GEORGIAN MATHEMATICAL JOURNAL: Vol. 4, No. 1, 1997, 65-80

GEOMETRICAL DECOMPOSITION OF THE FREE LOOP SPACE ON A MANIFOLD WITH FINITELY MANY CLOSED GEODESICS

THOMAS MORGENSTERN

Abstract. In Morse theory an isolated degenerate critical point can be resolved into a finite number of nondegenerate critical points by perturbing the totally degenerate part of the Morse function inside the domain of a generalized Morse chart. Up to homotopy we can admit pertubations within the whole characteristic manifold. Up to homotopy type a relative CW-complex is attached, which is the product of a big relative CW-complex, representing the degenerate part, and a small cell having the dimension of the Morse index.

Introduction

In [1] Gromoll and Meyer succeeded in extending the Morse theory to isolated critical points and orbits. Applying these methods in [2] they were able to show that the sequence of Betti numbers with rational coefficients of the free (Sobolev) loop space ΛM is bounded for a closed manifold M with only finitely many bona fide closed geodesics. The free loop space ΛM is homotopy equivalent to a CW-complex. The intention of this paper is to show that not only these homological numbers are bounded but also the number of cells in all but finitely many dimensions is bounded for some homotopy equivalent CW-complex. It also gives a geometrical insight into why the Gromoll–Meyer theorem is true and motivates the use of special homotopies adapted to the free loop space.

If, on the other hand, the sequence of Betti numbers is unbounded then there have to be infinitely many prime non-trivial closed geodesics for every differentiable structure and Riemannian metric on M. This was shown to be true for all rationally nonmonogenic simply connected manifolds by Vigué-Poirrier and Sullivan [3]. This answers the existence question for a lot

1991 *Mathematics Subject Classification.* 58E10.
Key words and phrases. Morse theory, closed geodesics, critical points, free loop space, Lusternik–Schnirelmann category.

1072-947X/97/0100-0065$12.50/0

of manifolds excluding, for example, spheres and projective spaces. There are also results for nonsimply connected manifolds but this case becomes more complicated the more complicated the fundamental group gets. The homological Gromoll–Meyer theorem is true for arbitrary field coefficients and one sees it to be true for integer coefficients as well, interpreting the Betti numbers as minimal number of generators. For simply connected ΛM our Theorem 1 thus follows from the abstract Proprosition [4, V. 8.3]. A nonsimply connected space is not determined by its homology, as the example of the Poincaré manifolds shows.

Theorem 1 (Geometrical Gromoll–Meyer Theorem). *If (M, g) is a closed differentiable Riemannian manifold, $I(M)$ is the isometry group of M, $G := I(M) \oplus \mathbb{O}(2)$, and if there are only finitely many G-orbits of geometrically distinct nontrivial closed geodesics, then there are numbers R and Q and a CW-complex which is homotopy equivalent to ΛM and has less than R cells of dimension greater than or equal to Q.*

The extension of Morse theory was given by three lemmas. The first is a generalization of the Morse lemma [1, Lemma 1] introducing characteristic manifolds. The second step is a localization argument [1, Lemma 3] using what is now called Gromoll–Meyer pairs. Finally, the last and most important one is the Shifting theorem [1, Theorem], which can also be regarded as a kind of product formula. These lemmas proved to be very fruitful and a great deal of work nowadays relies on them. Using some classical results by Bott and observations about iterated characteristic manifolds the homological proof is then given in [2].

The homological proof uses excision strongly, which we don't have at hand in homotopy theory. We will therefore introduce pairs of subsets very similar to the isolating neighborhoods of the Conley index. With the help of these and a telescope argument we are able to recover the homotopy type of the space under consideration. To do this we don't take the step towards a local index, we don't divide out the exit set. Consequently we have to find appropriate properties for these pairs enabling the step from local to global and conditions under which we can verify them. As in the original Gromoll–Meyer theory the product property turns out to be important and involved. We as well divide the proof into two parts, first setting up an appropriate general machinery and then using it in our special case.

1. h-Morse Theory

We are interested in the homotopy category of compactly generated pairs (X, A). From the homotopy type of filtration pairs (X_n, X_{n-1}) of a relative NDR filtration we can draw conclusions about the homotopy type of the pair (X, A) (see [5, Lemma 6] or [6, A.5.11]).

Definition 2. A *relative h-CW-decomposition* of the pair (X, A) is a relative NDR-filtration of (X, A) such that

[hCW]: for all n there is some relative CW-complex with $(X_n, X_{n-1}) \simeq (Y_n, B_n)$ CW-complex.

A relative *h-CW-complex* (X, A) is a pair of compactly generated spaces admitting a relative h-CW-decomposition.

Lemma 3. *If there is a relative h-CW-decomposition of* (X, A) *then the pair is homotopy equivalent to a relative CW-complex with one cell* (*of the same dimension*) *for every cell in* (Y_n, B_n) [6, A.4.12 and A.4.15].

The step from local to global will use the following proposition:

Proposition 4 ([12, 7.5.7]). *If* $X = X_1 \cup X_2$ *is the union of closed subsets, the inclusion* $i_2 : X_1 \cap X_2 \hookrightarrow X_2$ *is a closed cofibration and if* (Y, B) *is a closed cofibred pair, homotopy equivalent via* r *to* $(X_2, X_1 \cap X_2)$, *then there is a map* Φ *such that* $(\Phi, \mathrm{id}_{X_1}) : (X_1 \sqcup_r Y, X_1) \simeq (X, X_1)$.

Corollary 5. *If* $X = X_1 \cup X_2$ *as in Proposition* 4 *and in addition,* (Y, B) *is a relative CW-complex then* $(X_1 \sqcup_r Y, X_1)$ *is a relative CW-complex with one cell for each cell of* (Y, B).

1.1. Local h-Decomposition Pairs. A *strict Liapunov function* for the topological flow φ on the Hausdorff space M is a continuous function $f : M \to \mathbb{R}$, which is strictly decreasing along $\varphi(x)$ for all $x \notin K^\varphi$, the stationary points. As usual, the set $f^k = M^k := \{x \in M | f(x) \leq k\}$. A closed subset T has the *mean value property* with respect to the flow if for any $x \in T$ and two times t_1, t_2 with $t_1 < t_2$, $\varphi(x, t_1) \in T$ and $\varphi(x, t_2) \in T$ implies $\varphi(x, t) \in T$ for all $t_1 < t < t_2$. We define the following sets: the *exit set* $T_- := \{x \in T | \forall t > 0 : \varphi(x, t) \notin T\}$, the *entrance set* $T_+ := \{x \in T | \forall t > 0 : \varphi(x, -t) \notin T\}$, and the *vertical boundary set* $T_0 := \mathrm{cl}(\partial T - (T_- \cup T_+))$.

Lemma 6. *Let* $T \subseteq M$ *be a closed set with the mean value property with respect to the flow* φ *and let* $k_T \in \mathbb{R}$ *such that* $f(T_+) \geq k_T$ *and* $f(T_0 \cup T_-) \leq k_T$. *Then* $x \in T_+ - (T_0 \cup T_-)$ *if and only if there is* $t_1 > 0$ *with* $\varphi(x, t_1) \in \mathrm{Int}\, T$; *furthermore* $\varphi(x, t) \in \mathrm{Int}\, T$ *for all* $0 < t \leq t_1$.

Corollary 7. *Let the assumptions of Lemma* 6 *be valid and* $\partial T \cap K^\varphi = \varnothing$. *Denote by* $\partial_{f^k \cup T} T$ *the boundary of* T *in* $f^k \cup T$. *Then* $\partial_{f^k \cup T} T = T_0 \cup T_-$.

