|x|})$ be the conical domain lying in the half-space $t < 0$ and bounded by the surface $\partial K : t = |x| g(\frac{x}{|x|})$ with vertex at the origin $O(0, \dots, 0)$, where g is a completely defined negative smooth function given on the unit sphere in R^n. If $P_0(x_1^0, \dots, x_n^0, t^0) \in R^{n+1}$, then we denote by K_{P_0} the conical domain K drawn from the point P_0 towards the decreasing values of time, i.e., $K_{P_0} : -\infty < t \leq t^0 + |x - x^0| g(\frac{x - x^0}{|x - x^0|})$.

It will be assumed below that the characteristic matrix of system (1) is nonnegative on the cone ∂K, i.e.,

$$\Big(E\alpha_{n+1}^2 - \sum_{i,j=1}^{n} A_{ij} \alpha_i \alpha_j \Big) \eta\eta \geq 0, \quad \eta \in R^m, \tag{17}$$

where $(\alpha_1, \dots, \alpha_n, \alpha_{n+1})$ is the external unit normal vector to the cone ∂K at points different from the vertex.

We shall clarify the geometrical meaning of condition (17).

By condition (2) the symmetrical matrix $\sum_{i,j=1}^{n} A_{ij} \xi_i \xi_j$, $(\xi_1, \dots, \xi_n) \in R^n$ is positively definite. Therefore there is an orthogonal matrix $T = T(\xi_1, \dots, \xi_n)$ such that the matrix $T^{-1}(\sum_{i,j=1}^{n} A_{ij} \xi_i \xi_j) T$ is diagonal and its elements $\sigma_1, \dots, \sigma_m$ on the diagonal are positive, i.e., $\sigma_i = \lambda_i^2(\xi_1, \dots, \xi_n) > 0$, $\lambda_i > 0$, $i = 1, \dots, m$. Therefore the real numbers $\xi_{n+1} = \pm \lambda_i(\xi_1, \dots, \xi_n)$, $i = 1, \dots, m$, are the roots of the characteristic polynomial $p_0(\xi) = \det(E\xi_{n+1}^2 - \sum_{i,j=1}^{n} A_{ij} \xi_i \xi_j)$ of system (1). As one can easily verify, $\lambda_0(\xi_1, \dots, \xi_n) = \max_{1 \leq i \leq m} \lambda_i(\xi_1, \dots, \xi_n)$ is a continuous positive homogeneous function of first order with respect to the variables $\xi_1, \dots, \xi_n$. For system (1), the internal cavity of the cone of normals [10] lying in the half-space $\xi_{n+1} \geq 0$ is the convex cone [11]

$$\Gamma = \big\{ (\xi_1, \dots, \xi_{n+1}) \in R^{n+1} : \ \xi_{n+1} = \lambda_0(\xi_1, \dots, \xi_n) \big\}.$$

Since

$$\Big(E\alpha_{n+1}^2 - \sum_{i,j=1}^{n} A_{ij} \alpha_i \alpha_j \Big) \eta\eta =$$

$$= \Big(E\alpha_{n+1}^2 - T^{-1} \Big(\sum_{i,j=1}^{n} A_{ij} \alpha_i \alpha_j \Big) T \Big) T^{-1}\eta T^{-1}\eta,$$

where $(E\alpha_{n+1}^2 - T^{-1}(\sum_{i,j=1}^n A_{ij}\alpha_i\alpha_j)T)$ is a diagonal matrix with elements $(\alpha_{n+1}^2 - \lambda_i^2)$ on the diagonal, condition (17) is equivalent to the condition

$$|\alpha_{n+1}| \geq \lambda_0(\alpha_1, \dots, \alpha_n). \tag{18}$$

By virtue of the assumptions for the cone K we have

$$\alpha_{n+1}\big|_K > 0 \tag{19}$$

and therefore (18) can be rewritten as

$$\alpha_{n+1} \geq \lambda_0(\alpha_1, \dots, \alpha_n). \tag{20}$$

Denote by $\Gamma^* = \bigcap\limits_{\eta\in\Gamma}\{\xi \in R^{n+1} : \xi\eta \leq 0\}$ a conical domain dual to Γ. As is known, $\partial\Gamma^*$ is a convex shell of the cone of rays of system (1) lying in the direction of decreasing values of the time $t = \xi_{n+1}$ [10, 11].

