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ANALOGUES OF THE KOLOSOV–MUSKHELISHVILI GENERAL REPRESENTATION FORMULAS AND CAUCHY–RIEMANN CONDITIONS IN THE THEORY OF ELASTIC MIXTURES

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Abstract. Analogues of the well-known Kolosov–Muskhelishvili formulas of general representations are obtained for nonhomogeneous equations of statics in the case of the theory of elastic mixtures. It is shown that in this theory the displacement and stress vector components, as well as the stress tensor components, are represented through four arbitrary analytic functions.

The usual Cauchy–Riemann conditions are generalized for homogeneous equations of statics in the theory of elastic mixtures.

1. In this section we shall derive analogues of the Kolosov–Muskhelishvili general representation formulas for nonhomogeneous equations of statics in the theory of elastic mixtures. It will be shown that displacement and stress vector components, as well as stress tensor components, are represented in this theory by means of four arbitrary analytic functions.

The representations obtained here will be used in our next papers to investigate two-dimensional boundary value problems for the above-mentioned equations of an elastic mixture.

In the two-dimensional case the basic nonhomogeneous equations of the theory of elastic mixtures have the form (see [1] and [2])

$$\begin{aligned} &a_1\Delta u' + b_1 \operatorname{grad}\operatorname{div} u' + c\Delta u'' + d \operatorname{grad}\operatorname{div} u'' = -\rho_1 F' \equiv \psi', \\ &c\Delta u' + d \operatorname{grad}\operatorname{div} u' + a_2\Delta u'' + b_2 \operatorname{grad}\operatorname{div} u'' = -\rho_2 F'' \equiv \psi'', \end{aligned} \tag{1.1}$$

where Δ is the two-dimensional Laplacian, grad and div are the principal operators of the field theory, ρ_1 and ρ_2 are the partial densities (positive constants) of the mixture, F' and F'' are the mass force, $u' = (u_1', u_2')$ and

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$u'' = (u''_1, u''_2)$ are the displacement vectors, a_1,, b_1, c, d, a_2, b_2 are the known constants characterizing the physical properties of the mixture. We have

$$\begin{gathered} a_1 = \mu_1 - \lambda_5, \quad b_1 = \mu_1 + \lambda_1 + \lambda_5 - \rho^{-1}\alpha_2\rho_2, \quad a_2 = \mu_2 - \lambda_5, \\ c = \mu_3 + \lambda_5, \quad b_2 = \mu_2 + \lambda_2 + \lambda_5 + \rho^{-1}\alpha_2\rho_1, \\ d = \mu_3 + \lambda_3 - \lambda_5 - \rho^{-1}\alpha_2\rho_1 \equiv \mu_3 + \lambda_4 - \lambda_5 + \rho^{-1}\alpha_2\rho_2, \\ \rho = \rho_1 + \rho_2, \quad \alpha_2 = \lambda_3 - \lambda_4, \end{gathered} \tag{1.2}$$

where μ_1, μ_2, μ_3, λ_1, λ_2, λ_3, λ_4, λ_5 are new constants also characterizing the physical properties of the mixture and satisfying the definite conditions (inequalities) [2].

In what follows we shall need the homogeneous equations corresponding to equations (1.1); obviously, they have the form ($F' = F'' = 0$ or $\psi' = \psi'' = 0$)

$$\begin{aligned} &a_1\Delta u' + b_1 \operatorname{grad}\operatorname{div} u' + c\Delta u'' + d \operatorname{grad}\operatorname{div} u'' = 0, \\ &c\Delta u' + d \operatorname{grad}\operatorname{div} u' + a_2\Delta u'' + b_2 \operatorname{grad}\operatorname{div} u'' = 0. \end{aligned} \tag{1.3}$$

In the theory of elastic mixtures the displacement vector is usually denoted by $u = (u', u'')$. In this paper u is the four-dimensional vector, i.e., $u = (u_1, u_2, u_3, u_4)$ or $u_1 = u'_1$, $u_2 = u'_2$, $u_3 = u''_1$, $u_4 = u''_2$.

The system of basic equations (1.1) can (equivalently) be rewritten as

$$\begin{aligned} &a_1\Delta u' + c\Delta u'' + b_1 \operatorname{grad}\theta' + d \operatorname{grad}\theta'' = \psi', \\ &c\Delta u' + a_2\Delta u'' + d \operatorname{grad}\theta' + b_2 \operatorname{grad}\theta'' = \psi'', \end{aligned} \tag{1.4}$$

where

$$\theta' = \frac{\partial u'_1}{\partial x_1} + \frac{\partial u'_2}{\partial x_2}, \quad \theta'' = \frac{\partial u''_1}{\partial x_1} + \frac{\partial u''_2}{\partial x_2}. \tag{1.5}$$

For our further discussion we shall also need the functions

$$\omega' = \frac{\partial u'_2}{\partial x_1} - \frac{\partial u'_1}{\partial x_2}, \quad \omega'' = \frac{\partial u''_2}{\partial x_1} - \frac{\partial u''_1}{\partial x_2}. \tag{1.6}$$

As mentioned above, here we want to represent the solution (i.e., the displacement vector components) of (1.1) and the stress vector components (calculated by means of the displacement vector) and stress tensor components through analytic functions of a complex variable. To this end, for the basic equations of statics in the theory of elastic mixtures we shall generalize the method developed by Vekua and Muskhelishvili for nonhomogeneous equations of statics of an isotropic elastic body in the two-dimensional case (see [3] or [4]).

We introduce the following variables:

$$z = x_1 + ix_2, \quad \overline{z} = x_1 - ix_2,$$

i.e.,

$$x_1 = \frac{z+\overline{z}}{2}, \quad x_2 = \frac{z-\overline{z}}{2},$$

where

$$\begin{gathered}\frac{\partial}{\partial x_1} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}}, \quad \frac{\partial}{\partial x_2} = i\Big(\frac{\partial}{\partial z} - \frac{\partial}{\partial \overline{z}}\Big),\\ \frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\Big), \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x_2} + i\frac{\partial}{\partial x_2}\Big).\end{gathered} \tag{1.7}$$

After performing simple calculations we obtain

$$\begin{gathered}\Delta = 4\,\frac{\partial^2}{\partial z \partial \overline{z}}, \quad \theta' = \frac{\partial w'}{\partial z} + \frac{\partial \overline{w}'}{\partial \overline{z}}, \quad \theta'' = \frac{\partial w''}{\partial z} + \frac{\partial \overline{w}''}{\partial \overline{z}},\\ \omega' = -i\Big(\frac{\partial w'}{\partial z} - \frac{\partial \overline{w}'}{\partial \overline{z}}\Big), \quad \omega'' = -i\Big(\frac{\partial w''}{\partial z} - \frac{\partial \overline{w}''}{\partial \overline{z}}\Big),\end{gathered} \tag{1.8}$$

where

$$w' = u_1' + iu_2', \quad w'' = u_1'' + iu_2''. \tag{1.9}$$

On account of (1.7), (1.8), and (1.9) we can rewrite (1.4) as two complex equations

$$\begin{gathered}4a_1\,\frac{\partial^2 w'}{\partial \overline{z} \partial z} + 4c\,\frac{\partial^2 w''}{\partial \overline{z} \partial z} + 2b_1\,\frac{\partial \theta'}{\partial \overline{z}} + 2d\,\frac{\partial \theta''}{\partial \overline{z}} = \Psi',\\ 4c\,\frac{\partial^2 w'}{\partial \overline{z} \partial z} + 4a_2\,\frac{\partial^2 w''}{\partial \overline{z} \partial z} + 2d\,\frac{\partial \theta'}{\partial \overline{z}} + 2b_2\,\frac{\partial \theta''}{\partial \overline{z}} = \Psi'',\end{gathered} \tag{1.10}$$