Lemma 8. *If for some* $x \in T$ *with* $f(x) \geq k$ *there is* $t_1 \geq 0$ *with* $\varphi(x, t_1) \in T_- \cap f^{-1}[k, \infty)$ *and* $t_2 \geq 0$ *with* $f(\varphi(x, t_2)) = k$ *and* $\varphi(x, t_2) \in T$, *then* $t_1 = t_2$.

Lemma 9. *If $x \in M$ with $f(x) \geq k \geq k_T$ and there is $t_1 \geq 0$ with $\varphi(x, t_1) \in T_+ \cap f^{-1}[k, \infty)$ and $t_2 \geq 0$ with $f(\varphi(x, t_2)) = k$ and $\varphi(x, t_2) \in \mathrm{cl}(f^{-1}(k) - T)$, then $t_1 = t_2$.*

For a subset $A \subseteq T$ we define $T^+(A) := \{x \in T | \exists t \geq 0 : \varphi(x, t) \in A\}$. A closed subset A has in T the *retraction property from above* (with respect to φ), if the function $x \mapsto t_A(x) := \min_{\varphi(x,t) \in A}\{t \geq 0\}$ is continuous on $T^+(A)$.

Definition 10. Let M be a Hausdorff space, φ a topological flow, and K^φ its stationary points. Let $f : M \to \mathbb{R}$ be a strict Liapunov function for φ. A *local h-decomposition pair* of $K \subseteq K^\varphi$ corresponding to φ and f is a pair $(U, \tilde{U})$ together with a number k, the *separation level*, having the following properties:

- **[hm1]:** $U, \tilde{U} \subseteq M$ are closed;
- **[hm2]:** the top $T := \mathrm{cl}(U - \tilde{U})$ has the mean value property with respect to the flow φ and $K^\varphi \cap T = K$ and $K \subseteq \mathrm{Int}(T)$;
- **[hm3]:** T has the retraction property from above with respect to φ;
- **[hm4]:** $f(T_+) \geq k$, $f(\tilde{U}) \leq k$ and $T \cap \tilde{U} = T_0 \cup T_-$;
- **[hm5]:** the inclusion $i : \tilde{U} \hookrightarrow U$ is a cofibration.

Corollary 11. *By* **[hm4]** *$f^k - (U - \tilde{U})$ is closed, and by* **[hm5]** *$f^k - (U - \tilde{U}) \hookrightarrow f^k \cup U$ is a cofibration* [7, Satz 7.36].

Definition 12. A local h-decomposition pair has the following properties:

- **[hm6]:** if there is a relative CW-complex (Y, B) homotopy equivalent to $(U, \tilde{U})$;
- **[exc]:** if $T_0 = \varnothing$, $\kappa := \inf f(K) > k$, T_- has the retraction property from above in T and if for all $x \in T$ there is either $t \geq 0$ with $\varphi(x, t) \in T_-$ or otherwise $\varphi(x, t) \to K$ as $t \to \infty$.

1.2. h-Morse Decomposition.

Definition 13. Let f be a strict Liapunov function for φ and T a closed subset with the mean value property. φ has the property **(D)** for f on T if:

(i) for every open neighborhood O of the stationary points K^φ the level sets $T \cap f^{-1}(u) - O$ have the retraction property from above;
(ii) for every $x \in T$ there is $t > 0$ with $\varphi(x, t) \in T_-$ or, otherwise, for the flow line $\varphi(x) : (\alpha(x), \beta(x)) \to M$ either $f \circ \varphi(x) \to -\infty$ or $\varphi(x) \to K^\varphi$ as $t \to \beta(x)$.

If f is differentiable and satisfies the Palais–Smale condition (C) on T then the negative gradient flow φ has the property **(D)** for f on T. If f is bounded from below on T then the flow line either converges to the critical points or ends in T_-.

Definition 14. Let M be a metrizable space and A, X closed subsets with $A \subseteq X$. Let φ be a topological flow on M and $T := \mathrm{cl}(X - A)$ have the mean value property. Let $f : M \to \mathbb{R}$ be a strict Liapunov function for φ, and φ have the property **(D)** for f on T. Let $\{K_l\}_{l \in L}$ be a partition of the stationary points $K^\varphi \cap (X - A)$ such that for every $l \in L$ there is a local h-decomposition pair $(U_l, \tilde{U}_l)$, k_l of K_l with $U_l \subseteq X$ and the following properties:

- [**hm7**]**:** There is a countable index set $J = [0, \ldots, j_{\max}]$ or $J = \mathbb{N}_0$ and a partition $L = \bigcup_{j \in J} L_j$ such that for all $l, m \in L_j$ we have $k_l = k_m$. There is a nondecreasing function $u : J \to \mathbb{R}$ such that $u_j := u(j) = k_l$ for all $l \in L_j$. If $J = \mathbb{N}_0$ we have the convergence $u_j \to \sup_{x \in X - A} f(x)$ as $j \to \infty$.
- [**hm8**]**:** For $i < j$ we have $T_l \cap U_m = \varnothing$ if $l \in L_j$ and $m \in L_i$.
- [**hm9**]**:** For all l, $m \in L_j$ with $l \neq m$ we have $U_l \cap U_m = \varnothing$ and, moreover, $\bigcup_{l \in L_j} U_l$ is closed.
- [**hm10**]**:** For all $l \in L$ we have $T_l \subseteq X - A$. A has the retraction property from above in X, $T_- \cup T_0 = \partial_X A$, $f(T_0) < u_0$ and $A^{u_0} \hookrightarrow X^{u_0} - \bigcup_{l \in L}(U_l - \tilde{U}_l)$ is deformation retract.

The *relative h-Morse decomposition* of (X, A) is a relative NDR filtration of (X, A) defined by setting $X_{-1} := A$, and for $n = 2j$

$$X_n := X_{n-1} \cup (X^{u_j} - \bigcup_{l \in L_i,\, i \geq j} (U_l - \tilde{U}_l)) \ ,$$

$$X_{n+1} := X_n \cup \bigcup_{l \in L_j} U_l \ .$$

If $J = [0, \ldots, j_{\max}]$ is finite, we always set $X_n := X$ for $n \geq 2(j_{\max} + 1)$.

Proposition 15. *If all local h-decomposition pairs have the property* [**hm6**]*, then a relative h-Morse decomposition is a relative h-CW-decomposition of* (X, A).

Proof. Let $n = 2j$. Let $r_l : (Y_l, B_l) \to (U_l, \tilde{U}_l)$ be homotopy equivalences to relative CW-complexes, and $r_n := \coprod_{l \in L_j} r_l : \coprod_{l \in L_j} (Y_l, B_l) \to \bigcup_{l \in L_j} (U_l, \tilde{U}_l)$. All $\iota_j : \bigcup_{l \in L_j} \tilde{U}_l \hookrightarrow \bigcup_{l \in L_j} U_l$ are cofibrations, and by Proposition 4 $(X_{n+1}, X_n) = (X_n \cup \bigcup_{l \in L_j} U_l, X_n) \simeq (X_n \sqcup_{r_n} \coprod_{l \in L_j} Y_l, X_n)$, a relative CW-complex with cells e^λ_γ for every cell e^λ_γ of a relative CW-complex $\coprod_{l \in L_j} (Y_l, B_l)$.

We show that $X_{n-1} \hookrightarrow X_n$ is deformation retract. By [**hm10**] $X_{-1} \hookrightarrow X_0$ is deformation retract. If $u_{j-1} = u_j$ we have $X_n = X_{n-1}$. Let $u_{j-1} < u_j$. For every $x \in X_n - X_{n-1}$ there is $0 < t < +\infty$ with $\varphi(x, t) \in A$ or $f(\varphi(x, t)) \leq u_{j-1}$ or $\varphi(x, t) \in T_l$ for some $l \in L$.