Using inequality (20), one can prove, as in [9], that for a convex smooth cone K condition (17) is equivalent to the condition $K \supset \Gamma^*$.

It will always be assumed below that $k_1 < 0$, $k_2 > 0$ and (3) is fulfilled. Hence, as we have seen, it follows that inequalities (8) are valid. □

Lemma 2. *For a point $P_0(x^0, t^0) \in D$ of the solution $u(x,t)$ of problem* (16), (6) *of the class $C^2(\overline{D})$ or $W_2^2(D)$ the domain of dependence is contained in the conical domain K_{P_0} with vertex at the point P_0.*

Proof. We set

$$\Omega_{P_0} = D \cap K_{P_0}, \quad S_{iP_0} = S_i \cap \partial\Omega_{P_0}, \quad i = 1, 2.$$

To prove the lemma it is sufficient to show that if

$$f_i\big|_{S_{iP_0}} \equiv u\big|_{S_{iP_0}} = 0, \;\; i = 1, 2, \quad F\big|_{\Omega_{P_0}} \equiv L_0 u\big|_{\Omega_{P_0}} = 0, \tag{21}$$

then $u|_{\Omega_{P_0}} = 0$.

First we shall consider the case $u \in C^2(\overline{D})$. Let S_{3P_0} denote the remaining boundary of the domain Ω_{P_0}, i.e., $S_{3P_0} = \partial\Omega_{P_0} \backslash (S_{1P_0} \cup S_{2P_0})$. By virtue of (17) and (19) we have

$$\alpha_{n+1}\big|_{S_{3P_0}} > 0, \quad \alpha_{n+1}^{-1}\Big(E\alpha_{n+1}^2 - \sum_{i,j=1}^n A_{ij}\alpha_i\alpha_j\Big)\eta\eta\Big|_{S_{3P_0}} \geq 0. \tag{22}$$

After multiplying scalarly both sides of equations (16) by the vector $2u_t$ and integrating the obtained expression with respect to the domain Ω_{P_0}, by virtue of (2), (8), (21), (22) and the reasoning used in proving (9) we obtain the inequality

$$0 = 2\int_{\Omega_{P_0}} F u_t \, dx dt =$$

$$= \int_{\partial\Omega_{P_0}} \Big[\Big(u_t u_t + \sum_{i,j=1}^n A_{ij} u_{x_i} u_{x_j}\Big)\alpha_{n+1} - 2\sum_{i,j=1}^n A_{ij} u_t u_{x_j} \alpha_i\Big] ds =$$

$$= \int_{\partial\Omega_{P_0}} \alpha_{n+1}^{-1} \Big[\sum_{i,j=1}^n A_{ij}\big(\alpha_{n+1} u_{x_i} - \alpha_i u_t\big)\big(\alpha_{n+1} u_{x_j} - \alpha_j u_t\big)\Big] ds +$$

$$+ \int_{\partial\Omega_{P_0}} \alpha_{n+1}^{-1} \Big(E\alpha_{n+1}^2 - \sum_{i,j=1}^n A_{ij}\alpha_i\alpha_j\Big) u_t u_t ds \geq$$

$$\geq c_0 \int_{S_{3P_0}} \alpha_{n+1}^{-1} \sum_{i=1}^n \|\alpha_{n+1} u_{x_i} - \alpha_i u_t\|^2 ds. \tag{23}$$

In deriving (23) we used the fact that the operator $\alpha_{n+1}\frac{\partial}{\partial x_i} - \alpha_i \frac{\partial}{\partial t}$ is the internal differential operator on the boundary $\partial\Omega_{P_0}$ and, in particular, the equalities

$$\Big(\alpha_{n+1}\frac{\partial u}{\partial x_i} - \alpha_i \frac{\partial u}{\partial t}\Big)\Big|_{S_{1P_0}\cup S_{2P_0}} = 0, \quad i = 1, \dots, n,$$

hold due to (21).

