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THREE-DIMENSIONAL BOUNDARY VALUE PROBLEMS OF ELASTOTHERMODIFFUSION WITH MIXED BOUNDARY CONDITIONS

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ABSTRACT. We investigate the basic boundary value problems of the connected theory of elastothermodiffusion for three-dimensional domains bounded by several closed surfaces when the same boundary conditions are fulfilled on every separate boundary surface, but these conditions differ on different groups of surfaces. Using the results of papers [1–8], we prove theorems on the existence and uniqueness of the classical solutions of these problems.

1. Statement of the problem. Let $\mathbb{R}^3$ be the three-dimensional Euclidean space and let $x=(x_1,x_2,x_3), y=(y_1,y_2,y_3),\dots,$ be the points of this space. Let $D_k\subset\mathbb{R}^3$ be a finite domain bounded by a closed surface $S_k\in\Lambda_2(\alpha)$, $\alpha>0$, $k=0,1,\dots,m$ [1]; moreover, $S_k\cap S_j=\varnothing$, $k\neq j=\overline{0,m}$, S_0 covers all the others which in their turn do not cover each other, $\overline{D}_k=D_k\cup S_k$, $S=\cup_{k=0}^m S_k$, $D^+=D_0\setminus\cup_{k=1}^m\overline{D}_k$; that is, D^+ is a finite (bounded) domain with the boundary S; $D^-=\mathbb{R}^3\setminus\cup_{k=1}^m\overline{D}_k$ is an infinite (unbounded) bounded domain with the boundary $S'=\cup_{k=1}^m S_k$.

The principal (nonstationary) dynamic system of differential equations of the conjugate theory of thermodiffusion of homogeneous isotropic elastic body has the form [2, 3]:

$$\begin{aligned}
&A\Big(\frac{\partial}{\partial x}\Big)v(x,t)-\gamma_1\operatorname{grad}v_4(x,t)-\gamma_2\operatorname{grad}v_5(x_2,t)-\rho\frac{\partial^2 v}{\partial t^2}=q(x,t),\\
&\qquad\delta_1\Delta v_4(x,t)-a_1\frac{\partial v_4}{\partial t}-a_{12}\frac{\partial v_5}{\partial t}-\gamma_1\frac{\partial}{\partial t}\operatorname{div}v(x,t)=q_4(x,t),\\
&\qquad\delta_2\Delta v_5(x,t)-a_2\frac{\partial v_5}{\partial t}-a_{12}\frac{\partial v_4}{\partial t}-\gamma_2\frac{\partial}{\partial t}\operatorname{div}v(x,t)=q_5(x,t),
\end{aligned}\tag{1}$$

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where $v = (v_1, v_2, v_3)^T = \|v_k\|_{k=1}^3$ is the displacement vector, v_4 is the deviation of temperature, v_5 is "the chemical potential" of the medium, t is the time, $A(\frac{\partial}{\partial x}) = \|\mu\delta_{ik} + (\lambda+\mu)\frac{\partial^2}{\partial x_i \partial x_k}\|_{3\times 3}$ is the matrix Lamé differential (static) operator, Δ is the three-dimensional Laplace operator, δ_{ik} is the Kronecker symbol, $q = (q_1, q_2, q_3)^T$ is the vector of mass forces, q_4 is the heat source, q_5 is the diffusing mass source; λ, μ, γ_1, γ_2, δ_1, δ_2, a_1, a_2, a_{12}, ρ are the known real elastic, thermal, and diffusion constants, respectively, satisfying natural restrictions [3];

$$\mu > 0, \quad 3\lambda + 2\mu > 0, \quad \rho > 0,$$
$$a_k > 0, \quad \delta_k > 0, \quad k = 1, 2, \quad a_1 a_2 - a_{12}^2 > 0,$$

T as a superscript denotes transposition.