where

$$\Psi' = \psi_1' + i\psi_2', \quad \Psi'' = \psi_1'' + i\psi_2''.$$

Obviously, (1.10) can be rewritten as

$$\begin{gathered}\frac{\partial}{\partial \overline{z}}\Big(4a_1\,\frac{\partial w'}{\partial z} + 4c\,\frac{\partial w''}{\partial z} + 2b_1\theta' + 2d\theta''\Big) = \Psi',\\ \frac{\partial}{\partial \overline{z}}\Big(4c\,\frac{\partial w'}{\partial z} + 4a_2\,\frac{\partial w''}{\partial z} + 2d\theta' + 2b_2\theta''\Big) = \Psi'',\end{gathered}$$

which, after applying the Pompeiu formula [4], gives

$$\begin{gathered}4a_1\,\frac{\partial w'}{\partial z} + 4c\,\frac{\partial w''}{\partial z} + 2b_1\theta' + 2d\theta'' = 4\varphi_1'(z) + \frac{1}{\pi}\int\limits_D \frac{\Psi'(y_1,y_2)}{\sigma}\,dy_1dy_2,\\ 4c\,\frac{\partial w'}{\partial z} + 4a_2\,\frac{\partial w''}{\partial z} + 2d\theta' + 2b_2\theta'' = 4\varphi_2'(z) + \frac{1}{\pi}\int\limits_D \frac{\Psi''(y_1,y_2)}{\sigma}\,dy_1dy_2,\end{gathered} \tag{1.11}$$

where $\sigma = z - \zeta$, $\zeta = y_1 + iy_2$, $\varphi_1'(z)$ and $\varphi_2'(z)$ are arbitrary analytic functions which we have represented as derivatives of arbitrary analytic functions, while multiplier 4 has been introduced for convenience. In (1.11) D is a finite or infinite two-dimensional domain. In the case of an infinite domain the functions Ψ' and Ψ'' satisfy the definite conditions near the point at infinity.

Remark. The integral terms (partial solutions) appear in (1.11) by virtue of the fact that the Pompeiu formula

$$w(x) = \frac{1}{\pi} \int\limits_D \frac{F(y_1, y_2)}{\sigma} \, dy_1 \, dy_2$$

holds (under certain assumptions) for the equation

$$\frac{\partial w}{\partial \overline{z}} = F = F_1 + iF_2.$$

The proof of the Pompeiu formula for both a finite and an infinite domain D is given in [4].

We shall give one more proof of the Pompeiu formula. Let $w = u + iv$. Then, on separating the real and imaginary parts, the equation for w can be written as two equations:

$$\frac{\partial u}{\partial x_1} - \frac{\partial v}{\partial x_2} = 2F_1, \quad \frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1} = 2F_2.$$

If we now introduce new functions φ and ψ by

$$u = \frac{\partial \varphi}{\partial x_1} + \frac{\partial \psi}{\partial x_2}, \quad v = -\frac{\partial \varphi}{\partial x_2} + \frac{\partial \psi}{\partial x_1},$$

the previous system for φ and ψ can be rewritten as

$$\Delta \varphi = 2F_1, \quad \Delta \psi = F_2.$$

By the well-known formula for a partial solution of the Poisson equation, we obtain

$$\varphi = \frac{1}{\pi} \int\limits_D \ln r F_1 \, dy_1 \, dy_2, \quad \psi = \frac{1}{\pi} \int\limits_D \ln r F_2 \, dy_1 \, dy_2,$$

where

$$r = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} = |\sigma|.$$

By calculating the partial derivatives of first order for φ and ψ we obtain

$$u=\frac{1}{\pi}\int\limits_D\Big(\frac{x_1-y_1}{r^2}\,F_1+\frac{x_2-y_2}{r^2}\,F_2\Big)dy_1dy_2,$$
$$v=\frac{1}{\pi}\int\limits_D\Big(-\frac{x_2-y_2}{r^2}\,F_1+\frac{x_1-y_1}{r^2}\,F_2\Big)dy_1dy_2,$$

and hence

$$w=u+iv=\frac{1}{\pi}\int\limits_D\frac{F(y_1,y_2)}{\sigma}\,dy_1\,dy_2.$$

The latter formula coincides with the Pompeiu formula. □

Combining (1.11) with the formulas obtained from (1.11), passing to the conjugate values, and taking (1.8) into account, we obtain, after some transformations for θ' and θ'', the system of equations

$$\begin{aligned}(a_1+b_1)\theta'+(c+d)\theta''&=2\,\mathrm{Re}\Big[\varphi_1'(z)+\frac{1}{4\pi}\int\limits_D\frac{\Psi'(y_1,y_2)}{\sigma}\,dy_1\,dy_2\Big],\\(c+d)\theta'+(a_2+b_2)\theta''&=2\,\mathrm{Re}\Big[\varphi_2'(z)+\frac{1}{4\pi}\int\limits_D\frac{\Psi''(y_1,y_2)}{\sigma}\,dy_1\,dy_2\Big],\end{aligned}\tag{1.12}$$

where the symbol Re denotes the real part.

On subtracting the complex-valued values and again taking into account (1.8), we obtain in a manner similar to the above the following system for ω' and ω'':

$$\begin{aligned}a_1\omega'+c\omega''&=2\,\mathrm{Im}\Big[\varphi_1'(z)+\frac{1}{4\pi}\int\limits_D\frac{\Psi'(y_1,y_2)}{\sigma}\,dy_1\,dy_2\Big],\\c\omega'+a_2\omega''&=2\,\mathrm{Im}\Big[\varphi_2'(z)+\frac{1}{4\pi}\int\limits_D\frac{\Psi''(y_1,y_2)}{\sigma}\,dy_1\,dy_2\Big],\end{aligned}\tag{1.13}$$

where the symbol Im denotes the imaginary part.

By solving system (1.12) for θ' and θ'' we obtain

$$
\begin{aligned}
\theta' &= \frac{2}{d_1}\,\mathrm{Re}\bigg\{(a_2+b_2)\bigg[\varphi_1'(z)+\frac{1}{4\pi}\int\limits_D \frac{\Psi'(y_1,y_2)}{\sigma}\,dy_1\,dy_2\bigg]-\\
&\quad-(c+d)\bigg[\varphi_2'(z)+\frac{1}{4\pi}\int\limits_D \frac{\Psi''(y_1,y_2)}{\sigma}\,dy_1\,dy_2\bigg]\bigg\},\\
\theta'' &= \frac{2}{d_1}\,\mathrm{Re}\bigg\{-(c+d)\bigg[\varphi_1'(z)+\frac{1}{4\pi}\int\limits_D \frac{\Psi'(y_1,y_2)}{\sigma}\,dy_1\,dy_2\bigg]+\\
&\quad+(a_1+b_1)\bigg[\varphi_2'(z)+\frac{1}{4\pi}\int\limits_D \frac{\Psi''(y_1,y_2)}{\sigma}\,dy_1\,dy_2\bigg]\bigg\},
\end{aligned}
\tag{1.14}
$$

where $d_1=(a_1+b_1)(a_2+b_2)-(c+d)^2>0$.