Since $\inf_{|\alpha|=1} \lambda_0(\alpha_1, \dots, \alpha_n) > 0$, by (20) we have $\inf_{S_{3P_0}} \alpha_{n+1} > 0$, which by (23) implies

$$\big(\alpha_{n+1} u_{x_i} - \alpha_i u_t\big)\big|_{S_{3P_0}} = 0, \quad i = 1, \dots, n. \tag{24}$$

Taking into account that $u \in C^2(\overline{D})$ and the internal differential operators $\alpha_{n+1}\frac{\partial}{\partial x_i} - \alpha_i \frac{\partial}{\partial t}$, $i = 1, \dots, n$, are linearly independent on the n-dimensional connected surface S_{3P_0}, we immediately find from (24) that

$$u\big|_{S_{3P_0}} \equiv \text{const}. \tag{25}$$

But on account of (21) $u|_{S_{3P_0}\cap(S_{1P_0}\cup S_{2P_0})} = 0$, and hence by (25) we obtain

$$u\big|_{S_{3P_0}} \equiv 0. \tag{26}$$

From (26), in particular, it follows that $u(P_0) = 0$.

Taking now an arbitrary point $Q \in \Omega_{P_0}$, by (21) we conclude that the above equalities hold provided that the point P_0 is replaced by the point Q. Thus by repeating the above reasoning for the domain Ω_Q we obtain $u(Q) = 0$. Therefore for the case $u \in C^2(\overline{D})$ we have $u|_{\Omega_{P_0}} = 0$.

By a similar slightly modified reasoning and embedding theorem for the Sobolev spaces one can prove this lemma for the case $u \in W_2^2(D)$ [5]. □

Let $K^+ : |x| g_+(\frac{x}{|x|}) \leq t < +\infty$ denote a conical domain lying in the half-space $t > 0$ and bounded by the surface $\partial K_+ : t = |x| g_+(\frac{x}{|x|})$ with vertex at the origin $O(0, \dots, 0)$, where g_+ is a completely defined smooth function given on the unit sphere R^n. For $P_0(x^0,t^0)\in R^{n+1}$ we denote by $K^+_{P_0}$ a conical domain K^+ drawn from the point P_0 in the direction of increasing values of time, i.e.,

$$K^+_{P_0}:\ t^0+|x-x^0|g_+\Big(\frac{x-x^0}{|x-x^0|}\Big)\le t<\infty.$$

As in the case of the conical domain K, it will be assumed that the characteristic matrix of system (1) is non-negative on ∂K^+, i.e., that (17) holds. It is obvious that conical domains K^+ are obtained from conical domains K by the central symmetry with respect to the origin. If Γ^*_+ is a conical domain centrally symmetrical to Γ^* with respect to the origin, then, as in the case of the domain K, one can prove that for the convex smooth cone K^+ condition (17) is equivalent to the condition $K^+\supset\Gamma^*_+$.

Lemma 3. *Let D_0 be a bounded subdomain of D with a piecewise smooth boundary bounded from above by a hyperplane $t=t_0$, and from the sides by hyperplanes S_1, S_2 and by piecewise smooth hypersurfaces S_3, S_4 of temporal type on which the inequalities*

$$\alpha_{n+1}\big|_{S_3}<0,\quad \alpha_{n+1}\big|_{S_4}<0 \tag{27}$$

hold, where $\nu=(\alpha_1,\dots,\alpha_{n+1})$ is the external unit vector to ∂D_0 and $S_3\cap S_4=\varnothing$. Let $u_0\in C^\infty(\overline{D}_0)$ and $g_i=u_0|_{\partial D_0\cap S_i}$, $i=1,2$, $F_0=L_0u_0$, $X=\operatorname{supp}g_1\cup\operatorname{supp}g_2\cup\operatorname{supp}F_0$, $Y=\cup_{P_0\in X}K^+_{P_0}$. Denote by S_3^ε, S_4^ε the ε-neighborhoods of hypersurfaces S_3, S_4, where ε is a fixed sufficiently small number. Then if $u_0\big|_{S_3\cup S_4}=0$, $Y\cap(S_3^\varepsilon\cup S_4^\varepsilon)=\varnothing$, the function

$$u(P)=\begin{cases}u_0(P), & P\in D_0,\\ 0, & P\in D\backslash D_0\end{cases}$$

is a solution of problem (16), (6) *of the class $C^\infty_*(\overline{D})$ for*

$$f_i(P)=\begin{cases}g_i(P), & P\in\partial D_0\cap S_i,\\ 0, & P\in S_i\backslash(\partial D_0\cap S_i),\end{cases}\quad i=1,2,$$

$$F(P)=\begin{cases}F_0(P), & P\in D_0,\\ 0, & P\in D\backslash D_0.\end{cases}$$

The proof of Lemma 3 repeats in the main the proof of the corresponding lemma from [5].