From Lemmas 8 and 9 we see that $t_{(A\cup\bigcup_{k_l<u_j} T_l)-f^{<u_{j-1}}}(x) = t_{f^{-1}(u_{j-1})\cap X-\mathrm{Int}(A\cup\bigcup_{l\in L} T_l)}(x)$ when both are defined; therefore $t_{X_{n-1}}$ is continuous on X_n. We define the deformation retraction by $h : X_n\times I \to X_n$ $(x,s)\mapsto \varphi(x, st_{X_{n-1}}(x))$. The homotopy stays in $X^{u_j} - \bigcup_{l\in L_i,\, i\geq j}(U_l - \tilde{U}_l)$ because for every $x \in X_n - X_{n-1}$ we have $\varphi(x,t) \notin \bigcup_{l\in L_i,\, i\geq j} U_l - \tilde{U}_l$ for all $t > 0$. □

1.3. Existence of Local h-Decomposition Pairs. Let G be a compact Lie group operating continuously on a differentiable manifold $\mu : G\times M \to M$, and let μ_g, defined by $\mu_g(x) := \mu(g,x)$, be differentiable for every $g \in G$. We say that G *operates differentiably on* c if the map $\mu(c)c : G \to M$, $g \mapsto g.c$ is differentiable. Let (B,g) be a Riemannian manifold, $\pi : E \to B$ a differentiable Rimannian vector bundle with a bundle metric $\langle\ ,\ \rangle_p$. Let G operate differentiably on $c \in B$ and $B \cong G/G_c$; moreover, let G operate continuously fiber-preserving and orthogonal on $\pi : E \to B$. Let $\tilde{O} \subseteq E$ be an open neighborhood of the zero section $0_{G.c}$. Let $\tilde{g} : \tilde{O} \to \mathrm{Pos}\ T(\tilde{O})$ and $\langle\ ,\ \rangle_{\xi_p}$ be Riemannian metrics on $\tilde{O}$. There is a closed invariant neighborhood $D \subseteq \tilde{O}$ of the zero section on which $\langle\ ,\ \rangle_{\xi_p}$ and $\tilde{g}(\xi_p)$ are uniformly equivalent, i.e., $a^2\langle\ ,\ \rangle_{\xi_p} \leq \tilde{g}(\xi_p) \leq b^2\langle\ ,\ \rangle_{\xi_p}$ for some $0 < a < 1 < b$ and all $\xi_p \in D$. Let $\tilde{f} : \tilde{O} \to \mathbb{R}$ be an invariant differentiable function. Let $\tilde{O}_p := \tilde{O} \cap \pi^{-1}(p)$, $\tilde{f}_p := \tilde{f}|\tilde{O}_b$ and $\mathrm{grad}\,\tilde{f}_p \in E_p$ be the gradient of $\tilde{f}_p$ formed by the Hilbert metric $\langle\ ,\ \rangle_p$ on E_p. Let $\mathrm{grad}\,\tilde{f}$ be the gradient formed by $\langle\ ,\ \rangle_{\xi_p}$ and φ be the flow of $-\mathrm{grad}\,\tilde{f}$. Assume that $\mathrm{grad}\,\tilde{f} = \mathrm{grad}\,\tilde{f}_p$ under the canonical identification of the vertical tangential bundle with the bundle itself. Let $\widetilde{\mathrm{grad}}\,\tilde{f}$ be the gradient formed by $\tilde{g}$ and $\tilde{\varphi}$ the flow of $-\widetilde{\mathrm{grad}}\,\tilde{f}$.

Lemma 16. *If* $(U_c,\tilde{U}_c)$, k *is a* G_c*-invariant local h-decomposition pair of* K_c *corresponding to* $\tilde{f}_c$ *and the flow* φ_c *of* $-\mathrm{grad}\,\tilde{f}_c$ *with* $U_c \subseteq O_c$*, then* $(U,\tilde{U}) := (G.U_c, G.\tilde{U}_c)$, k *is a local h-decomposition pair of* $K := G.K_c$ *corresponding to* $\tilde{f}$ *and* φ*. If* $(U_c,\tilde{U}_c)$ *has* **[exc]** *so does* $(U,\tilde{U})$*.*

Proof. The flow φ preserves fibres, hence **[hm2]** and **[hm4]** are fulfilled. $\xi_p \mapsto \inf\{t \geq 0|\varphi(\xi_p,t) \in U\}$ is invariant because the flow is. Choosing continuous bundle charts over V with the help of the G-operation one sees that **[hm3]** holds and also that $V\times\tilde{U}_c \hookrightarrow V\times U_c$ is a cofibration. That it is globally a cofibration can be seen by partitions of unity and [5], hence **[hm5]** holds. □

Let the gradient $\mathrm{grad}\,e(\xi_p)$ of $e(\xi_p) = \langle\xi_p,\xi_p\rangle_p$ formed by $\langle\ ,\ \rangle_{\xi_p}$ be bounded on D, and let $\tilde{f}$ satisfy the Palais–Smale condition (C) with respect to $\tilde{g}$ on D. Let $0_{G.c}$ be an isolated critical submanifold of $\tilde{f}$ and $\tilde{f}(0_{G.c}) = \kappa$. $K^{\tilde{f}} = K^{\tilde{\varphi}} = K^{\varphi}$.

Lemma 17. *For constants* ε, $\varrho > 0$ *with* $D_\varrho(0_{G.c}) \subseteq D$ *and* $\tilde{f}^{-1}[\kappa - \varepsilon, \kappa + \varepsilon] \cap D_\varrho(0_{G.c}) \cap K^{\tilde{f}} = 0_{G.c}$ *there is a* (classical) G-invariant Gromoll–Meyer pair (W, W_-), $\kappa - \delta$ *of* $0_{G.c}$ *corresponding to* φ, $\tilde{\varphi}$ *and* $\tilde{f}$ *inside* $\tilde{f}^{-1}[\kappa - \varepsilon, \kappa + \varepsilon] \cap D_\varrho(0_{G.c})$. *Every Gromoll–Meyer pair is a local h-decomposition pair.*

Proof. Choose $\varrho \geq \varrho_1 > \varrho_0 > 0$, set $\beta^2 := \inf\{\tilde{g}(\xi_p)(\widetilde{\operatorname{grad}}\tilde{f}(\xi_p), \widetilde{\operatorname{grad}}\tilde{f}(\xi_p)) \mid \xi_p \in D_{\varrho_1}(0_{G.c}) - B_{\varrho_0}(0_{G.c})\} > 0$ because condition (C) is fulfilled and set, finally, $\alpha^2 := \sup_{\xi_p \in D_{\varrho_1}(0_B)} \langle \operatorname{grad} e(\xi_p), \operatorname{grad} e(\xi_p)\rangle_{\xi_p} < \infty$. We define a function $w(\xi_p) := \langle \xi_p, \xi_p \rangle_p + \eta(\tilde{f}(\xi_p) - \kappa)$ and

$$W := w^\omega \cap \tilde{f}^{-1}[\kappa - \delta, \kappa + \delta'],$$
$$W_- := W \cap \tilde{f}^{-1}(\kappa - \delta),$$

where $\eta > \frac{\alpha}{a\beta}$, $\varrho_0^2 + \eta\delta' \leq \omega \leq \varrho_1^2 - \eta\delta$ and $0 < \delta, \delta' < \min\{\varepsilon, \frac{\varrho_1^2 - \varrho_0^2}{2\eta}\}$. (cf. [2, §2], [8, Theorem 5.3 and p. 74]). The level set $f^{-1}(\kappa - \delta) \cap W$ is regular, hence one can deform a small neighborhood of W_- in W with the help of the flow φ inside of W onto W_- and with [7, Satz 3.13, Satz 3.9, Lemma 3.4, Satz 3.26] it follows that $W_- \hookrightarrow W$ is a cofibration. □

On a differentiable Riemannian manifold (M, g) let there be a differentiable function $f : M \to \mathbb{R}$, its negative gradient $-\operatorname{grad} f$ formed with g and its flow φ. For a diffeomorphism $\psi : \tilde{O} \to \psi(\tilde{O}) \subseteq M$ set $\tilde{f} := \psi^* f : \tilde{O} \to \mathbb{R}$ and $\tilde{g} := \psi^* g$.