For the unknown $\frac{\partial w'}{\partial z}$ and $\frac{\partial w''}{\partial z}$ system (1.11) gives

$$
\begin{aligned}
\frac{\partial w'}{\partial z} &= e_1\varphi_1'(z)+e_2\varphi_2'(z)+\frac{1}{4\pi}\int\limits_D (e_1\Psi'+e_2\Psi'')\frac{dy_1\,dy_2}{\sigma}+\\
&\quad+\frac{1}{2d_2}\big[(cd-b_1a_2)\theta'+(cb_2-da_2)\theta''\big],\\
\frac{\partial w''}{\partial z} &= e_2\varphi_1'(z)+e_3\varphi_2'(z)+\frac{1}{4\pi}\int\limits_D (e_2\Psi'+e_3\Psi'')\frac{dy_1\,dy_2}{\sigma}+\\
&\quad+\frac{1}{2d_2}\big[(cb_1-da_1)\theta'+(cd-a_1b_2)\theta''\big],
\end{aligned}
\tag{1.15}
$$

where

$$
e_1=\frac{a_2}{d_2},\;\; e_2=-\frac{c}{d_2},\;\; e_3=\frac{a_1}{d_2},\;\; d_2=a_1a_2-c^2>0. \tag{1.16}
$$

From (1.14) we obtain by elementary calculations

$$
\begin{aligned}
&\frac{1}{2d_2}\big[(cd-b_1a_2)\theta'+(cb_2-da_2)\theta''\big]=\\
&\qquad=\mathrm{Re}\bigg[e_4\varphi_1'(z)+e_5\varphi_2'(z)+\frac{1}{4\pi}\int\limits_D (e_4\Psi'+e_5\Psi'')\frac{dy_1\,dy_2}{\sigma}\bigg],\\
&\frac{1}{2d_2}\big[(cb_1-da_1)\theta'+(cd-a_1b_2)\theta''\big]=\\
&\qquad=\mathrm{Re}\bigg[e_5\varphi_1'(z)+e_6\varphi_2'(z)+\frac{1}{4\pi}\int\limits_D (e_5\Psi'+e_6\Psi'')\frac{dy_1\,dy_2}{\sigma}\bigg],
\end{aligned}
\tag{1.17}
$$

where

$$\begin{aligned} e_4 &= \frac{(c+d)(da_2-cb_2)+(a_2+b_2)(cd-b_1a_2)}{d_1d_2}, \\ e_5 &= \frac{(a_1+b_1)(cb_2-da_2)+(c+d)(a_2b_1-cd)}{d_1d_2} = \\ &= \frac{(c+d)(a_1b_2-cd)+(a_2+b_2)(cb_1-da_1)}{d_1d_2}, \\ e_6 &= \frac{(a_1+b_1)(cd-a_1b_2)+(c+d)(da_1-cb_1)}{d_1d_2}. \end{aligned} \tag{1.18}$$

After substituting (1.17) into (1.15), we can rewrite $\frac{\partial w'}{\partial z}$ and $\frac{\partial w''}{\partial z}$ in a simpler form

$$\begin{aligned} \frac{\partial w'}{\partial z} &= m_1\varphi_1'(z)+m_2\varphi_2'(z)+\frac{e_4}{2}\,\overline{\varphi_1'(z)}+\frac{e_5}{2}\,\overline{\varphi_2'(z)}+ \\ &\quad+\frac{1}{4\pi}\int\limits_D(m_1\Psi'+m_2\Psi'')\frac{dy_1\,dy_2}{\sigma}+\frac{1}{8\pi}\int\limits_D(e_4\overline{\Psi}'+e_5\overline{\Psi}'')\frac{dy_1\,dy_2}{\overline{\sigma}}, \\ \frac{\partial w''}{\partial z} &= m_2\varphi_1'(z)+m_3\varphi_2'(z)+\frac{e_5}{2}\,\overline{\varphi_1'(z)}+\frac{e_6}{2}\,\overline{\varphi_2'(z)}+ \\ &\quad+\frac{1}{4\pi}\int\limits_D(m_2\Psi'+m_3\Psi'')\frac{dy_1\,dy_2}{\sigma}+\frac{1}{8\pi}\int\limits_D(e_5\overline{\Psi}'+e_6\overline{\Psi}'')\frac{dy_1\,dy_2}{\overline{\sigma}}, \end{aligned} \tag{1.19}$$

where

$$m_1=e_1+\frac{e_4}{2},\quad m_2=e_2+\frac{e_5}{2},\quad m_3=e_3+\frac{e_6}{2}. \tag{1.20}$$

Since $\frac{1}{\sigma}=\frac{\partial}{\partial z}(\ln\sigma+\ln\overline{\sigma})=2\,\frac{\partial}{\partial z}\,\ln|\sigma|$, we obtain from (1.19) by integration

$$\begin{aligned} w' &= m_1\varphi_1(z)+m_2\varphi_2(z)+\frac{z}{2}\big[e_4\overline{\varphi_1'(z)}+e_5\overline{\varphi_2'(z)}\big]+ \\ &\quad+\overline{\psi_1(z)}+\frac{1}{2\pi}\int\limits_D(m_1\Psi'+m_2\Psi'')\ln|\sigma|dy_1\,dy_2+ \\ &\quad+\frac{1}{8\pi}\int\limits_D\frac{\sigma}{\overline{\sigma}}(e_4\overline{\Psi}'+e_5\overline{\Psi}'')dy_1\,dy_2, \\ w'' &= m_2\varphi_1(z)+m_3\varphi_2(z)+\frac{z}{2}\big[e_5\overline{\varphi_1'(z)}+e_6\overline{\varphi_2'(z)}\big]+ \\ &\quad+\overline{\psi_2(z)}+\frac{1}{2\pi}\int\limits_D(m_2\Psi'+m_3\Psi'')\ln|\sigma|dy_1\,dy_2+ \\ &\quad+\frac{1}{8\pi}\int\limits_D\frac{\sigma}{\overline{\sigma}}(e_5\overline{\Psi}'+e_6\overline{\Psi}'')dy_1\,dy_2, \end{aligned} \tag{1.21}$$

where $\psi_1(z)$ and $\psi_2(z)$ are new arbitrary analytic functions.

In the theory of elastic mixtures, formulas (1.21) obtained for the displacement vector components are analogues of Kolosov–Muskhelishvili general representation formulas.

If the system of equations (1.1) is homogeneous, i.e., $\psi' = \psi'' = 0$ or $\Psi' = \Psi'' = 0$, then the integral terms in (1.21) vanish and we obtain the formulas

$$\begin{aligned} w' &= m_1\varphi_1(z) + m_2\varphi_2(z) + \frac{z}{2}\big[e_4\overline{\varphi_1'(z)} + e_5\overline{\varphi_2'(z)}\big] + \overline{\psi_1(z)}, \\ w'' &= m_2\varphi_1(z) + m_3\varphi_2(z) + \frac{z}{2}\big[e_5\overline{\varphi_1'(z)} + e_6\overline{\varphi_2'(z)}\big] + \overline{\psi_2(z)}, \end{aligned} \tag{1.22}$$

which are anlogues of Kolosov–Muskhelishvili formulas for the displacement vector components of equation (1.3).

The integral terms in (1.21) are one particular solution of system (1.1). To rewrite these terms in a different form we introduce the vectors $u^{(0)}(x) = (u_1', u_2', u_1'', u_2'')$ and $\psi(x) = (\psi_1', \psi_2', \psi_1'', \psi_2'')$. Now, after separating the real parts, from (1.21) we have

$$u^{(0)}(x) = \frac{1}{2\pi}\int\limits_D \phi(x-y)\psi(y)\,dy_1\,dy_2, \tag{1.23}$$

where

$$\phi(x-y) = \operatorname{Re}\Gamma(x-y), \tag{1.24}$$

$$\Gamma(x-y) =$$

$$= \left\| \begin{matrix} m_1\ln\sigma + \frac{e_4}{4}\frac{\overline{\sigma}}{\sigma}, & \frac{ie_4}{4}\frac{\overline{\sigma}}{\sigma}, & m_2\ln\sigma + \frac{e_5}{2}\frac{\overline{\sigma}}{\sigma}, & \frac{ie_5}{4}\frac{\overline{\sigma}}{\sigma} \\ \frac{ie_4}{4}\frac{\overline{\sigma}}{\sigma}, & m_1\ln\sigma - \frac{e_4}{4}\frac{\overline{\sigma}}{\sigma}, & \frac{ie_5}{4}\frac{\overline{\sigma}}{\sigma}, & m_2\ln\sigma - \frac{e_5}{4}\frac{\overline{\sigma}}{\sigma} \\ m_2\ln\sigma + \frac{e_5}{4}\frac{\overline{\sigma}}{\sigma}, & \frac{ie_5}{4}\frac{\overline{\sigma}}{\sigma}, & m_3\ln\sigma + \frac{e_6}{4}\frac{\overline{\sigma}}{\sigma}, & \frac{ie_6}{4}\frac{\overline{\sigma}}{\sigma} \\ \frac{ie_5}{4}\frac{\overline{\sigma}}{\sigma}, & m_2\ln\sigma - \frac{e_5}{4}\frac{\overline{\sigma}}{\sigma}, & \frac{ie_6}{4}\frac{\overline{\sigma}}{\sigma}, & m_3\ln\sigma - \frac{e_6}{4}\frac{\overline{\sigma}}{\sigma} \end{matrix} \right\|.$$

Here $\phi(x-y)$ is a fundamental matrix. Each term of matrix (1.24) is a single-valued function on the entire plane and has at most a logarithmic singularity at the point $x = y$. By direct calculations it can be proved that each column of the matrix $\phi(x-y)$ (considered as a vector) is a solution of system (1.3) with respect to the cordinates of the point x for $x \neq y$. It is obvious from (1.24) that $\phi(x-y)$ is a symmetric matrix.