Remark 1. Note that Lemma 3 remains valid in the case where conditions (27) are not fulfilled on some set $\omega\subset S_3\cup S_4$ of zero n-dimensional measure, i.e., $\alpha_{n+1}|_\omega=0$. In particular, Lemma 3 remains true if $\omega=\cup_{i=1}^p\gamma_i$ is the union of a finite number of smooth $(n-1)$-dimensional submanifolds

$\gamma_i \subset S_3 \cup S_4$ and $\alpha_{n+1}|_\omega = 0$, $\alpha_{n+1}|_{(S_3\cup S_4)\setminus\omega} < 0$. This fact will be used below in proving the existence theorem for problem (16), (6).

Remark 2. Also note that Lemmas 2 and 3 actually suggest a method for constructing a solution of problem (16), (6) which will be described below in proving Theorem 1. The method consists in reducing the initial problem (16), (6) to a mixed-type problem for a second-order hyperbolic system in a cylinder.

Let $\xi_{n+1} = \pm\lambda_i(\xi_1,\dots,\xi_n)$, $\lambda_i > 0$, $i = 1,\dots,m$, be the roots of the characteristic polynomial $\det(E\xi_{n+1}^2 - \sum_{i,j=1}^n A_{ij}\xi_i\xi_j)$ of system (1) with respect to ξ_{n+1}. The functions $\lambda_0(\xi_1,\dots,\xi_n) = \max_{1\le i\le m}\lambda_i(\xi_1,\dots,\xi_n)$, $\lambda_0^-(\xi_1,\dots,\xi_n) = \min_{1\le i\le m}\lambda_i(\xi_1,\dots,\xi_n)$ are obviously continuous positive homogeneous functions of first order with respect to the variables $\xi_1,\dots,\xi_n$. Consider the cones

$$\Gamma = \big\{(\xi_1,\dots,\xi_{n+1}) \in R^{n+1} : \xi_{n+1} = \lambda_0(\xi_1,\dots,\xi_n)\big\},$$
$$\Gamma_- = \big\{(\xi_1,\dots,\xi_{n+1}) \in R^{n+1} : \xi_{n+1} = -\lambda_0^-(\xi_1,\dots,\xi_n)\big\}.$$

As is known, the interior of the convex cone Γ is a set of all spatial type normals lying in the half-plane $\xi_{n+1} \ge 0$, while $\partial\Gamma_-^*$ is the interior of the cone of rays of system (1) lying in the direction of increasing values of the time $t = \xi_{n+1}$, where $\Gamma_-^* = \bigcap_{\eta\in\Gamma_-}\{\xi \in R^{n+1} : \xi\eta \le 0\}$ [10]. Let $\overset{\circ}{\Gamma}$ denote the domain bounded by the cone Γ, i.e., the interior of Γ. Below it will be assumed without loss of generality that

$$\Gamma_-^* \subset \overset{\circ}{\Gamma}, \tag{28}$$

since otherwise this can be achieved by a subsequent change of the independent variables $x_i' = \varepsilon x_i$, $i = 1,\dots,n$, $t' = t$ in system (1) for a sufficiently small $\varepsilon = \text{const}$.

For $k_1 \le k \le k_2$ we denote by l_k a ray coming out of the origin with the direction vector $(0,\dots,0,k,1)$, i.e., $l_k : \tau(0,\dots,0,k,1)$, $0 < \tau < +\infty$. By virtue of the assumptions made for the supports of the data of problem (1), (6) we have

$$l_k \subset \overset{\circ}{\Gamma}{}_-^*. \tag{29}$$

Denote by H_k an arbitrary noncharacteristic hyperplane of temporal type containing the ray l_k. Take an arbitrary point $P \in l_k$ and choose a Cartesian system $x_1^0,\dots,x_n^0,t^0$ which is connected with this point and has the vertex at the point P so that the t^0-axis could be directed along the ray l_k, and the x_n^0-axis along the normal to H_k at this point towards increasing values of time. Denote by H_k^+ that part of the half-space R^{n+1} with the boundary H_k which contains the positive x_n^0-semiaxis. Denote by $Q^0(\xi)$

and $\widetilde{p}_0(\xi) = \det Q^0(\xi)$ respectively the characteristic matrix and the polynomial of system (1) written in terms of the coordinate system $x_1^0, \dots, x_n^0, t^0$ connected with an arbitrarily chosen point $P \in l_k$.