Corollary 18. *If* $K \subseteq \psi(\tilde{O})$ *is a critical set and* $(U, \tilde{U})$, k *a local h-decomposition pair corresponding to* $\tilde{\varphi}$ *and* $\tilde{f}$, *then the pair* $(\psi(U), \psi(\tilde{U}))$, k *is a local h-decomposition pair of* K *corresponding to* φ *and* f.

Proof. Flow lines entering $\psi(U)$ in a finite time enter $\psi(\tilde{O}) - \psi(U)$ beforehand. But there $t_{\psi(T)}(x)$ is continuous, hence it is continuous everywhere. ψ is a homeomorphism, hence the claim. □

Let $H = H^+ \oplus H^-$ be a direct sum of Hilbert spaces, O an open neighborhood of 0 and $f : (x, y) \mapsto \frac{1}{2}(\|x\|^2 - \|y\|^2)$. Let $D^\lambda \subseteq H^-$ be the closed unit disc with $\lambda = \dim H^-$. Then the following is true:

Lemma 19. $(W, W_-) \simeq (W_- \sqcup_\alpha D^\lambda, W_-) \simeq (D^\lambda, S^{\lambda-1})$, *where* $\alpha : S^{\lambda-1} \to W$ *is given by* $\bar{y} \mapsto (0, \sqrt{2\delta}\bar{y})$ [9, I §3].

Let $H = \mathbb{R}^\nu$ be finite-dimensional and $f : O \to \mathbb{R}$ be differentiable, 0 an isolated critical point, $(U, \tilde{U})$, k a local h-decomposition pair of 0 corresponding to f and φ. Furthermore, let $\varrho > 0$ with $B_\varrho(0) \subseteq U - \tilde{U}$. Then there is a differentiable function $\bar{f} : O \to \mathbb{R}$ with only finitely many nondegenerate critical points and $f|O - B_\varrho(0) = \bar{f}|O - B_\varrho(0)$ [10, Lemma

8.6], [1, eq. (3)] or [8, Theorem 5.7]. $(U, \tilde{U})$, k is also a local h-decomposition pair corresponding to $\bar{f}$ and its negative gradient flow $\bar{\varphi}$.

Corollary 20. *If $(U, \tilde{U})$, k has* [**exc**] *it has* [**hm6**], *i.e., there is a homotopy equivalent finite relative CW-complex $(Y, \tilde{U})$.*

Proof. Construct a relative h-Morse decomposition of $(U, \tilde{U})$ with the help of $\bar{f}$. Take open disjoint Morse charts for the finitely many critical points of $\bar{f}$ in the interior of $U - \tilde{U}$ and the images of Gromoll–Meyer-pairs, using Corollary 18 and Lemma 19. The claim follows from Lemma 19, Proposition 15 and Lemma 3. □

Proposition 21 ([11]). *If $(p, \tilde{p}) : (E, \tilde{E}) \to B$ is a fibration pair of Hurewicz fibrations (including the property of it being a closed cofibred pair), if B is 0-connected and homotopy equivalent to a CW-complex P, and if the fibres $(F, \tilde{F})$ are homotopy equivalent to some relative CW-complex (K, L), then $(E, \tilde{E})$ is homotopy equivalent to a relative CW-complex having the same numbers of cells of the same dimension as the product of CW-complexes $P \times (K, L)$.*

Proof. We can assume that B is already a CW-complex. $(E, \tilde{E})$ is a cofibred pair over B. Look first at $B' \cup_\varphi D^n \cong B' \cup e^n = B' \cup \chi(D^n)$. Set $E' := p^{-1}(B')$ and $\tilde{E}' := E' \cap \tilde{E}$. $(E' \cup \tilde{E}) \cap p^{-1}(e^n) = \tilde{p}^{-1}(e^n) \cup p^{-1}(\partial e^n) \hookrightarrow p^{-1}(e^n)$ is a closed cofibration (use [12, 7.3.9]). $(\chi^* E, \chi^* \tilde{E})$ is a cofibred pair, hence because $D^n \simeq *$ there is a fiber homotopy equivalence of pairs $\Phi : D^n \times (F, \tilde{F}) \to (\chi^* E, \chi^* \tilde{E})$. Using a homotopy equivalence $i : (K, L) \to (F, \tilde{F})$, set $\mu := p^*\chi \circ \Phi \circ (\mathrm{id} \times i) | D^n \times A \cup S^{n-1} \times X$ and define

$$(K', L') := ((E' \cup \tilde{E}) \sqcup_\mu (D^n \times X), E' \cup \tilde{E}) ,$$

which is a relative CW-complex with one cell for every cell in $(D^n, S^{n-1}) \times (K, L)$. From Corollary 5 we have $(K', L') \simeq (E' \cup \tilde{E} \cup p^{-1}(e^n), E' \cup \tilde{E})$. If B has only countably many cells we can define by this process a relative h-Morse decomposition of $(E, \tilde{E})$ (otherwise proceed by skeletons) and use Lemma 3. □

Corollary 22. *If in Lemma 16 $(U_c, \tilde{U}_c)$ is homotopy equivalent to the relative CW-complex (K, L) and G/G_c homotopy equivalent to the finite CW-complex P, then $(U, \tilde{U}) := (G.U_c, G.\tilde{U}_c)$ is homotopy equivalent to a relative CW-complex with the same number of cells as $P \times (K, L)$, hence it has the property* [**hm6**].

1.4. Excision of Local h-Decomposition Pairs. Cf. [1, §2 (V_1, V_1^-) and equation (9)].

Proposition 23 (Excision and Change of Metric). *If $(U, \tilde{U})$, k is a local h-decomposition pair of the isolated critical orbit $0_{G.c}$ corresponding to φ and $\tilde{f}$ with the property* [**exc**], *if $U \subseteq \tilde{O}$ and if $\tilde{f}$ has (C) on T_U with respect to $\langle\ ,\ \rangle_{\xi_p}$, then for all $\varrho, \varepsilon > 0$ there is a homotopy equivalent local h-decomposition pair $(V, \tilde{V})$, k_V of $0_{G.c}$ corresponding to φ, $\tilde{\varphi}$ and $\tilde{f}$ with $T_V \subseteq B_\varrho(0_{G.c}) \cap \tilde{f}^{-1}[\kappa - \varepsilon, \kappa + \varepsilon]$ and again with* [**exc**]. *If $(U, \tilde{U})$ and $\tilde{f}$ are G-invariant, then $(V, \tilde{V})$ can be chosen G-invariant as well and homotopy G-equivariant.*

Let $W \subseteq T$ be a closed subset with the mean value property, W_+ have the retraction property from above, and $K \subseteq \mathrm{Int}(W)$. Let $\delta > 0$ with $\tilde{f}(\partial W - W_+) \leq \kappa - \delta$ and $\tilde{f}(T_{U-}) \leq \kappa - \delta$. Choose $\delta \geq \tilde{\delta} > 0$, denote $\tilde{\varepsilon} := \delta - \tilde{\delta}$, set $k_V := \kappa - \tilde{\delta}$ and define the pair as follows:

$$\begin{aligned} S &:= T^{k_V - \tilde{\varepsilon}} \cup W, \\ \tilde{S} &:= T^{k_V - \tilde{\varepsilon}} \cup W^{k_V}. \end{aligned}$$

Claim 24. $(S, \tilde{S}) \simeq (T, T_-)$ *rel* T_-.