Now we shall derive general complex representations for the components of the stress tensor and stress vector in the theory of elastic mixtures. As is known from [2], using the displacement vector $u = (u_1', u_2'', u_1'', u_2'')$ the stress vector components can be written as follows:

$$\begin{aligned} (Tu)_1 &= \tau_{11}'n_1 + \tau_{21}'n_2, \quad (Tu)_2 = \tau_{12}'n_1 + \tau_{22}'n_2, \\ (Tu)_3 &= \tau_{11}''n_1 + \tau_{21}''n_2, \quad (Tu)_4 = \tau_{12}''n_1 + \tau_{22}''n_2, \end{aligned} \tag{1.25}$$

where $n=(n_1,n_2)$ is an arbitrary unit vector and

$$\left.\begin{aligned}
\tau'_{11}&=\Big(\lambda_1-\frac{\alpha_2\rho_2}{\rho}\Big)\theta'+\Big(\lambda_3-\frac{\alpha_2\rho_1}{\rho}\Big)\theta''+\\
&\quad+2\mu_1\frac{\partial u'_1}{\partial x_1}+2\mu_3\frac{\partial u''_1}{\partial x_1},\\
\tau'_{21}&=(\mu_1-\lambda_5)\frac{\partial u'_1}{\partial x_2}+(\mu_1+\lambda_5)\frac{\partial u'_2}{\partial x_1}+\\
&\quad+(\mu_3+\lambda_5)\frac{\partial u''_1}{\partial x_2}+(\mu_3-\lambda_5)\frac{\partial u''_2}{\partial x_1},\\
\tau'_{12}&=(\mu_1+\lambda_5)\frac{\partial u'_1}{\partial x_2}+(\mu_1-\lambda_5)\frac{\partial u'_2}{\partial x_1}+\\
&\quad+(\mu_3-\lambda_5)\frac{\partial u''_1}{\partial x_2}+(\mu_3+\lambda_5)\frac{\partial u''_2}{\partial x_1},\\
\tau'_{22}&=\Big(\lambda_1-\frac{\alpha_2\rho_2}{\rho}\Big)\theta'+\Big(\lambda_3-\frac{\alpha_2\rho_1}{\rho}\Big)\theta''+\\
&\quad+2\mu_1\frac{\partial u'_2}{\partial x_2}+2\mu_3\frac{\partial u''_2}{\partial x_2},
\end{aligned}\right\}\tag{1.26}$$

$$\left.\begin{aligned}
\tau''_{11}&=\Big(\lambda_4+\frac{\alpha_2\rho_2}{\rho}\Big)\theta'+\Big(\lambda_2+\frac{\alpha_2\rho_1}{\rho}\Big)\theta''+\\
&\quad+2\mu_3\frac{\partial u'_1}{\partial x_1}+2\mu_2\frac{\partial u''_1}{\partial x_1},\\
\tau''_{21}&=(\mu_3+\lambda_5)\frac{\partial u'_1}{\partial x_2}+(\mu_3-\lambda_5)\frac{\partial u'_2}{\partial x_1}+\\
&\quad+(\mu_2-\lambda_5)\frac{\partial u''_1}{\partial x_2}+(\mu_2+\lambda_5)\frac{\partial u''_2}{\partial x_1},\\
\tau''_{12}&=(\mu_3-\lambda_5)\frac{\partial u'_1}{\partial x_2}+(\mu_3+\lambda_5)\frac{\partial u'_2}{\partial x_1}+\\
&\quad+(\mu_2+\lambda_5)\frac{\partial u''_1}{\partial x_2}+(\mu_2-\lambda_5)\frac{\partial u''_2}{\partial x_1},\\
\tau''_{22}&=\Big(\lambda_4+\frac{\alpha_2\rho_2}{\rho}\Big)\theta'+\Big(\lambda_2+\frac{\alpha_2\rho_1}{\rho}\Big)\theta''+\\
&\quad+2\mu_3\frac{\partial u'_2}{\partial x_2}+2\mu_2\frac{\partial u''_2}{\partial x_2},
\end{aligned}\right\}\tag{1.27}$$

where θ' and θ'' are defined by (1.5).

Using (1.2), (1.8) and (1.9) and performing some simple transformation, we obtain

$$\tau'_{11}+\tau'_{22}=4\,\mathrm{Re}\Big[(a_1+b_1-\mu_1)\frac{\partial w'}{\partial z}+(c+d-\mu_3)\frac{\partial w''}{\partial z}\Big],$$

$$
\begin{aligned}
&\tau_{11}' - \tau_{22}' + i(\tau_{21}' + \tau_{12}') = 4\mu_1 \frac{\partial w'}{\partial \overline{z}} + 4\mu_3 \frac{\partial w''}{\partial \overline{z}},\\
&\tau_{21}' - \tau_{12}' = 4\lambda_5 \operatorname{Im}\Big(\frac{\partial w'}{\partial z} - \frac{\partial w''}{\partial z}\Big),\\
&\tau_{11}'' + \tau_{22}'' = 4\operatorname{Re}\Big[(c+d-\mu_3)\frac{\partial w'}{\partial z} + (a_2+b_2-\mu_2)\frac{\partial w''}{\partial z}\Big],\\
&\tau_{11}'' - \tau_{22}'' + i(\tau_{21}'' + \tau_{12}'') = 4\mu_3 \frac{\partial w'}{\partial \overline{z}} + 4\mu_2 \frac{\partial w''}{\partial \overline{z}},\\
&\tau_{21}'' - \tau_{12}'' = -4\lambda_5 \operatorname{Im}\Big(\frac{\partial w'}{\partial z} - \frac{\partial w''}{\partial z}\Big).
\end{aligned}
$$

After substituting the expressions for w' and w'' from (1.21) into the above formulas we have

$$
\begin{aligned}
\tau_{11}' + \tau_{22}' = {} & 2\operatorname{Re}\Big\{(2-A_1-B_1)\varphi_1'(z) - (A_2+B_2)\varphi_2'(z) + \\
& + \frac{1}{4\pi}\int\limits_D \big[(2-A_1-B_1)\Psi' - (A_2+B_2)\Psi''\big]\frac{dy_1\,dy_2}{\sigma}\Big\},\\
\tau_{11}' - \tau_{22}' - i(\tau_{21}' + \tau_{12}') = {} & 2\overline{z}\big[B_1\varphi_1''(z) + B_2\varphi_2''(z)\big] + 4\mu_1\psi_1'(z) + \\
& + 4\mu_3\psi_2'(z) + \frac{1}{2\pi}\int\limits_D \big(A_1\overline{\Psi}' + A_2\overline{\Psi}''\big)\frac{dy_1\,dy_2}{\sigma} - \\
& - \frac{1}{2\pi}\int\limits_D \frac{\overline{\sigma}}{\sigma^2}(B_1\Psi' + B_2\Psi'')dy_1\,dy_2,\\
\tau_{21}' - \tau_{12}' = {} & 4\lambda_5 \operatorname{Im}\Big\{(e_1-e_2)\varphi_1'(z) + (e_2-e_3)\varphi_2'(z) + \\
& + \frac{1}{4\pi}\int\limits_D \big[(e_1-e_2)\Psi' + (e_2-e_3)\Psi''\big]\frac{dy_1\,dy_2}{\sigma}\Big\}, \qquad (1.28)\\
\tau_{11}'' + \tau_{22}'' = {} & 2\operatorname{Re}\Big\{-(A_3+B_3)\varphi_1'(z) + (2-A_4-B_4)\varphi_2'(z) + \\
& + \frac{1}{4\pi}\int\limits_D \big[-(A_3+B_3)\Psi' + (2-A_4-B_4)\Psi''\big]\frac{dy_1\,dy_2}{\sigma}\Big\},\\
\tau_{11}'' - \tau_{22}'' - i(\tau_{21}'' + \tau_{12}'') = {} & 2\overline{z}\big[B_3\varphi_1''(z) + B_4\varphi_2''(z)\big] + 4\mu_3\psi_1'(z) + \\
& + 4\mu_2\psi_2'(z) + \frac{1}{2\pi}\int\limits_D \big(A_3\overline{\Psi}' + A_4\overline{\Psi}''\big)\frac{dy_1\,dy_2}{\sigma} - \\
& - \frac{1}{2\pi}\int\limits_D \frac{\overline{\sigma}}{\sigma^2}(B_3\Psi' + B_4\Psi'')dy_1\,dy_2,
\end{aligned}
$$