By (29) there is a convex cone Γ_1 which is tangential to the hyperplane H_k along the ray l_k, lies in the half-space H_k^+, and is contained in the set $\overset{\circ}{\Gamma}{}_-^* \cup O$, where $O = O(0, \dots, 0)$ is the vertex of Γ_-. Hence by (28) and the arguments from [12] it follows that when system (1) is strictly hyperbolic, exactly m characteristic hyperplanes of system (1) pass inwards the angle $t^0 > 0$, $x_n^0 > 0$ through the $(n-1)$-dimensional plane $t^0 = x_n^0 = 0$ connected with an arbitrary point $P \in l_k$. Hence in turn it follows that for $\operatorname{Re} s > 0$ the number of roots $\lambda_j(\xi_1, \dots, \xi_{n-1}, s)$ is equal to m, $i = \sqrt{-1}$, provided that we take into account the multiplicity of the polynomial $\widetilde{p}_0(i\xi_1, \dots, i\xi_{n-1}, \lambda, s)$ with $\operatorname{Re} \lambda_j < 0$ [12]. The polynomial $\widetilde{p}_0(i\xi_1, \dots, i\xi_{n-1}, \lambda, s)$ can be written in the form of product $\Delta_-(s)\Delta_+(s)$, where for $\operatorname{Re} s > 0$ the roots of the polynomials $\Delta_-(\lambda)$ and $\Delta_+(\lambda)$ lie respectively to the left and to the right of the imaginary axis, while the coefficients are continuous for s, $\operatorname{Re} s \geq 0$, $(\xi_1, \dots, \xi_{n-1}) \in R^{n-1}$, $\xi_1^2 + \dots + \xi_{n-1}^2 + |s|^2 = 1$ [13]. Denote by $\widetilde{Q}^0(i\xi_1, \dots, i\xi_{n-1}, \lambda, s)$ an $m \times m$ matrix whose element $\widetilde{Q}^0_{pq}(i\xi_1, \dots, i\xi_{n-1}, \lambda, s)$ is equal to an algebraic complement to the element $Q^0_{pq}(i\xi_1, \dots, i\xi_{n-1}, \lambda, s)$ in the characteristic matrix $Q^0(i\xi_1, \dots, i\xi_{n-1}, \lambda, s)$ of system (1).

Let us consider

Condition 1. System (1) is strictly hyperbolic. For an arbitrary noncharacteristic hyperplane of temporal type H_k, passing through the ray $l_k : \tau(0, \dots, 0, k, 1)$, $0 < \tau < +\infty$ when $k_1 \leq k \leq k_2$, and for an arbitrary point $P \in l_k$ and any s, $\operatorname{Re} s \geq 0$, and $(\xi_1, \dots, \xi_{n-1}) \in R^{n-1}$ such that $\xi_1^2 + \dots + \xi_{n-1}^2 + |s|^2 = 1$, the rows of the matrix $\widetilde{Q}^0(i\xi_1, \dots, i\xi_{n-1}, \lambda, s)$ are linearly independent as polynomials of λ modulo the polynomial $\Delta_-(\lambda)$.

It will be assumed below that the functions f_1 and f_2 in the boundary conditions (6) vanish on $\gamma_0 = S_1 \cap S_2$, i.e.,

$$f_i\big|_{\gamma_0} = 0, \quad i = 1, 2. \tag{30}$$

Denote by $\overset{\circ}{W}{}_2^1(S_i, \gamma_0)$ functions of the class $W_2^1(S_i)$ which satisfy equality (30), i.e., $\overset{\circ}{W}{}_2^1(S_i, \gamma_0) = \{f \in W_2^1(S_i) : f|_{\gamma_0} = 0\}$, $i = 1, 2$.