Proof. $t_S(x) := \min\{t \geq 0 | \varphi(x,t) \in S\}$ is continuous on T according to Lemma 9. Define $i : (T, T_-) \to (S, \tilde{S})$ by $x \mapsto \varphi(x, t_S(x))$ and

$$\begin{array}{rcl} r : (S, \tilde{S}) & \to & (T, T_-) \\ x & \mapsto & \left\{ \begin{array}{lll} \varphi(x, t_{T_-}(x)) & for & f(x) \leq k_V \\ \varphi\left(x, \frac{2(\kappa - f(x)) - \tilde{\delta}}{\tilde{\delta}} t_{T_-}(x)\right) & for & f(x) \in [k_V, \kappa - \tilde{\delta}/2] \\ x & for & f(x) \geq \kappa - \tilde{\delta}/2 \end{array} \right. . \end{array}$$

All maps are the identity on T_-. □

Corollary 25. *The pair $(V, \tilde{V}) := (S \cup \tilde{U}, \tilde{S} \cup \tilde{U})$, k_V is a local h-decomposition pair with $(U, \tilde{U}) \simeq (V, \tilde{V})$ and $T_V = \mathrm{cl}(S - \tilde{S}) \subseteq W$.*

Proof of Proposition 23. Choose an invariant Gromoll–Meyer pair (W, W_-) of $0_{G.c}$ with $W \subseteq B_{\varrho'}(0_{G.c}) \cap f^{-1}[\kappa - \varepsilon', \kappa + \varepsilon']$, where $\varepsilon' := \min\{\varepsilon, \kappa - k_U\}$ and $\varrho > \varrho' > 0$ is such that $B_{\varrho'}(0_{G.c}) \cap f^{-1}[\kappa - \varepsilon', \kappa + \varepsilon'] \subseteq U - \tilde{U}$. Choose $\tilde{\delta} > 0$ with $\varepsilon' > \tilde{\delta}$ and set $k_V := \kappa - \tilde{\delta}$. Then use Corollary 25. □

1.5. Product Proposition. Let there be given an equivariant orthogonal bundle sum $E \cong E^+ \oplus E^- \oplus E^0$, $\xi_p = (\xi_p^+, \xi_p^-, \xi_p^0)$, and an *equivariant characteristic manifold*

$$\begin{aligned} W_{\text{char}} : (D^0_{\varrho_2}(0_{G.c}), 0_{G.c}) &\to (\tilde{O}, 0_{G.c}) \\ \xi_p^0 &\mapsto (h(\xi_p^0), \xi_p^0) \end{aligned}$$

as in the generalized equivariant Morse Lemma. Then there are $\varrho > 0$ such that $D_\varrho(W_{\text{char}}(D_{\varrho_2}(0_{G.c}))) \subseteq \tilde{O}$, some numbers $\varrho_1, \varrho_3 > 0$ and a continuous equivariant fiber preserving homeomorphism, a *generalized equivariant Morse chart*,

$$\begin{aligned} \Phi : (D^{\pm}_{\varrho_3}(0_{G.c}) \times D^0_{\varrho_1}(0_{G.c}), 0_{G.c}) &\longrightarrow D_\varrho(W_{\text{char}}(D_{\varrho_2}(0_{G.c}))) \\ (\xi_p^{\pm}, \xi_p^0) &\mapsto \eta(\xi_p^{\pm}, \xi_p^0) + \xi_p^0 + h(\xi_p^0) \ , \end{aligned}$$

which extends W_{char} (which means $\eta(0_p^{\pm}, \xi_p^0) = 0_p$), with $\eta(\xi_p^{\pm}, \xi_p^0) \in E^+ \oplus E^-$, and

$$\tilde{f}_p \circ \Phi(\xi_p) = \frac{1}{2}(\|\xi_p^+\|_p^2 - \|\xi_p^-\|_p^2) + \tilde{f}_p(h(\xi_p^0), \xi_p^0) =: \tilde{f}_p^1(\xi_p^+, \xi_p^-) + \tilde{f}_p^0(\xi_p^0) \ .$$

If $\tilde{f}$ is sufficiently differentiable, then Φ is a local diffeomorphism [13, Prop. 4.2.1], [10, Theorem 8.3], [8, Corollary 7.1].

To clarify here the dark points in [2] and some other places we should make the following point clearer. The Whitney sum representation of E and Morse-charts as above exist under conditions on the spectrum of the second derivative $d^2\tilde{f}(0_p)$. The subbundles are G-invariant if the map induced by every element $g \in G$ is differentiable and isometric. If this is the case and E_p^0 is finite-dimensional, then E^0 is a finite-dimensional manifold on which G acts differentiably [14, V.1. Corollary]. In our case the map is fiberwise linear and orthogonal. Choose some linear connection $C^{\pm}$ on $E^{\pm} := E^+ \oplus E^-$.

Lemma 26. *If E^0 is finite dimensional and the action of G is fiber preserving and orthogonal, then there are a canonical linear connection $C^0 : T(B) \oplus E^0 \to T(E^0)$ and a canonical Riemannian metric $\langle\ ,\ \rangle_{\xi_p}$ on E. The gradient formed by this metric* $\operatorname{grad}(\tilde{f} \circ \Phi)$ *is in the vertical tangent space $T^v(E)$.*

Proof. Represent $X_p \in T_p(B)$ by a differentiable curve $\gamma : (-\varepsilon, \varepsilon) \to G$ and for $\xi_p^0 \in E_p^0$ set $C_p^0(X_p, \xi_p^0) = (\gamma(t).\xi_p^0)'(0)$. This map is linear in the second variable because G operates linearly on fibers and is continuous in both variables because G operates differentiably. Let $C := C^{\pm} \oplus C^0$ be the product connection. The canonical horizontal tangent space is given by $T^h_{\xi_p}(E) := C_p(T_p(B), \xi_p)$ and the canonical metric by [15, Def. 1.9.12, 1.11.9]. $\square$

Therefore $\operatorname{grad}(\tilde{f} \circ \Phi) = \operatorname{grad}(\tilde{f} \circ \Phi)_p$. It also follows that 0_p is an isolated critical point of $\tilde{f}_p$ iff $0_{G.c}$ is an isolated critical manifold of $\tilde{f}$.

Lemma 27. *Given constants $0 < \varrho_0 < \varrho_1$, there is an equivariant diffeomorphism $\tilde{\Phi} : D^{\pm}_{\varrho_3}(0_{G.c}) \times D^0_{\varrho_2}(0_{G.c}) \to D_\varrho(W_{char}(D_{\varrho_2}(0_{G.c})))$ extending W_{char} such that $\tilde{\Phi}|D_{\varrho_3} \times D_{\varrho_0} = \Phi|D_{\varrho_3} \times D_{\varrho_0}$.*

Proof. Choose $\chi \in C^\infty([0, \varrho_2], \mathbb{R}^+_0)$ with $\chi|[0, \varrho_0] \equiv 1$ and $\chi|[\varrho_1, \varrho_2] \equiv 0$, define $\tilde{\Phi} : (\xi^\pm_p, \xi^0_p) \mapsto \xi^0_p + h(\xi^0_p) + \eta(\xi^\pm_p, \chi(\|\xi^0_p\|) \cdot \xi^0_p + (1 - \chi(\|\xi^0_p\|))\varrho_1 \cdot \frac{\xi^0_p}{\|\xi^0_p\|})$. If $\|\xi^0_p\|$ is invariant, so is $\chi(\|\xi^0_p\|)$. $\square$

Let $H^1 \times H^0$ be the product of Hilbert spaces, $O^1 \times O^0$ an open neighborhood of 0, and $f : O^1 \times O^0 \to \mathbb{R}$ a differentiable function. Let there be $D^1 \times D^0 \subseteq O^1 \times O^0$, a closed neighborhood of 0, on which $f = f^1 + f^0 : (x, z) \mapsto f^1(x) + f^0(z)$. Denote by φ^1 (respectively φ^0) the partial negative gradient flows of f^1 (respectively of f^0) and by φ that of f.