$$\tau''_{21} - \tau''_{12} = 4\lambda_5 \operatorname{Im}\Big\{(e_2 - e_1)\varphi'_1(z) + (e_3 - e_2)\varphi'_2(z) + \\ + \frac{1}{4\pi}\int_D \big[(e_2 - e_1)\Psi' + (e_3 - e_2)\Psi''\big]\frac{dy_1\,dy_2}{\sigma}\Big\},$$

where

$$\begin{array}{ll} A_1 = 2(\mu_1 m_1 + \mu_3 m_2), & A_2 = 2(\mu_1 m_2 + \mu_3 m_3),\\ A_3 = 2(\mu_3 m_1 + \mu_2 m_2), & A_4 = 2(\mu_3 m_2 + \mu_2 m_3),\\ B_1 = \mu_1 e_4 + \mu_3 e_5, & B_2 = \mu_1 e_5 + \mu_3 e_6,\\ B_3 = \mu_3 e_4 + \mu_2 e_5, & B_4 = \mu_3 e_5 + \mu_2 e_6, \end{array} \tag{1.29}$$

and the constants e_1, e_2, e_3 are defined by (1.16).

It is easy to calculate the stress tensor components τ'_{11}, τ'_{22}, τ'_{21}, τ'_{12}, τ''_{11}, τ''_{22}, τ''_{21}, τ''_{12} by (1.28). They are expressed through four arbitrary analyic functions and their derivatives. Since for the time being we do not need the specific values of the stress tensor components, we shall not write them out.

For the stress tensor components formulas (1.28) are the generalized Kolosov–Muskhelishvili formulas in the theory of elastic mixtures.

As said above, the four-dimensional vector $u^{(0)}(x)$ defined by (1.23) is a particular solution of system (1.1). By using this solution one can always reduce, without loss of generaliy, the nonhomogeneous equation (1.1) to the homogeneous equation (1.3). Hence in what follows we shall consider only equation (1.3).

Now the expressions for the stress vector components from (1.25) can be rewritten in a simpler form. By virtue of (1.2), (1.5) and (1.6), we rewrite (1.26) and (1.27) as

$$\left.\begin{aligned} \tau'_{11} &= (a_1 + b_1)\theta' + (c+d)\theta'' - 2\mu_1\frac{\partial u'_2}{\partial x_2} - 2\mu_3\frac{\partial u''_2}{\partial x_2},\\ \tau'_{21} &= -a_1\omega' - c\omega'' + 2\mu_1\frac{\partial u'_2}{\partial x_1} + 2\mu_3\frac{\partial u''_2}{\partial x_1},\\ \tau'_{12} &= a_1\omega' + c\omega'' + 2\mu_1\frac{\partial u'_1}{\partial x_2} + 2\mu_3\frac{\partial u''_1}{\partial x_2},\\ \tau'_{22} &= (a_1 + b_1)\theta' + (c+d)\theta'' - 2\mu_1\frac{\partial u'_1}{\partial x_1} - 2\mu_3\frac{\partial u''_1}{\partial x_1}, \end{aligned}\right\} \tag{1.30}$$

$$\left.\begin{aligned} \tau''_{11} &= (c+d)\theta' + (a_2 + b_2)\theta'' - 2\mu_3\frac{\partial u'_2}{\partial x_2} - 2\mu_2\frac{\partial u''_2}{\partial x_2},\\ \tau''_{21} &= -c\omega' - a_2\omega'' + 2\mu_3\frac{\partial u'_2}{\partial x_1} + 2\mu_2\frac{\partial u''_2}{\partial x_1},\\ \tau''_{12} &= c\omega' + a_2\omega'' + 2\mu_3\frac{\partial u'_1}{\partial x_2} + 2\mu_2\frac{\partial u''_1}{\partial x_2},\\ \tau''_{22} &= (c+d)\theta' + (a_2 + b_2)\theta'' - 2\mu_3\frac{\partial u'_1}{\partial x_1} - 2\mu_2\frac{\partial u''_1}{\partial x_1}, \end{aligned}\right\} \tag{1.31}$$

Now, applying (1.12) and (1.13) and introducing the notation

$$\frac{\partial}{\partial s(x)} = n_1 \frac{\partial}{\partial x_2} - n_2 \frac{\partial}{\partial x_1}, \tag{1.32}$$

which expresses the derivative with respect to the tangent when the point $x(x_1, x_2)$ is on the boundary, we obtain from (1.25) with (1.30) and (1.31) taken into account

$$\begin{aligned}
(Tu)_1 &= 2\operatorname{Re}\varphi_1'(z)n_1 - 2\operatorname{Im}\varphi_1'(z)n_2 - 2\mu_1 \frac{\partial u_2'}{\partial s(x)} - 2\mu_3 \frac{\partial u_2''}{\partial s(x)},\\
(Tu)_2 &= 2\operatorname{Im}\varphi_1'(z)n_1 + 2\operatorname{Re}\varphi_1'(z)n_2 + 2\mu_1 \frac{\partial u_1'}{\partial s(x)} + 2\mu_3 \frac{\partial u_1''}{\partial s(x)},\\
(Tu)_3 &= 2\operatorname{Re}\varphi_2'(z)n_1 - 2\operatorname{Im}\varphi_2'(z)n_2 - 2\mu_3 \frac{\partial u_2'}{\partial s(x)} - 2\mu_2 \frac{\partial u_2''}{\partial s(x)},\\
(Tu)_4 &= 2\operatorname{Im}\varphi_2'(z)n_1 + 2\operatorname{Re}\varphi_2'(z)n_2 + 2\mu_3 \frac{\partial u_1'}{\partial s(x)} + 2\mu_2 \frac{\partial u_1''}{\partial s(x)}.
\end{aligned}$$

Hence, using notation (1.9) and (1.32) and performing some simple transformations, we can write

$$\begin{aligned}
i(Tu)_1 - (Tu)_2 &= \frac{\partial}{\partial s(x)}\big(2\varphi_1(z) - 2\mu_1 w' - 2\mu_3 w''\big),\\
i(Tu)_3 - (Tu)_4 &= \frac{\partial}{\partial s(x)}\big(2\varphi_2(z) - 2\mu_3 w' - 2\mu_2 w''\big).
\end{aligned}$$

After substituting the expressions for w' and w'' from (1.22) into the above formulas we easily obtain

$$\begin{aligned}
i(Tu)_1 - (Tu)_2 &= \frac{\partial}{\partial s(x)}\big\{(2-A_1)\varphi_1(z) - A_2\varphi_2(z) -\\
&\quad - z[B_1\overline{\varphi_1'(z)} + B_2\overline{\varphi_1'(z)}] - 2\mu_1\overline{\psi_1(z)} - 2\mu_3\overline{\psi_2(z)}\big\},\\
i(Tu)_3 - (Tu)_4 &= \frac{\partial}{\partial s(x)}\big\{ - A_3\varphi_1(z) + (2-A_4)\varphi_2(z) -\\
&\quad - z[B_3\overline{\varphi_1'(z)} + B_4\overline{\varphi_2'(z)}] - 2\mu_3\overline{\psi_1(z)} - 2\mu_2\overline{\psi_2(z)}\big\},
\end{aligned} \tag{1.33}$$

where A_1, A_2, A_3, A_4 and B_1, B_2, B_3, B_4 are defined by (1.29).