Let us introduce a five-component vector $V(x,t) = (v_1, v_2, \dots, v_5)^T = (v, v_4, v_5)^T$ and investigate the system (1) in the two cases:

I. When $V(x,t)$ is represented in the form $p > 0$, stationary oscillations with frequency $V(x,t) = \mathrm{Re}[e^{-ipt} U(x,p)]$;

II. When $V(x,t)$ is the Laplace–Mellin integral

$$V(x,t) = \frac{1}{2\pi i} \int\limits_{\sigma - i\infty}^{\sigma + i\infty} e^{rt} U(x,\tau) d\tau, \quad \tau = \sigma + iq,$$

$\sigma > 0$, the general dynamical case.

It is easy to see that in both cases the system (1) is reduced to the form (for $U(x,\omega) = (u_1, u_2, \dots, u_5)^T = (u, u_4, u_5)^T$)

$$\begin{aligned} A\Big(\frac{\partial}{\partial x}\Big)u - \gamma_1 \operatorname{grad} u_4 - \gamma_2 \operatorname{grad} u_5 + \rho\omega^2 u = \varphi, \\ \delta_1 \Delta u_4 + i\omega a_1 u_4 + i\omega a_{12} u_5 + i\omega\gamma_1 \operatorname{div} u = \varphi_4, \\ \delta_2 \Delta u_5 + i\omega a_2 u_5 + i\omega a_{12} u_4 + i\omega\gamma_2 \operatorname{div} u = \varphi_5, \end{aligned} \tag{2}$$

where $\omega = p > 0$ in the first case and $\omega = i\tau = -q + i\sigma$, $\sigma > 0$, in the second one.

We rewrite (2) in matrix-vector form

$$\mathbf{B}\Big(\frac{\partial}{\partial x}, \omega\Big) U(x,\omega) = \Phi(x), \tag{3}$$

where $\mathbf{B}(\frac{\partial}{\partial x}, \omega)$ is a 5×5 matrix differential operator:

$$\mathbf{B}\Big(\frac{\partial}{\partial x}, \omega\Big) = \Big\|\mathbf{B}_{jk}\Big(\frac{\partial}{\partial x}, \omega\Big)\Big\|_{5\times 5},$$
$$\mathbf{B}_{jk}\Big(\frac{\partial}{\partial x}, \omega\Big) = (\mu\Delta + \rho\omega^2)\delta_{jk} + (\lambda+\mu)\frac{\partial^2}{\partial x_j \partial x_k}, \quad j,k = \overline{1,3},$$

$$\mathbf{B}_{jk}\Big(\frac{\partial}{\partial x},\omega\Big)=i\omega\gamma_{j-3}\frac{\partial}{\partial x_k},\quad k=\overline{1,3},\quad j=4,5,$$
$$\mathbf{B}_{kj}\Big(\frac{\partial}{\partial x},\omega\Big)=-\gamma_{j-3}\frac{\partial}{\partial x_k},\quad k=\overline{1,3},\quad j=4,5,$$
$$\mathbf{B}_{jj}\Big(\frac{\partial}{\partial x},\omega\Big)=\delta_{j-3}\Delta+i\omega a_{j-3},\quad j=4,5,\quad \mathbf{B}_{45}=4_{j4}=i\omega a_{12};$$

$\Phi=(\varphi,\varphi_4,\varphi_5)^T$ is a given vector.

We shall consider the following boundary value problems.

Problem $M^+(\omega)$**.** In the domain D^+ find a regular vector $U=(u,u_4,u_5)^T$ $(U\in C^1(\overline{D}^+)\cap C^2(D^+))$, the solution of the system (3), by the boundary conditions:

$$\begin{aligned}
U^+(y)&=F^{(k)}(y),\quad y\in S_k,\quad k=\overline{0,m_1},\\
\Big[R\Big(\frac{\partial}{\partial y},n\Big)U(y)\Big]^+&=F^{(k)}(y),\quad y\in S_k,\quad k=\overline{m_1+1,m_2},\\
\Big[Q\Big(\frac{\partial}{\partial y},n\Big)U(y)\Big]^+&=F^{(k)}(y),\quad y\in S_k,\quad k=\overline{m_2+1,m_3},\\
\Big[P\Big(\frac{\partial}{\partial y},n\Big)U(y)\Big]^+&=F^{(k)}(y),\quad y\in S_k,\quad k=\overline{m_3+1,m_4},\\
\Big[N_1\Big(\frac{\partial}{\partial y},n\Big)U(y)\Big]^+&=F^{(k)}(y),\quad y\in S_k,\quad k=\overline{m_4+1,m_5},\\
\Big[\Omega_1\Big(\frac{\partial}{\partial y},n\Big)U(y)\Big]^+&=F^{(k)}(y),\quad y\in S_k,\quad k=\overline{m_5+1,m_6},\\
\Big[L_1\Big(\frac{\partial}{\partial y},n\Big)U(y)\Big]^+&=F^{(k)}(y),\quad y\in S_k,\quad k=\overline{m_6+1,m_7},\\
\Big[I_1\Big(\frac{\partial}{\partial y},n\Big)U(y)\Big]^+&=F^{(k)}(y),\quad y\in S_k,\quad k=\overline{m_7+1,m}.
\end{aligned}$$

Here m_i, $i=\overline{1,7}$, are arbitrary natural numbers such that $0\le m_1\le m_2\le\cdots\le m_7\le m$,

$$\begin{aligned}
RU&=\Big(HU,\frac{\partial u_4}{\partial n},\frac{\partial u_5}{\partial n},\Big)^T,\\
QU&=\Big(U,\frac{\partial u_4}{\partial n},\frac{\partial u_5}{\partial n}\Big)^T,\quad PU=\big(HU,u_4,u_5\big)^T,\\
N_1U&=\Big((u\cdot n),HU-n(HU\cdot n),\frac{\partial u_4}{\partial n},\frac{\partial u_5}{\partial n}\Big)^T,\\
\Omega_1U&=\big(u-n(u\cdot n),(HU\cdot n),u_4,u_5\big)^T,\\
L_1U&=\big((u\cdot n),HU-n(HU\cdot n),u_4,u_5\big)^T,
\end{aligned}$$

$$I_1U=\Big(u-n(u\cdot n),(HU\cdot n),\frac{\partial u_4}{\partial n},\frac{\partial u_5}{\partial n}\Big)^T;$$

$HU = Tu - \gamma_1 nu_4 - \gamma_2 nu_5$ is the thermodiffusion stress vector, Tu is the elastic stress vector; $(u\cdot n)$, $HU-n(HU\cdot n)$ are the normal components of the displacement vector and the thermodiffusion stress vectors, respectively; $F^{(k)}(y)\in C^{1,\beta}(S_k)$, $\beta>0$, $k=\overline{0,m}$, are the tangential components of the displacement vector and the thermodiffusion stress vector; $\Phi=(\varphi,\varphi_4,\varphi_5)^T\in C^{0,\alpha}(\overline{D^+})$, $\alpha>0$, are the known vectors, and $n=n(y)=(n_1(y),n_2(y),n_3(y))^T=(n_1,n_2,n_3)^T$ is the unit normal vector at the point $y\in S$ (outer with respect to D^+) or at the point $y\in S'$ (outer with respect to D^-) [1, 2].

Problem $M^-(\omega)$**.** In the infinite domain D^- find a regular vector $u=(u,u_4,u_5)$, the solution of the system (3), satisfying both the boundary conditions:

$$\begin{aligned}
U^-(y)&=F^{(k)}(y),\quad y\in S_k,\quad k=\overline{1,m_1},\\
\Big[R\Big(\frac{\partial}{\partial y},n\Big)U(y)\Big]^-&=F^{(k)}(y),\quad y\in S_k,\quad k=\overline{m_1+1,m_2},\\
\Big[Q\Big(\frac{\partial}{\partial y},n\Big)U(y)\Big]^-&=F^{(k)}(y),\quad y\in S_k,\quad k=\overline{m_2+1,m_3},\\
\Big[P\Big(\frac{\partial}{\partial y},n\Big)U(y)\Big]^-&=F^{(k)}(y),\quad y\in S_k,\quad k=\overline{m_3+1,m_4},\\
\Big[N_1\Big(\frac{\partial}{\partial y},n\Big)U(y)\Big]^-&=F^{(k)}(y),\quad y\in S_k,\quad k=\overline{m_4+1,m_5},\\
\Big[\Omega_1\Big(\frac{\partial}{\partial y},n\Big)U(y)\Big]^-&=F^{(k)}(y),\quad y\in S_k,\quad k=\overline{m_5+1,m_6},\\
\Big[L_1\Big(\frac{\partial}{\partial y},n\Big)U(y)\Big]^-&=F^{(k)}(y),\quad y\in S_k,\quad k=\overline{m_6+1,m_7},\\
\Big[I_1\Big(\frac{\partial}{\partial y},n\Big)U(y)\Big]^-&=F^{(k)}(y),\quad y\in S_k,\quad k=\overline{m_7+1,m},
\end{aligned}$$

and the following asymptotic equalities at infinity [2, 3]: the radiation condition of the thermodiffusion theory:

$$\begin{gathered}
u^{(1)}(x)=o(r^{-1}),\quad \frac{\partial u^{(1)}(x)}{\partial x_k}=O(r^{-2}),\quad k=1,2,3,\\
u_j(x)=o(r^{-1}),\quad \frac{\partial u_j(x)}{\partial x_k}=O(r^{-2}),\quad j=1,2,\quad k=1,2,3,\\
u^{(2)}(x)=O(r^{-1}),\quad \frac{\partial u^{(2)}}{\partial r}-i\lambda_4u^{(2)}(x)=o(r^{-1})
\end{gathered}\tag{4}$$

for $\omega = p > 0$ and

$$\begin{aligned} &u^{(j)}(x) = o(r^{-1}), \quad \frac{\partial u^{(j)}(x)}{\partial x_k} = O(r^{-2}), \quad j = 1,2, \quad k = 1,2,3, \\ &u_j(x) = o(r^{-1}), \quad \frac{\partial u_j(x)}{\partial x_k} = O(r^{-2}), \quad j = 4,5, \quad k = 1,2,3, \end{aligned} \tag{5}$$

for $\omega = i\tau$ $(\sigma > 0)$. Here

$$U = \big(u^{(1)} + u^{(2)}, u_4, u_5\big)^T \quad \operatorname{rot} u^{(1)} = 0, \quad \operatorname{div} u^{(2)} = 0,$$

$$\prod_{k=1}^{3}(\Delta + \lambda_k^2)u^{(1)} = 0, \quad (\Delta + \lambda_4^2)u^{(2)} = 0, \quad \prod_{k=1}^{3}(\Delta + \lambda_k^2)u_j = 0, \quad j = 4,5,$$

r is the radius-vector of the point x and λ_k^2, $k = \overline{1,4}$, are the characteristic parameters [3]; $\Phi \in C^{0,\beta}(\overline{D}^-)$ is the given finitary vector.

For our purpose it is convenient to rewrite the boundary conditions in the equivalent form. Therefore instead of the above-stated problems we shall consider the following equivalent problems [2, 5, 6].

Problem $\widehat{M}^+(\omega)$. Find in D^+ a regular vector $U = (u, u_4, u_5)^T$, the solution of the system (3), by the boundary conditions