Proposition 28 (Product Proposition). *If H^0 is finite dimensional, if we are given a local h-decomposition pair $(U^0, \tilde{U}^0)$, k_{U^0} of 0^0 corresponding to φ^0 and f^0 which is compact and has the property* [**exc**], *and if $0^1 \in D^1$ is an isolated critical point of f^1 which satisfies the Palais–Smale condition (C) on D^1, then there are a local h-decomposition pair $(U^1, \tilde{U}^1)$, k_1 of 0^1 corresponding to f^1 and φ^1, a local h-decomposition pair $(U, \tilde{U})$, k of 0 corresponding to φ and f and a homotopy equivalence to the product*

$$(U, \tilde{U}) \simeq (U^1, \tilde{U}^1) \times (U^0, \tilde{U}^0) \ .$$

$(U, \tilde{U})$, k again has the property [**exc**]. *All pairs and homotopies can be chosen to be equivariant.*

We prove a technical lemma first.

Let $(V^1, \tilde{V}^1)$, k_1 and $(V^0, \tilde{V}^0)$, k_0 be local h-decomposition pairs of K^1, K^0 corresponding to φ^1, φ^0 and f^1, f^0 respectively with $V^1 \subseteq O^1$ and $V^0 \subseteq O^0$. We use the following notations: $T^1 := T_{V^1}$ and $T^0 := T_{V^0}$. Let $T^1 \subseteq \operatorname{Int} D^1$ and $T^0 \subseteq \operatorname{Int} D^0$; then $(T^1 \times T^0)$ has the mean value property, and $K = K^1 \times K^0$ are the critical points of f in $T^1 \times T^0$. Let $f(T^i) \subseteq [k_i - \varepsilon_i, k_i + \varepsilon'_i]$ and $f(K^i) \geq \kappa_i$ for $i = 0, 1$; then we have $f(K) \geq \kappa_1 + \kappa_0$ and $f(\partial(T^1 \times T^0) - (T^1 \times T^0)_+) \leq \max\{k_1 + k_0 + \varepsilon'_0, k_0 + k_1 + \varepsilon'_1\} = k_1 + k_0 + \max\{\varepsilon'_1, \varepsilon'_0\}$.

Lemma 29. *If $k_1 + k_0 + \max\{\varepsilon'_1, \varepsilon'_0\} < \kappa_1 + \kappa_0$, if both pairs have the property* [**exc**], *if there exists a number $k \in [k_1 + k_0 + \max\{\varepsilon'_1, \varepsilon'_0\}, \kappa_1 + \kappa_0)$, such that $f((V^1 - D^1) \times V^0 \cup V^1 \times (V^0 - D^0)) \leq k$ and if condition (C) is satisfied by f^0 on T^0 and f^1 on T^1, then there exists (equivariant) local h-decomposition pair $(U, \tilde{U})$, k of K corresponding to φ and f with* *the property* [**exc**], *and there is* (*equivariant*) *homotopy equivalence to the product*

$$(U,\tilde{U}) \simeq (V^1,\tilde{V}^1) \times (V^0,\tilde{V}^0)\ .$$

Proof. It immediately follows that $f(\tilde{V}^1 \times V^0 \cup V^1 \times \tilde{V}^0) \le k$. Define

$$U := V^1 \times V^0,$$
$$\tilde{U} := (V^1 \times V^0)^k\ ,$$

and [**hm1**], [**hm2**], [**hm4**], [**hm5**] and [**exc**] follow easily. [**hm3**] is true, because $t_{T_1\times T_0}(x,z) = \max\{t_{T_1}(x), t_{T_0}(z)\}$ and $T^1 \times T^0 \subseteq \mathrm{Int}(D^1 \times D^0)$. It only remains to show the existence of homotopy equivalence. f satisfies (C) on $T^1 \times T^0$ and $(T_1 \times T_0)_-$ has the retraction property from above in $T^1 \times T^0$; hence we can define a deformation retraction of pairs $r : (T^1 \times T^0, (T^1 \times T^0)^k) \to (T^1 \times T^0, (T^1 \times T^0)_-)$ as in Claim 24 and Proposition 23. $\square$

Proof of Proposition 28. By hypothesis $B_{\varrho_0}(0^0) \subseteq D^0$ for some $\varrho_0 > 0$; hence by Proposition 23 there is a compact local h-decomposition pair $(V^0, \tilde{V}^0)$, $k_0 = \kappa_0 - \delta_0$ with [**exc**], $T_{V^0} \subseteq B_{\varrho_0}(0^0) \cap (f^0)^{-1}[k_0 := \kappa_0 - \delta_0, \kappa_0 + \delta_0']$, where δ_0, $\delta_0' > 0$ and δ_0' can later be chosen arbitrarily small. There is $\varrho > 0$ with $f(B_\varrho(\{0^1\} \times V^0) - \mathrm{Int}\, D^1 \times B_{\varrho_0}(0^0)) \le \kappa_1 + \kappa_0 - \delta_0/2$ and $B_\varrho(0^1) \subseteq D^1$. Choose a Gromoll–Meyer pair (W^1, W^1_-) of 0^1 for f^1 with $W^1 \subseteq B_\varrho(0^1)$, $f^1(W^1) \subseteq [k_1 := \kappa_1 - \delta_1, \kappa_1 + \delta_1']$, $W^1_- = W^1 \cap (f^1)^{-1}(\kappa_1 - \delta_1)$, where δ_1, $\delta_1' > 0$ and $\delta_1' < \delta_0$. $\varepsilon_i' := \delta_1 + \delta_i'$ and therefore $k_1 + k_0 + \max\{\varepsilon_1', \varepsilon_0'\} = \kappa_1 - \delta_1 + \kappa_0 - \delta_0 + \max\{\delta_1 + \delta_1', \delta_0 + \delta_0'\} = \kappa_1 + \kappa_0 - \min\{\delta_0 - \delta_1', \delta_1 - \delta_0'\}$. We see, that if we shrink δ_0' such that $\delta_0' < \delta_1$, then the sum becomes smaller than $\kappa_1 + \kappa_0$. Choose $\kappa_1 + \kappa_0 > k > \kappa_1 + \kappa_0 - \min\{\delta_0/2, \delta_1 - \delta_0', \delta_0 - \delta_1'\}$ and use Lemma 29. $\square$

2. The Geometric Gromoll–Meyer Theorem

Let (M,g) be a closed differentiable Riemannian manifold and $\Lambda M := H^1(S^1, M)$ the space of free closed Sobolev loops. If c is a differentiable closed loop then $T_c(\Lambda M) \cong H^1(S^1, c^*T(M))$, where $H^1(S^1, c^*T(M))$ is the space of H^1-vector fields along c on which $\mathbb{O}(2)$ acts orthogonally. On differentiable curves $\mathbb{O}(2)$ acts differentiably. Even though ΛM is not a differentiable $\mathbb{O}(2)$-manifold, there is a kind of equivariant exponential map $\widetilde{\exp} : T(\Lambda M) \supseteq \tilde{O} \to \Lambda M$ fibrewise induced by the exponential map $\exp : TM \supseteq O \to M$ of the manifold [15, 2.3.12]. We define fibrewise an energy integral $\tilde{E}$ on $T\Lambda M$ by $\xi_c \mapsto \frac{1}{2}\int_0^1 \|\nabla_1 \xi_c(t)\|^2_{c(t)} dt$. $\tilde{E}$ coincides on $\tilde{O}$ with the pull back $\widetilde{\exp}^* E$. Denote by φ the negative gradient flow of E on ΛM.