Thus for the stress vector components we have obtained a general representation expressed in terms of analytic functions. For the stress vector components in the theory of elastic mixtures these formulas are the generalization of the Kolosov–Muskhelishvili formulas.

Formulas (1.33) imply that M. Levy's theorem does not hold in the theory of elastic mixtures, i.e., the stress vector components depend on constants characterizing the physical properties of an elastic mixture.

In the theory of elastic mixtures, in addition to the stress vector, much importance is also attached to the so-called generalized stress vector

$$\overset{\varkappa}{T}u = Tu + \varkappa\frac{\partial u}{\partial s}, \tag{1.34}$$

where Tu is the stress vector, $\frac{\partial}{\partial s}$ is defined by (1.32), u is the four-dimensional displacement vector, and $\varkappa$ is the constant matrix:

$$\varkappa = \begin{Vmatrix} 0 & \varkappa_1 & 0 & \varkappa_3 \\ -\varkappa_1 & 0 & -\varkappa_3 & 0 \\ 0 & \varkappa_3 & 0 & \varkappa_2 \\ -\varkappa_3 & 0 & -\varkappa_2 & 0 \end{Vmatrix}, \tag{1.35}$$

where $\varkappa_1, \varkappa_2, \varkappa_3$ take arbitrary real values. We have written an arbitrary matrix $\varkappa$ in form (1.35) (some of its term are zero) due to system (1.3); this is the highest arbitrariness that system (1.3) can provide. Note that analogous generalized stress vectors were introduced by us for equations of statics of isotropic and anisotropic elastic bodies in our earlier studies.

Let us consider some particular values of the constant matrix $\varkappa$. In (1.35) we write $\varkappa_1 = \varkappa_2 = \varkappa_3 = 0$, i.e., $\varkappa = 0$. In that case $\overset{\circ}{T} \equiv T$ and the generalized stress vector coincides with the stress vector. Assuming now that $\varkappa_1 = 2\mu_1$, $\varkappa_2 = 2\mu_2$, $\varkappa_3 = 2\mu_3$, we obtain $\varkappa = \varkappa_L$ and denote the generalized stress vector by L. In view of the above calculations we have

$$\begin{aligned} (Lu)_1 &= \big[(a_1+b_1)\theta' + (c+d)\theta''\big]n_1 - (a_1\omega' + c\omega'')n_2 = \\ &= 2\operatorname{Re}\varphi_1'(z)n_1 - 2\operatorname{Im}\varphi_1'(z)n_2, \\ (Lu)_2 &= (a_1\omega' + c\omega'')n_1 + \big[(a_1+b_1)\theta' + (c+d)\theta''\big]n_2 = \\ &= 2\operatorname{Im}\varphi_1'(z)n_1 + \operatorname{Re}\varphi_1'(z)n_2, \\ (Lu)_3 &= \big[(c+d)\theta' + (a_2+b_2)\theta''\big]n_1 - (c\omega' + a_2\omega'')n_2 = \\ &= 2\operatorname{Re}\varphi_2'(z)n_1 - 2\operatorname{Im}\varphi_2'(z)n_2, \\ (Lu)_4 &= (c\omega' + a_2\omega'')n_1 + \big[(c+d)\theta' + (a_2+b_2)\theta''\big]n_2 = \\ &= 2\operatorname{Im}\varphi_2'(z)n_1 + \operatorname{Re}\varphi_2'(z)n_2, \end{aligned}$$

where

$$i(Lu)_1 - (Lu)_2 = 2\frac{\partial\varphi_1(z)}{\partial s(x)}, \quad i(Lu)_3 - (Lu)_4 = 2\frac{\partial\varphi_2(z)}{\partial s(x)}. \tag{1.36}$$

Since (1.34) implies

$$Lu = Tu + \varkappa_L\,\frac{\partial u}{\partial s},$$

the generalized stress vector can be rewritten as

$$\overset{\varkappa}{T}u = Lu + (\varkappa - \varkappa_L)\frac{\partial u}{\partial s},$$

which by virtue of (1.36) yields

$$\begin{aligned} i(\overset{\varkappa}{T}u)_1 - (\overset{\varkappa}{T}u)_2 &= \frac{\partial}{\partial s(x)}\big[2\varphi_1(z) + (\varkappa_1 - 2\mu_1)w' + (\varkappa_3 - 2\mu_3)w''\big], \\ i(\overset{\varkappa}{T}u)_3 - (\overset{\varkappa}{T}u)_4 &= \frac{\partial}{\partial s(x)}\big[2\varphi_2(z) + (\varkappa_3 - 2\mu_3)w' + (\varkappa_2 - 2\mu_2)w''\big]. \end{aligned} \tag{1.37}$$

Let us consider one more specific value of the constant matrix $\varkappa$ which is important for studying the first boundary value problem in the theory of elastic mixtures. Assume that in (1.37)

$$\begin{aligned} &\varkappa_1 - 2\mu_1 = -\frac{m_3}{\Delta_0}, \quad \varkappa_2 - 2\mu_2 = -\frac{m_1}{\Delta_0}, \\ &\varkappa_3 - 2\mu_3 = \frac{m_2}{\Delta_0}, \quad \Delta_0 = m_1 m_3 - m_2^2. \end{aligned} \tag{1.38}$$

Then we have $\varkappa \equiv \varkappa_N$, and we denote the generalized stress vector by N. Performing simple calculations we obtain from (1.37)

$$\begin{aligned} i(Nu)_1 - (Nu)_2 &= \frac{\partial}{\partial s(x)}\Big\{\varphi_1(z) + z[\varepsilon_1\overline{\varphi_1'(z)} + \varepsilon_2\overline{\varphi_2'(z)}] - \\ &\quad - \frac{m_3}{\Delta_0}\,\overline{\psi_1(z)} + \frac{m_2}{\Delta_0}\,\overline{\psi_2(z)}\Big), \\ i(Nu)_3 - (Nu)_4 &= \frac{\partial}{\partial s(x)}\Big\{\varphi_2(z) + z[\varepsilon_3\overline{\varphi_1'(z)} + \varepsilon_4\overline{\varphi_2'(z)}] + \\ &\quad + \frac{m_2}{\Delta_0}\,\overline{\psi_1(z)} - \frac{m_1}{\Delta_0}\,\overline{\psi_2(z)}\Big), \end{aligned} \tag{1.39}$$

where

$$\begin{aligned} \varepsilon_1 &= \frac{e_5 m_2 - e_4 m_3}{2\Delta_0}, \quad \varepsilon_3 = \frac{e_4 m_2 - e_5 m_1}{2\Delta_0}, \\ \varepsilon_2 &= \frac{e_6 m_2 - e_5 m_3}{2\Delta_0}, \quad \varepsilon_4 = \frac{e_5 m_2 - e_6 m_1}{2\Delta_0}, \end{aligned} \tag{1.40}$$

and e_4, e_5, e_6 and m_1, m_2, m_3 are defined by (1.18) and (1.20), respectively.

Using the values m_1, m_2, m_3 and e_4, e_5, e_6, we obtain, after obvious calculations, the following new expressions for the coefficients ε_k $(k = \overline{1,4})$:

$$\begin{aligned} &\delta_0\varepsilon_1 = b_1(2a_2 + b_2) - d(2c + d), \quad \delta_0\varepsilon_3 = 2(da_2 - cb_2), \\ &\delta_0\varepsilon_2 = 2(da_1 - cb_1), \quad \delta_0\varepsilon_4 = b_2(2a_1 + b_1) - d(2c + d), \\ &\delta_0 = (2a_1 + b_1)(2a_2 + b_2) - (2c + d)^2. \end{aligned}$$

The application of the generalized stress vector we have introduced here will be discussed in our future works.