$$\begin{aligned} U^+(y) &= f^{(k)}(y), \quad y \in S_k, \quad k = \overline{0, m_1}, \\ \Big[R\Big(\frac{\partial}{\partial y}, n\Big)U(y)\Big]^+ &= f^{(k)}(y), \quad y \in S_k, \quad k = \overline{m_1+1, m_2}, \\ \Big[Q\Big(\frac{\partial}{\partial y}, n\Big)U(y)\Big]^+ &= f^{(k)}(y), \quad y \in S_k, \quad k = \overline{m_2+1, m_3}, \\ \Big[P\Big(\frac{\partial}{\partial y}, n\Big)U(y)\Big]^+ &= f^{(k)}(y), \quad y \in S_k, \quad k = \overline{m_3+1, m_4}, \\ \Big[N\Big(\frac{\partial}{\partial y}, n\Big)U(y)\Big]^+ &= f^{(k)}(y), \quad y \in S_k, \quad k = \overline{m_4+1, m_5}, \\ \Big[\Omega\Big(\frac{\partial}{\partial y}, n\Big)U(y)\Big]^+ &= f^{(k)}(y), \quad y \in S_k, \quad k = \overline{m_5+1, m_6}, \\ \Big[L\Big(\frac{\partial}{\partial y}, n\Big)U(y)\Big]^+ &= f^{(k)}(y), \quad y \in S_k, \quad k = \overline{m_6+1, m_7}, \\ \Big[I\Big(\frac{\partial}{\partial y}, n\Big)U(y)\Big]^+ &= f^{(k)}(y), \quad y \in S_k, \quad k = \overline{m_7+1, m}, \end{aligned}$$

where $f^{(k)}(y) = (f_1^{(k)}(y), f_2^{(k)}(y), f_3^{(k)}(y), f_4^{(k)}(y), f_5^{(k)}(y))^T$, $y \in S_k$, $k = \overline{0, m_4}$, $f^{(k)}(y) = (f_1^{(k)}(y), \dots, f_6^{(k)}(y))^T$, $y \in S_k$, $k = \overline{m_4+1, m}$, are the vectors written, obviously, in terms of the boundary data of the problem $M^+(\omega)$ [7]:

$$NU = \Big\{2\mu D(u\cdot n) - (HU - n(HU\cdot n)), (u\cdot n), \frac{\partial u_4}{\partial n}, \frac{\partial u_5}{\partial n}\Big\}^T,$$

$$\Omega U = \Big\{u - n(u\cdot n), (HU\cdot n) + 2\mu\sum_{j=1}^{3} D_j(u_j - n_j(u\cdot n)), u_4, u_5\Big\}^T,$$

$$LU = \Big\{2\mu D(u\cdot n) - ((HU - n(HU\cdot n)), (u\cdot n), u_4, u_5\Big\}^T,$$

$$IU = \Big\{u - n(u\cdot n), (HU\cdot n) + \\ + 2\mu\sum_{j=1}^{3} D_j(u_j - n_j(u\cdot n)), \frac{\partial u_4}{\partial n}, \frac{\partial u_5}{\partial n}\Big\}^T,$$

$$D\Big(\frac{\partial}{\partial y}, n\Big) = \operatorname{grad} - n\frac{\partial}{\partial n}\,, \quad D^j\Big(\frac{\partial}{\partial y}, n\Big) = \frac{\partial}{\partial y_j} - n_j\frac{\partial}{\partial n}$$

(see [1, 2]).

We consider also the corresponding exterior problem $\widehat{M}^-(\omega)$. Denote the homogeneous problems by $\widehat{M}_0^{\pm}(\omega)$. Use will be made of the conjugate operators $\widetilde{N}, \widetilde{\Omega}, \dots$ which, as is easily seen, can be obtained from the corresponding operators by substituting the parameters γ_1 and γ_2 by $i\omega\gamma_1$ and $i\omega\gamma_2$, respectively.

It should be noted that in [2, 4] the theorems on the existence and uniqueness of the solution of the problems $M^{\pm}(\omega)$ have been proved for $0 \le m_1 \le m_2 \le m_3 \le m$ $(m_4 = m_5 = m_6 = m_7 = m)$.

2. Uniqueness Theorems.

Theorem 1. *In the class of regular vector-functions the homogeneous problem* $\widehat{M}_0^+(i\tau)$, $\sigma > 0$, *has only trivial solution.*