We have

Theorem 1. *Let condition* 1 *be fulfilled. Then for any* $f_i \in \overset{\circ}{W}{}_2^1(S_i, \gamma_0)$, $i = 1, 2$, $F \in L_2(D)$ *there exists a unique strong solution* u *of problem* (16), (6) *of the class* W_2^1, *for which estimate* (7) *is valid.*

Proof. Here only a short scheme of the proof is given. The detailed proof of Theorem 1 for one equation of hyperbolic type is given in [5].

Denote by $S_i^0 : k_i t - x_n = 0$, $0 \le t < +\infty$, $i = 1, 2$, a half-plane containing the support S_i of the boundary conditions (6), and by D_0 the dihedral angle between the half-planes S_1^0 and S_2^0. As is known, the function $f_i \in \overset{\circ}{W}{}_2^1(S_i, \Gamma)$ can be extended in the half-plane S_i^0 as a function $\widetilde{f}_i$ of the class $\overset{\circ}{W}{}_2^1(S_i^0)$, i.e., $(f_i - \widetilde{f}_i)|_{S_i} = 0$, $\widetilde{f}_i \in \overset{\circ}{W}{}_2^1(S_i^0)$, $i = 1, 2$. We set

$$\widetilde{F}(P) = \begin{cases} F(P), & P \in D, \\ 0, & P \in D_0 \backslash D. \end{cases}$$

It is obvious that $\widetilde{F} \in L_2(D_0)$. If $C_0^\infty(D_0)$, $C_0^\infty(S_i^0)$, $i = 1, 2$, are the spaces of finite infinitely differentiable functions, then, as we know, these spaces will be dense everywhere in the spaces $L_2(D_0)$, $\overset{\circ}{W}{}_2^1(S_i^0)$, $i = 1, 2$, respectively. Therefore there are sequences $F_n \in C_0^\infty(D_0)$ and $f_{in} \in C_0^\infty(S_i^0)$, $i = 1, 2$, such that

$$\lim_{n\to\infty} \|\widetilde{F} - F_n\|_{L_2(D)} = \lim_{n\to\infty} \|\widetilde{f}_i - \widetilde{f}_{in}\|_{W_2^1(S_i^0)} = 0, \quad i = 1, 2. \tag{31}$$

In the domain of the variables x_n, t we introduce the polar coordinates r, φ, taking the t-axis as a polar axis, and counting the polar angle φ from the polar axis and assuming it positive in the clockwise direction. Let φ_i be the dihedral angle formed by the half-planes S_i^0 and $x_n = 0$, $0 \le t < +\infty$, $i = 1, 2$.

In passing from the Cartesian system $x_1, x_2, \dots, x_n, t$ to the system $x_1, ...,$ x_{n-1}, $\tau = \log r$, φ the dihedral angle D_0 becomes an infinite layer

$$H = \big\{ -\infty < x_i < \infty,\ i = 1, \dots, n-1,\ -\infty < \tau < \infty,\ -\varphi_1 < \varphi < \varphi_2 \big\},$$

and equation (16) in the previous terms for the functions u and F takes the form

$$e^{-2\tau} L_1(\tau, \varphi, \partial) u = F, \tag{32}$$

where $\partial = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}}, \frac{\partial}{\partial \tau}, \frac{\partial}{\partial \varphi})$, $L_1(\tau, \varphi, \partial)$ is a second-order matrix differential operator of hyperbolic type with respect to τ with infinitely differentiable coefficients depending on τ and φ.

Denote by $H_{\beta_k} \subset H$, $\beta_k = \text{const} > 0$, a cylindrical domain $\Omega_k \times (-\infty, \infty)$ of the class C^∞, where $(-\infty, \infty)$ is the τ-axis, and by ∂H_{β_k} its lateral surface $\partial\Omega_k \times (-\infty, \infty)$. It is assumed that the cylinder $H_{\beta_k}^* = \{|x_i| < \beta_k, i = 1, \dots, n-1, -\infty < \tau < \infty, -\varphi_1 < \varphi < \varphi_2\}$ is contained in H_{β_k}. Under the inverse transformation $(x_1, \dots, x_{n-1}, \tau, \varphi) \to (x_1, \dots, x_n, t)$ the cylindrical domain H_{β_k} will transform to the infinite domain $G_{\beta_k} \subset D_0$ bounded by the surfaces $\widetilde{S}_i = S_i^0 \cap \partial G_{\beta_k}$, $i = 1, 2$, and also by the surfaces