2. In this section the usual Cauchy–Riemann conditions will be generalized for homogeneous equations of statics in the theory of elastic mixtures.

First we introduce the vectors

$$\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6, \varphi_7, \varphi_8)$$

and

$$\psi = (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \psi_7, \psi_8),$$

where

$$\begin{aligned}
&\varphi_1 = \frac{\partial u_1}{\partial x_1}, \quad \varphi_2 = \frac{\partial u_2}{\partial x_2}, \quad \varphi_3 = \frac{\partial u_3}{\partial x_1}, \quad \varphi_4 = \frac{\partial u_4}{\partial x_2},\\
&\varphi_5 = \frac{\partial u_1}{\partial x_2}, \quad \varphi_6 = \frac{\partial u_2}{\partial x_1}, \quad \varphi_7 = \frac{\partial u_3}{\partial x_2}, \quad \varphi_8 = \frac{\partial u_4}{\partial x_1},\\
&\psi_1 = \frac{\partial v_1}{\partial x_1}, \quad \psi_2 = \frac{\partial v_2}{\partial x_2}, \quad \psi_3 = \frac{\partial v_3}{\partial x_1}, \quad \psi_4 = \frac{\partial v_4}{\partial x_2},\\
&\psi_5 = \frac{\partial v_1}{\partial x_2}, \quad \psi_6 = \frac{\partial v_2}{\partial x_1}, \quad \psi_7 = \frac{\partial v_3}{\partial x_2}, \quad \psi_8 = \frac{\partial v_4}{\partial x_1},
\end{aligned} \tag{2.1}$$

where $u = (u_1, u_2, u_3, u_4)$ and $v = (v_1, v_2, v_3, v_4)$ are arbitrary differentiable vectors.

Definition. The vectors u and v will be said to be conjugate vectors or to satisfy the generalized Cauchy–Riemann conditions if the following conditions are fulfilled:

$$a\varphi = b\psi, \tag{2.2}$$

where

$$a = \begin{pmatrix} A, & 0\\ 0, & B \end{pmatrix}, \quad b = \begin{pmatrix} 0, & C\\ D, & 0 \end{pmatrix}, \tag{2.3}$$

$$A = \begin{Vmatrix} a_1+b_1, & a_1+b_1-\frac{m_3}{\Delta_0}, & c+d, & c+d+\frac{m_2}{\Delta_0}\\ a_1+b_1-\frac{m_3}{\Delta_0}, & a_1+b_1, & c+d+\frac{m_2}{\Delta_0}, & c+d\\ c+d, & c+d+\frac{m_2}{\Delta_0}, & a_2+b_2, & a_2+b_2-\frac{m_1}{\Delta_0}\\ c+d+\frac{m_2}{\Delta_0}, & c+d, & a_2+b_2-\frac{m_1}{\Delta_0}, & a_2+b_2 \end{Vmatrix},$$

$$B = \begin{Vmatrix} -a_1, & a_1-\frac{m_3}{\Delta_0}, & -c, & c+\frac{m_2}{\Delta_0}\\ -a_1+\frac{m_3}{\Delta_0}, & -a_1, & -c-\frac{m_2}{\Delta_0}, & c\\ -c, & c+\frac{m_2}{\Delta_0}, & -a_2, & a_2-\frac{m_1}{\Delta_0}\\ -c-\frac{m_2}{\Delta_0}, & c, & -a_2+\frac{m_1}{\Delta_0}, & a_2 \end{Vmatrix},$$

$$C=\frac{1}{\Delta_0}\begin{Vmatrix} m_3, & 0, & -m_2, & 0\\ 0, & -m_3, & 0, & m_2\\ -m_2, & 0, & m_1, & 0\\ 0, & m_2, & 0, & -m_1\end{Vmatrix},$$

$$D=\frac{1}{\Delta_0}\begin{Vmatrix} m_3, & 0, & -m_2, & 0\\ 0, & m_3, & 0, & -m_2\\ -m_2, & 0, & m_1, & 0\\ 0, & -m_2, & 0, & m_1\end{Vmatrix}.$$

In (2.3) the symbol 0 denotes a four-dimensional matrix all of whose terms are zero.

The matrix equation (2.2) immediately implies that if v a twice continuously differentiable vector, then the vector u satisfies the homogeneous equations (1.3).

Let us now prove

Theorem. *Conditions* (2.2) *remain valid if the vectors* φ *is replaced by* ψ*, and the vector* ψ *by* $(-\varphi)$*. This means that the vector* u *must be replaced by* v*, and the vector* v *by* $(-u)$*.*

Proof. By (2.2) we have

$$\psi=b^{-1}a\varphi,$$

where

$$b^{-1}=\begin{pmatrix}0, & m\\ E, & 0\end{pmatrix},$$

$$m=\begin{Vmatrix} m_1, & 0, & m_2, & 0\\ 0, & m_1, & 0, & m_2\\ m_2, & 0, & m_3, & 0\\ 0, & m_2, & 0, & m_3\end{Vmatrix},$$

$$E=\begin{Vmatrix} m_1, & 0, & m_2, & 0\\ 0, & -m_1, & 0, & -m_2\\ m_2, & 0, & m_3, & 0\\ 0, & -m_2, & 0, & -m_3\end{Vmatrix}. \tag{2.4}$$

After long but simple calculations we obtain

$$ab^{-1}a=-b,$$

which implies that

$$a\psi=-b\varphi. \qquad \square$$

From the proven theorem it follows that the vector v is also a solution of system (1.3).

Now multiply the first equality of (2.2) by n_1, the fifth equality by $(-n_2)$, and combine them. Then, recalling the definition of the operator N by (1.38), we obtain

$$(Nu)_1 = \frac{m_3}{\Delta_0}\frac{\partial v_1}{\partial s} - \frac{m_2}{\Delta_0}\frac{\partial v_3}{\partial s}. \tag{2.5}$$

Quite similarly, with the operator N taken into account, we find from (2.2) that

$$\begin{aligned}(Nu)_2 &= \frac{m_3}{\Delta_0}\frac{\partial v_2}{\partial s} - \frac{m_2}{\Delta_0}\frac{\partial v_4}{\partial s},\\ (Nu)_3 &= -\frac{m_2}{\Delta_0}\frac{\partial v_1}{\partial s} + \frac{m_1}{\Delta_0}\frac{\partial v_3}{\partial s},\\ (Nu)_4 &= -\frac{m_2}{\Delta_0}\frac{\partial v_2}{\partial s} + \frac{m_1}{\Delta_0}\frac{\partial v_4}{\partial s}.\end{aligned} \tag{2.6}$$

Formulas (2.5) and (2.6) can be rewritten as

$$Nu = m^{-1}\frac{\partial v}{\partial s}, \tag{2.7}$$

where m is defined by (2.4) and

$$\begin{gathered}m^{-1} = \frac{1}{\Delta_0}\begin{Vmatrix} m_3, & 0, & -m_2, & 0\\ 0, & m_3, & 0, & -m_2\\ -m_2, & 0, & m_1, & 0\\ 0, & -m_2, & 0, & m_1\end{Vmatrix},\\ \Delta_0 = m_1m_3 - m_2^2.\end{gathered} \tag{2.8}$$

By virtue of the above-proven theorem one can easily verify that, along with (2.7), the formula

$$Nv = -m^{-1}\frac{\partial u}{\partial s} \tag{2.9}$$

is also valid.

Therefore the conjugate vectors u and v satisfy conditions (2.7) and (2.9).