Proof. The following generalized Green's formula is known [2]:

$$\int\limits_{D^+} \Big\{v^T\Big[A\Big(\frac{\partial}{\partial x}\Big)u - \gamma_1 \operatorname{grad} u_4 - \gamma_2 \operatorname{grad} u_5\Big] + \\ + E(v,u) - \gamma_1 u_4 \operatorname{div} v - \gamma_2 u_5 \operatorname{div} v\Big\}dx = \int\limits_S v^T HU ds, \qquad (6)$$

where $U = (u, u_4, u_5)^T$, $V = (v, v_4, v_5)^T$ are the regular vectors,

$$E(v,u) = \frac{3\lambda + 2\mu}{3} \operatorname{div} v \operatorname{div} u + \frac{\mu}{2} \sum_{k \neq q=1}^{3} \Big(\frac{\partial v_k}{\partial x_q} + \frac{\partial v_q}{\partial x_k}\Big)\Big(\frac{\partial u_k}{\partial x_q} + \frac{\partial u_q}{\partial x_k}\Big) +$$

$$+ \frac{\mu}{3} \sum_{k,q=1}^{3} \Big(\frac{\partial v_k}{\partial x_k} - \frac{\partial v_q}{\partial x_q}\Big)\Big(\frac{\partial u_k}{\partial x_k} - \frac{\partial u_q}{\partial x_q}\Big).$$

Let $U(x)$ be the solution of the problem $\widehat{M}_0^+(i\tau)$ and let $\overline{U}(x)$ be a complex-conjugate vector. Then from (6) we easily obtain

$$\int_{D^+} \Big\{\rho\tau^2|u|^2 + E(u,\overline{u}) + a_1|u_4|^2 + a_{12}(u_4\overline{u}_5 + \overline{u}_4 u_5) +$$

$$+ a_2|u_5|^2 + \frac{1}{\overline{\tau}} \sum_{k=1}^{2} \delta_k |\operatorname{grad} u_{3+k}|^2\Big\} dx =$$

$$= \int_S \Big[\overline{u}^T HU + \frac{1}{\overline{\tau}} \sum_{k=1}^{2} \delta_k u_{3+k} \frac{\partial \overline{u}_{3+k}}{\partial n}\Big] ds. \qquad (7)$$

Taking into account the homogeneity of the boundary conditions, the integral with respect to S in (7) vanishes. Moreover, $E(u,\overline{u}) \geq 0$, $\operatorname{Im} E(u,\overline{u}) = 0$, $a_1|u_4|^2 + a_{12}(u_4\overline{u}_5 + \overline{u}_4 u_5) + a_2|u_5|^2 = |\sqrt{a}_1 u_4 + \frac{a_{12}}{\sqrt{a_1}} u_5|^2 + \frac{a_1a_2 - a_{12}^2}{a_1}|u_5|^2 \geq 0$. Separation of the real and imaginary parts in (7) results in

$$\int_{D^+} \Big\{\rho(\sigma^2 - q^2)|u|^2 + E(u,\overline{u}) + \Big|\sqrt{a_1}u_4 + \frac{a_{12}}{\sqrt{a_1}}u_5\Big|^2 +$$

$$+ \frac{a_1a_2 - a_{12}^2}{a_1}|u_5|^2 + \frac{\sigma}{|\tau|^2} \sum_{k=1}^{2} \delta_k |\operatorname{grad} u_{3+k}|^2\Big\} dx = 0,$$

$$q \int_{D^+} \Big[2\rho\sigma|u|^2 + \frac{1}{|\tau|^2} \sum_{k=1}^{2} \delta_k |\operatorname{grad} u_{3+k}|^2\Big] dx = 0.$$

Consequently, $u(x) \equiv 0$, $u_4(x) \equiv 0$, $u_5(x) \equiv 0$, that is, $U(x) \equiv 0$, $x \in D^+$. □

Theorem 2. *The homogeneous problem* $\widehat{M}_0^-(i\tau)$ *has only trivial solution.*

This theorem follows directly from formulas (6) and (7) which, owing to (5), are likewise valid for an infinite domain D^-.