$\widetilde{S}_3$, $\widetilde{S}_4$. It can be shown that the surfaces $\widetilde{S}_3$ and $\widetilde{S}_4$ are of temporal type and the following conditions are fulfilled on them:

$$\alpha_{n+1}\big|_{(\widetilde{S}_3\cup\widetilde{S}_4)\setminus\omega}<0,\quad \alpha_{n+1}\big|_{\omega}=0, \tag{33}$$

where $\omega\subset\widetilde{S}_3\cup\widetilde{S}_4$ is some set of zero n-dimensional measure and $\nu=(\alpha_1,\dots,\alpha_n,\alpha_{n+1})$ is the external unit normal vector to ∂G_{β_k}.

Choose a number β_k so large that

$$\operatorname{supp} f_{ik}\subset\widetilde{S}_i,\ \ i=1,2,\quad \operatorname{supp} F_k\subset G_{\beta_k}.$$

Hence the function $g_k(x,t)$ defined on the boundary ∂G_{β_k} of the domain G_{β_k} as

$$g_k\big|_{\widetilde{S}_i}=f_{ik},\ \ i=1,2,\quad g_k\big|_{\widetilde{S}_3}=g_k\big|_{\widetilde{S}_4}=0 \tag{34}$$

will belong to the class $C_0^\infty(\partial G_{\beta_k})$.

In passing to the variables $x_1,\dots,x_{n-1},\tau,\varphi$, the functions g_k and F_k will transform to some functions which will be denoted as previously. It is obvious that

$$g_k\in C_0^\infty(\partial H_{\beta_k}),\quad F_k\in C_0^\infty(H_{\beta_k}). \tag{35}$$

For the hyperbolic equation (32) with $F=F_k$ we consider, in the cylinder H_{β_k}, the following mixed-type problem with the "Cauchy zero conditions" for $\tau=-\infty$:

$$e^{-2\tau}L_1(\tau,\varphi,\partial)v=F_k, \tag{36}$$

$$v\big|_{\partial H_{\beta_k}}=g_k. \tag{37}$$

Using condition 1 and the results from [13], [14], the mixed-type problem (36), (37) has, by virtue of (35), a unique solution $v=v_k$ of the class $C^\infty(\overline{H}_{\beta_k})$ which identically vanishes for $\tau<-M$, where $M=\text{const}$ is a sufficiently large positive number.

Returning to the initial variables $x_1,\dots,x_n,t$ and using the previous notations for the functions v_k and F_k, we obtain the following facts:

(1) the function $u_k^0=v_k|_{G^*_{\beta_k}}$, where $G^*_{\beta_k}=G_{\beta_k}\cap D$, belongs to the class $C^\infty(\overline{G}^*_{\beta_k})$ and satisfies the equation $L_0u_k^0=F_k$;

(2) on the lateral part $\cup_{i=1}^4\widetilde{S}_i^0$ of the boundary $G^*_{\beta_k}$ the function u_k^0 satisfies the conditions $u_k^0|_{\widetilde{S}_3^0\cup\widetilde{S}_4^0}=0$, $u_k^0|_{\widetilde{S}_i^0}=f_{ik}$, $i=1,2$, where, as one can easily verify, the surface $\widetilde{S}_i^0$ is a part of the surface S_i for $i=1,2$, and a part of the surface $\widetilde{S}_i$ for $i=3,4$ figuring in conditions (33).

Therefore, choosing a value of β_k sufficiently large and applying (33), (34), Lemma 3, and Remark 1 we conclude that the function

$$u_k(P)=\begin{cases} u_k^0(P), & P\in G^*_{\beta_k},\\ 0, & P\in D\backslash G^*_{\beta_k},\end{cases}$$

belongs to the class $C^\infty_*(\overline{D})$ and is a solution of problem (16), (6) for $f_i = f_{ik}$, $i=1,2$, and $F=F_k$.