We introduce the notation

$$w = u + iv; \tag{2.10}$$

then (2.8) and (2.9) can be written as

$$Nw = -im^{-1}\frac{\partial w}{\partial s},$$

which enables us to rewrite (1.34) as

$$\overset{\varkappa}{T}w=\big[-im^{-1}+(\varkappa-\varkappa_{N})\big]\frac{\partial w}{\partial s}.$$

For the stress vector we have

$$Tw=(-im^{-1}-\varkappa_{N})\frac{\partial w}{\partial s}.$$

The generalized Cauchy–Riemann conditions written as (2.2) are rather cumbersome as they contain eight scalar equations. Let us rewrite (2.2) in a more convenient form. For this we solve the first, third, sixth, and eighth equations of (2.2) with respect to the vector $\frac{\partial u}{\partial x_1}$ and obtain

$$\frac{\partial u}{\partial x_1}=\begin{Vmatrix} 0 & A_{11}-1, & 0, & -A_{12}\\ 1-A_{21}, & 0, & A_{22}, & 0\\ 0 & -A_{31}, & 0, & A_{32}-1\\ A_{41}, & 0, & 1-A_{42}, & 0\end{Vmatrix}\frac{\partial u}{\partial x_2}+$$

$$+\begin{Vmatrix} A_{11}, & 0, & -A_{12} & 0\\ 0, & A_{21}, & 0, & -A_{12}\\ -A_{31}, & 0, & A_{32}, & 0\\ 0, & -A_{41}, & 0, & A_{42}\end{Vmatrix}\frac{\partial v}{\partial x_1}, \tag{2.11}$$

where

$$A_{11}=\frac{m_2(c+d)+m_3(a_2+b_2)}{\Delta_0 d_1},\quad A_{21}=\frac{m_3a_2+m_2c}{\Delta_0 d_2},$$
$$A_{12}=\frac{m_2(a_2+b_2)+m_1(c+d)}{\Delta_0 d_1},\quad A_{22}=\frac{m_2a_2+m_1c}{\Delta_0 d_2},$$
$$A_{31}=\frac{m_2(a_1+b_1)+m_3(c+d)}{\Delta_0 d_1},\quad A_{41}=\frac{m_2a_1+m_3c}{\Delta_0 d_2},$$
$$A_{32}=\frac{m_1(a_1+b_1)+m_2(c+d)}{\Delta_0 d_1},\quad A_{42}=\frac{m_1a_1+m_2c}{\Delta_0 d_2}.$$

In quite a similar manner, from system (2.11), which is valid by the above-proven theorem, we obtain

$$\frac{\partial v}{\partial x_1}=\begin{Vmatrix} 0 & A_{11}-1, & 0, & -A_{12}\\ 1-A_{21}, & 0, & A_{22}, & 0\\ 0 & -A_{31}, & 0, & A_{32}\\ A_{41}, & 0, & 1-A_{42}, & 0\end{Vmatrix}\frac{\partial v}{\partial x_2}+$$

$$+\begin{Vmatrix} -A_{11}, & 0, & A_{12} & 0\\ 0, & -A_{21}, & 0, & A_{22}\\ A_{31}, & 0, & -A_{32}, & 0\\ 0, & A_{41}, & 0, & -A_{42}\end{Vmatrix}\frac{\partial u}{\partial x_1}. \tag{2.12}$$

Using (1.2), (1.18) and the formulas

$$e_1 + e_4 = \frac{a_2 + b_2}{d_1}, \quad e_2 + e_5 = -\frac{c + d}{d_1}, \quad e_3 + e_6 = \frac{a_1 + b_1}{d_1},$$

obtained from (1.16) and (1.18), we have with obvious transformations

$$A_{11} + A_{21} = 2, \quad A_{12} + A_{22} = 0,$$
$$A_{31} + A_{41} = 0, \quad A_{32} + A_{42} = 2.$$

We introduce the notation

$$A_{11} = 1 - \varepsilon_1, \quad A_{12} = \varepsilon_3, \quad A_{31} = \varepsilon_2, \quad A_{32} = 1 - \varepsilon_4,$$

where ε_k $(k = \overline{1,4})$ is defined by (1.40). Now, by virtue of (2.11), (2.12) with notation (2.10) taken into account we obtain

$$\frac{\partial w}{\partial x_1} = \mathcal{P}\frac{\partial w}{\partial x_2}, \tag{2.13}$$

where

$$\mathcal{P} = \begin{Vmatrix} i(\varepsilon_1 - 1), & -\varepsilon_1, & i\varepsilon_3, & -\varepsilon_3 \\ -\varepsilon_1, & -i(\varepsilon_1 + 1), & -\varepsilon_3, & -i\varepsilon_3 \\ i\varepsilon_2, & -\varepsilon_2, & i(\varepsilon_4 - 1), & -\varepsilon_4 \\ -\varepsilon_2, & -i\varepsilon_2, & -\varepsilon_4, & -i(\varepsilon_4 + 1) \end{Vmatrix}.$$

The solution of system (2.13) gives

$$\frac{\partial w}{\partial x_2} = Q\,\frac{\partial w}{\partial x_1}, \tag{2.14}$$

where

$$Q = \begin{Vmatrix} i(\varepsilon_1 + 1), & -\varepsilon_1, & i\varepsilon_3, & -\varepsilon_3 \\ -\varepsilon_1, & i(1 - \varepsilon_1), & -\varepsilon_3, & -i\varepsilon_3 \\ i\varepsilon_2, & -\varepsilon_2, & i(1 + \varepsilon_4), & -\varepsilon_4 \\ -\varepsilon_2, & -i\varepsilon_2, & -\varepsilon_4, & i(1 - \varepsilon_4) \end{Vmatrix}.$$

Direct calculations prove that

$$\mathcal{P}Q = Q\mathcal{P} = E, \quad \det\mathcal{P} = 1,$$

where E is the unit four-dimensional matrix.

In what follows by the generalized Cauchy–Riemann condition we mean (2.13) or (2.14).

Finally, note that for the vector $u = (u_1', u_2', u_1'', u_2'')$ we have written a general representation through four analytic functions in for (1.22). To write a general representation for the conjugate vector $v=(v_1,v_2,v_3,v_4)$ it is sufficient to compare (1.39) and (2.7). We obtain

$$\begin{aligned}
iv_1-v_2&=m_1\varphi_1(z)+m_2\varphi_2(z)-\frac{z}{2}\big[e_4\overline{\varphi_1'(z)}+e_5\overline{\varphi_2'(z)}\big]-\overline{\psi_1(z)},\\
iv_3-v_4&=m_2\varphi_1(z)+m_3\varphi_2(z)-\frac{z}{2}\big[e_5\overline{\varphi_1'(z)}+e_6\overline{\varphi_2'(z)}\big]-\overline{\psi_2(z)}.
\end{aligned}\tag{2.15}$$

Now, using (1.22) and (2.15), for the components of the vector $w=u+iv$ we have the following expressions:

$$\begin{aligned}
w_1&=m_1\varphi_1(z)+m_2\varphi_2(z)+\frac{\overline{z}}{2}\big[e_4\varphi_1'(z)+e_5\varphi_2'(z)\big]+\psi_1(z),\\
w_2&=(-i)\Big\{m_1\varphi_1(z)+m_2\varphi_2(z)-\frac{\overline{z}}{2}\big[e_4\varphi_1'(z)+e_5\varphi_2'(z)\big]-\psi_1(z)\Big\},\\
w_3&=m_2\varphi_1(z)+m_3\varphi_2(z)+\frac{\overline{z}}{2}\big[e_5\varphi_1'(z)+e_6\varphi_2'(z)\big]+\psi_2(z)\\
w_4&=(-i)\Big\{m_2\varphi_1(z)+m_3\varphi_2(z)-\frac{\overline{z}}{2}\big[e_5\varphi_1'(z)+e_5\varphi_2'(z)\big]-\psi_2(z)\Big\}.
\end{aligned}$$

Hence we immediately conclude that

$$w_1+iw_2=2[m_1\varphi_1(z)+m_2\varphi_2(z)],\quad w_3+iw_4=2[m_2\varphi_1(z)+m_3\varphi_2(z)]$$

are analytic functions.

The application of the formulas derived in this section will be given in our next works.

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