2.1. Local h-Decomposition Pairs for Iterated Geodesics. Denote by $I(M)$ the isometry group of M and set $G := I(M) \oplus \mathbb{O}(2)$. If c is a closed geodesic, then the G-orbit $G.c$ is a differentiable submanifold [16, I.5.4]. If m_c is the order of c, choose, on $T_c\Lambda M$, the G_c-equivariant metric

$$\langle \xi_c, \xi_c' \rangle_{c,m_c} := \langle \xi_c, \xi_c' \rangle_0 + \frac{1}{m_c^2} \langle \nabla_1 \xi_c, \nabla_1 \xi_c' \rangle_0$$

[2, §3 eqn. (14)], [13, §4.2] and denote its norm by $\| \ \|_{1,m_c}$. The iteration map $m_* : T\Lambda M|G.c \longrightarrow T\Lambda M|G.c^m$ is then an isometric embedding. Identify the normal bundle $\pi_c : N(G.c) \to G.c$ with the orthogonal complement of $TG.c$. For arbitrary m we choose the domain of the exponential map $\tilde{O} \cap N(G.c^m)$ as tubular neighborhoods $(\tilde{O}(G.c^m), \pi_{c^m}, \widetilde{exp})$; then $m_*(\tilde{O}(G.c)) \hookrightarrow \tilde{O}(G.c^m)$ is a linear bundle map.

Lemma 30 ([2, Lemma 5], [13, Lemma 4.2.5]). *Let c be a (iterated) nontrivial closed geodesic. If $\nu(c) = \nu(c^m)$, then $m_*|N_c^0$ is an isomorphism onto $N_{c^m}^0$. If W_{char} is a characteristic manifold for $\tilde{E}_c$, then $m_* \circ W_{char} \circ m_*^{-1}$ is a characteristic manifold for $\tilde{E}_{c^m}$.*

Corollary 31 ([2, Theorem 3]). *If $0_c \in (W_c^0, W_{c-}^0) \subseteq \tilde{O}_c^0$ is a Gromoll–Meyer pair for the degenerate part $\tilde{E}_c^0$, then $m_*(W_c^0, W_{c-}^0)$ is a homeomorphic compact Gromoll–Meyer pair for $\tilde{E}_{c^m}^0$ (formed by the iterated characteristic manifold) and the chosen Riemannian metric.*

Proof. We have $\|\xi_c^m\|_{1,mm_c}^2 + \frac{1}{m^2}\eta \tilde{E}_{c^m}^0(\xi_c^m) = \|\xi_c\|_{1,m_c}^2 + \eta \tilde{E}_c^0(\xi_c)$ and, moreover, $\xi_c^m + \frac{1}{m^2}\eta \operatorname{grad}_{mm_c} \tilde{E}_{c^m}^0(\xi_c^m) = \xi_c^m + \eta m_* \operatorname{grad}_{m_c} \tilde{E}_c^0(\xi_c)$. $\square$

Lemma 32. *There is an arbitrarily small invariant Gromoll–Meyer pair $(W_{c^m}^1, W_{c^m-}^1)$ of $0_{c^m}^{\pm}$ corresponding to the nondegenerate part $\tilde{E}_{c^m}^1$, an invariant local h-decomposition pair $(U_{c^m}, \tilde{U}_{c^m}) \simeq (W_{c^m}^1, W_{c^m-}^1) \times (W_c^0, W_{c-}^0)$ of 0_{c^m} and an invariant local h-decomposition pair $(V, \tilde{V})$, k_V of the zero section $0_{G.c^m}$ with $(V, \tilde{V}) \simeq G \times_{G_{c^m}} (U_{c^m}, \tilde{U}_{c^m})$. Furthermore, there exists a homeomorphic invariant local h-decomposition pair $(V', \tilde{V}')$, k_V of the orbit $G.c^m$. Given ϱ, $\varepsilon > 0$, there is a homotopy equivalent local h-decomposition pair $(U, \tilde{U})$, k_U of the orbit with $T_U \subseteq B_\varrho(G.c^m) \cap [m^2\kappa - \varepsilon, m^2\kappa + \varepsilon]$ corresponding to E and φ.*

Proof. By Lemma 27 there is an equivariant extended Morse chart $\tilde{\Phi}$: $(D_{\varrho_3}^{\pm}(0_{G.c^m}) \times D_{\varrho_2}^0(0_{G.c^m}), 0_{G.c^m}) \to (\tilde{O}, 0_{G.c^m})$ extending $m_* \circ W_{\text{char}} \circ m_*^{-1}$. Denote by $\tilde{E}_{c^m}^1 := E \circ \widetilde{\exp} \circ \tilde{\Phi}|D_{\varrho_3}^{\pm}(0_{c^m})$. The function $\tilde{E}_{c^m}^1(\xi^{\pm}) = \frac{1}{2}(\|\xi_c^+\|_{m_c}^2 - \|\xi_c^-\|_{m_c}^2)$ satisfies (C) with respect to $\langle \ , \ \rangle_{m_c}$ on every closed neighborhood $D_{c^m}^{\pm} \subseteq D_{\varrho_3}^{\pm}(0_{c^m})$.

From Product Proposition 28 with $D^1 = D_{c^m}^{\pm}$ and $D^0 = D_{\varrho_0}^0(0_{c^m})$ it follows that there are a Gromoll–Meyer pair $(W_{c^m}^1, W_{c^m-}^1)$ of $0_{c^m}^{\pm}$ with $W_{c^m}^1 \subseteq$

$B^{\pm}_{\varrho_3}(0_{c^m})$ corresponding to $\tilde{E}^1_{c^m}$ and $\langle\ ,\ \rangle^{\pm}_{mm_c}$ and a local h-decomposition pair $(U_{c^m}, \tilde{U}_{c^m})$, k of 0_{c^m} with $U_{c^m} \subseteq B^{\pm}_{\varrho_3}(0_{c^m}) \times B^0_{\varrho_2}(0_{c^m})$ corresponding to $(\tilde{E} \circ \tilde{\Phi})_{c^m}$ and $\langle\ ,\ \rangle_{mm_c}$, with $(U_{c^m}, \tilde{U}_{c^m}) \simeq (W^1_{c^m}, \tilde{W}^1_{c^m-}) \times (W^0_c, W^0_{c-})$.