By virtue of inequality (7) we obtain

$$\|u_k-u_p\|_{W_2^1(D)}\leq$$
$$\leq c\Big(\sum_{i=1}^2\|f_{ik}-f_{ip}\|_{W_2^1(S_i)}+\|F_k-F_p\|_{L_2(D)}\Big). \tag{38}$$

It follows from (31) and (38) that the sequence of functions u_k is fundamental in the space $W_2^1(D)$. The completeness of the space $W_2^1(D)$ implies that there exists a function $u\in W_2^1(D)$ such that $u_k\to u$ in the space $W_2^1(D)$, $L_0u_k\to F$ in the space $L_2(D)$, and $u_k|_{S_i}\to f_i$ in $W_2^1(S_i)$, $i=1,2$, for $k\to\infty$. Therefore the function u is a strong solution of problem (16), (6) of the class W_2^1. The uniqueness of this strong solution follows from inequality (7). □

Let us now turn to problem (1), (6). In the space $W_2^1(D)$ we introduce a norm depending on the parameter λ

$$\|u\|^2_{D,1,\lambda}=\int_D e^{-\lambda t}\Big(uu+u_tu_t+\sum_{i=1}^n u_{x_i}u_{x_i}\Big)dxdt,\quad \lambda>0.$$

In a similar manner we introduce norms $\|F\|_{D,0,\lambda}$ and $\|f_i\|_{S_i,1,\lambda}$ in the spaces $L_2(D)$ and $W_2^1(S_i)$, $i=1,2$.

Arguments similar to those in [4] enable us to prove

Lemma 4. *For any $u\in W_2^2(D)$ the a priori estimate*

$$\|u\|_{D,1,\lambda}\leq\frac{c_1}{\sqrt{\lambda}}\Big(\sum_{i=1}^2\|f_i\|_{S_i,1,\lambda}+\|F\|_{D,0,\lambda}\Big) \tag{39}$$

holds, where $f_i=u|_{S_i}$, $F=L_0u$, and the positive constant c_1 does not depend on u and the parameter λ.

By virtue of estimate (39), for a sufficiently large value of λ the lowest terms in equation (1) give arbitrarily small perturbations in the sense of the above-introduced equivalent norms of the spaces $L_2(D)$, $W_2^1(D)$, $W_2^1(S_i)$, $i=1,2$, which due to Theorem 1 and the resuls of [4] enables us to prove the unique solvability of problem (1), (6) in the class W_2^1.

Theorem 2. *Let condition* 1 *be fulfilled. Then for any* $f_i \in \overset{\circ}{W}{}_2^1(S_i,\Gamma)$, $i=1,2$, $F \in L_2(D)$ *there exists a unique strong solution* u *of problem* (1), (6) *of the class* W_2^1, *for which estimate* (7) *is true.*

Remark 3. In Theorems 1 and 2 it is required that system (1) or (16) be strictly hyperbolic, and also that the other part of condition 1 be fulfilled. These requirements can be left out for one class of hyperbolic systems of form (1). These are systems of form (1) for which inequality (1) holds and the symmetrical constant matrices A_{ij} are pairwise permutable, i.e., $A_{ij}A_{ks} = A_{ks}A_{ij}$. Then there exists an orthogonal matrix T_0 with constant elements such that $T_0^{-1}A_{ij}T_0$ is diagonal for any $i,j=1,\dots,n$. Therefore for the new unknown function $v = T_0^{-1}u$ we shall have, instead of system (1), a second-order hyperbolic system with the split principal part. But a problem of the Darboux type with boundary conditions of form (6) on temporal hyperplanes is uniquely solvable for one hyperbolic equation of second-order with constant coefficients at higher derivatives and estimate (39) holds for its solution [5]. Therefore the same arguments that enabled us to prove Theorem 2 on the basis of Lemma 4 and Theorem 1 make it possible to prove

Theorem 3. *Let the matrices* A_{ij} *in system* (1) *be pairwise permutable. Then for any* $f_i \in \overset{\circ}{W}{}_2^1(S_i,\Gamma)$, $i=1,2$, $F \in L_2(D)$ *there exists a unique strong solution* u *of problem* (1), (6) *of the class* W_2^1, *for which estimate* (7) *holds.*

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