From Lemma 16 it follows that $(U, \tilde{U}) := (G.U_{c^m}, G.\tilde{U}_{c^m})$, k is a local h-decomposition pair of $0_{G.c^m}$ corresponding to $\tilde{E}$ and the canonical metric $\langle\ ,\ \rangle_{\xi_p}$ of Lemma 26 formed by $\langle\ ,\ \rangle_{m_c}$. From Proposition 23 (change of metric) it follows that there is a local h-decomposition pair $(V, \tilde{V})$, k_V corresponding to $\tilde{E} \circ \tilde{\Phi}$ and to the pull back metric $\tilde{g}$. $(V', \tilde{V}') := (\widetilde{\exp} \circ \tilde{\Phi}(V), \widetilde{\exp} \circ \tilde{\Phi}(\tilde{V}))$, k_V is a local h-decomposition pair of $G.c^m$ corresponding to φ and E with properties [**exc**] by Corollary 18. The assertion of the lemma follows from Proposition 23. □

Lemma 33. *If $\lambda := \lambda(c^m)$ is the index of the iterated geodesic and $(W^0_c, W^0_{c-}) \simeq (K, L)$ is homotopy equivalent to some relative CW-complex and $G.c^m \simeq P$ to some CW-complex, then $(U, \tilde{U})$ is homotopy equivalent to a relative CW-complex (Y, B) having as many cells of the same dimension as $P \times (D^\lambda, S^{\lambda-1}) \times (K, L)$.*

Proof. From Lemma 19 it follows that $(W^1_c, W^1_{c-}) \simeq (D^\lambda, S^{\lambda-1})$. The claim follows from Corollary 22. □

Note that the iteration map m_* induces a homeomorphism between the orbits $G.c$ and $G.c^m$ because it is equivariant.

2.2. h-Morse Decomposition of ΛM.

Proof of Theorem 1. Let there be n prime distinct nontrivial closed geodesics $\{c_n\}_{n\in\{1,\dots,n_{\max}\}}$ on M and denote $\kappa_n := E(c_n)$. For every prime geodesic choose a partition

$$\{l_{in} m_{ijn}\} \quad \text{of} \quad \mathbb{N}^*, \quad \{l_{in}\}_{i\in\{1,\dots,r_n\}},$$

$(m_{ijn})_{i\in\{1,\dots,r_n\}, j\in\mathbb{N}^*}$ with $\nu(c_n^{l_i m_{ij}}) = \nu(c_n^{l_i})$ as in [2, Lemma 2] or [13, Proposition 4.2.6]. For every n and i choose a characteristic manifold $W^{in}_{char} : D^0(G.c_n^{l_{in}}) \to \tilde{O}(G.c_n^{l_{in}})$ and a Gromoll–Meyer pair $(W^0_{in}, W^0_{in-}) \subseteq D^0(0_{c_n^{l_{in}}})$ for $0_{c_n^{l_{in}}}$ corresponding to $\tilde{E}^0_{c_n^{l_{in}}}$ with $\tilde{E}^0_{c_n^{l_{in}}}(W^0_{in}) \geq \kappa_{in} - \delta_{in}$, and $0 < 3\delta_{in} < \kappa_{in} := l_{in}^2 \kappa_n$.

There are only finitely many critical orbits in E^u for all u and W^0_{in} is compact. From Lemma 32 with $m = l_{in} m_{ijn}$ it follows that there is a local h-decomposition pair $(V_{nm}, \tilde{V}_{nm})$, k'_{nm} of $0_{G.c_n^m}$ containing only this critical orbit and having the property [**exc**] and $(V_{nm}, \tilde{V}_{nm}) \cap N_{c_n^m} \simeq (W^1_{nm}, W^1_{nm-}) \times (W^0_{in}, W^0_{in-})$ for some Gromoll–Meyer pair (W^1_{nm}, W^1_{nm-}). The pair can be chosen such that $\tilde{E}(V_{nm}) \geq m^2_{ijn}(\kappa_{in} - 2\delta_{in})$. Set $(V'_{nm}, \tilde{V}'_{nm}) := \widetilde{\exp}(V_{nm}, \tilde{V}_{nm})$, k'_{nm}. Because E^u intersects only finitely many V'_{nm} and $G.c_n^m$ is compact there are homotopy equivalent local h-decomposition pairs $(U_{nm}, \tilde{U}_{nm})$, k_{nm} of the orbits $G.c_n^m$ with $k_{nm} \geq k'_{nm}$ and tops T_{nm} disjoint from all other $U_{n'm'}$. Set $K_0 := \Lambda^0 M \cong M$ and choose $k_0 < \min_{n,i}\{\kappa_{in} - 2\delta_{in}\}$ which is not critical. Then $k_0 < k_{nm}$, $K_0 \hookrightarrow \Lambda^{k_0} M$ is a deformation retract and $(U_0, \varnothing) := (\Lambda^{k_0} M, \varnothing)$, k_0 is a local h-decomposition pair of K_0. The set $\{K_0, K_{nm} := G.c_n^m\}_{n,m}$ is a partition of the critical points of E. Choose $J = \mathbb{N}$, enumerate $\{k_0, k_{nm}\}$ in nondecreasing order and obtain a (absolute) h-Morse decomposition of ΛM.

Choosing fixed relative CW-complexes $(K_{in}, L_{in}) \simeq (W_{in}^0, W_{in-}^0)$ and $P_n \simeq G.c_n$ and $(Y_{nm}, B_{nm}) := P_n \times (D^{\lambda(c_n^m)}, S^{\lambda(c_n^m)-1}) \times (K_{in}, L_{in}) \simeq (U_{nm}, \tilde{U}_{nm})$, the claim follows from combinatorial considerations as in [2, Corollary 2, Theorem 4] and [13, Lemma 4.2.5]. □

References

1. D. Gromoll and W. Meyer, On differentiable functions with isolated critical points. *Topology* **8**(1969), 361–369.

2. D. Gromoll and W. Meyer, Periodic geodesics on compact Riemannian manifolds. *J. Differential Geom.* **3**(1969), 493–510.

3. M. Vigué-Poirrier and D. Sullivan, The homology theory of the closed geodesics problem. *J. Differential Geom.* **11**(1976), 633–644.

4. A. T. Lundell and S. Weingram, The topology of CW complexes. *van Nostrand,* 1969.

5. T. tom Dieck, Partitions of unity in homotopy theory. *Compositio Math.* **23**(1971), 159–167.

6. R. Fritsch and R. A. Piccinini, Cellular structures in topology. *Cambridge Stud. Adv. Math.* **19**, *Cambridge Univ. Press, Cambridge,* 1990.

7. T. tom Dieck, K. H. Kamps and D. Puppe, Homotopietheorie. *Lecture Notes in Math., vol.* 157, *Springer, Heidelberg,* 1970.

8. K. C. Chang, Infinite dimensional Morse theory and multiple solution problems. *Birkhäuser, Boston,* 1993.

9. J. Milnor, Morse theory. *Ann. of Math. Stud.* **51**, *Princeton Univ. Press, Princeton,* 1963.

10. J. Mawhin and M. Willem, Critical point theory and hamiltonian systems. *Appl. Math. Sci.* **74**, *Springer, Heidelberg,* 1989.

11. J. Stasheff, A classification theorem for fibre spaces. *Topology* **2**(1963), 239–246.

12. R. Brown, Topology, 2.ed. *John Wiley and Sons, Brisbane,* 1988.

13. W. Klingenberg, Lectures on closed geodesics. *Grundlehren Math. Wiss.* **230**, *Springer, Heidelberg,* 1978.

14. D. Montgomery and L. Zippin, Topological transformation groups. *John Wiley & Sons,* 1966.

15. W. Klingenberg, Riemannian geometry. *de Gruyter Stud. Math.* **1**, *de Gruyter, Berlin,* 1982.

16. T. tom Dieck, Transformation groups. *de Gruyter Stud. Math.* **8**, *de Gruyter, Berlin,* 1987.

17. T. Morgenstern, Homotopietyp des freien Schleifenraums einer Mannigfalhgkeit mit endlich vielen Geodätischen. (Thesis) *Universität Heidelberg,* 1995.

(Received 7.04.1995)

Author's address:
Mathematisches Institut der Universität Heidelberg
Im Neuenheimer Feld 288, 69122 Heidelberg
Germany