Theorem 3. *The homogeneous problem $\widehat{M}_0^-(p)$, $p>0$, has only trivial solution.*

We give an outline of the proof here. In [2,3] for the solution of the homogeneous problem we have the following Green's formula:

$$\frac{2}{ip}\sum_{k=1}^{2}\delta_k\int\limits_{D^+}|\operatorname{grad}u_{3+k}|^2dx=\int\limits_{S}\Big\{\big[\overline{u}^THU-uH\overline{U}\big]+$$
$$+\frac{1}{ip}\sum_{k=1}^{2}\delta_k\Big(u_{3+k}\frac{\partial\overline{u}_{3+k}}{\partial n}+\overline{u}_{3+k}\frac{\partial u_{3+k}}{\partial n}\Big)\Big\}ds \tag{8}$$

written for the domain $D^-\cap\mathcal{H}(0,R)$, where $\mathcal{H}(0,R)$ is a sphere centered at the origin, of radius R. Owing to this formula the proof of the theorem is analogous to that for the case of thermoelasticity [1, 2]. We have (see [2, 3, 4])

Theorem 4. *The homogeneous problem $\widehat{M}_0^+(p)$, $p>0$ has the discrete spectrum of real (positive) eigenvalues.*

In the sequel we shall need the uniqueness theorem whose proof is obvious.

Theorem 5. *Homogeneous boundary value problems*

$$B\Big(\frac{\partial}{\partial x},i\tau\Big)U(x)=0,\quad x\in D_k,\quad k=\overline{1,m},$$

have only trivial solutions when one of the homogeneous boundary conditions of the problem $\widehat{M}_0^+(i\tau)$ is satisfied on the boundary of S_k.

3. Existence theorems for the problems $\widehat{M}^\pm(i\tau)$, $\widehat{M}^\pm(p)$, $p>0$. Consider the problem $\widehat{M}^\pm(i\tau)$. Let $D^{(m_4)}=D_0\setminus\cup_{k=1}^{m_4}\overline{D}_k$; denote by $G(x,y,D^{(m_4)},i\tau)$ the Green's tensor of the boundary value problem $\widehat{M}^+(i\tau)$ for $0\le m_1\le m_2\le m_3\le m_4$. Represent it in the form

$$G(x,y)=G(x,y,D^{(m_4)},i\tau)=\Gamma(x-y,i\tau)-g(x,y,D^{(m_4)},i\tau),$$

where $\Gamma(x-y,i\tau)$ is the matrix of fundamental solutions of the operator $B(\frac{\partial}{\partial x},i\tau)$, and $g(x,y)=g(x,y,D^{m_4},i\tau)$ is the regular (matrix) solution of the boundary value problem

$$B\Big(\frac{\partial}{\partial x},i\tau\Big)g(x,y)=0,\quad x,y\in D^{(m_4)},$$
$$g^+(z,y)=\Gamma(z-y,i\tau),\quad z\in S_k,\quad k=\overline{0,m_1},$$
$$\Big[R\Big(\frac{\partial}{\partial z},n\Big)g(z,y)\Big]^+=R\Big(\frac{\partial}{\partial z},n\Big)\Gamma(z-y,i\tau),\quad z\in S_k,\quad k=\overline{m_1+1,m_2},$$

$$\Big[Q\Big(\frac{\partial}{\partial z},n\Big)g(z,y)\Big]^+ = Q\Big(\frac{\partial}{\partial z},n\Big)\Gamma(z-y,i\tau), \quad z\in S_k, \quad k=\overline{m_2+1,m_3},$$
$$\Big[P\Big(\frac{\partial}{\partial z},n\Big)g(z,y)\Big]^+ = P\Big(\frac{\partial}{\partial z},n\Big)\Gamma(z-y,i\tau), \quad z\in S_k, \quad k=\overline{m_3+1,m_4}.$$

According to the results obtained in [4], this nonhomogeneous problem is always solvable. We seek the solution of the problem $\widehat{M}^+(i\tau)$ in the form

$$\begin{aligned}
U(x) = &-\frac{1}{2}\int\limits_{D^+} G(x,y)\Phi(y)dy - \\
&-\frac{1}{2}\sum_{k=0}^{m_1}\int\limits_{S_k}\Big[\widetilde{R}\Big(\frac{\partial}{\partial y},n\Big)G^T(x,y)\Big]^T f(y)ds + \\
&+\frac{1}{2}\sum_{k=m_1+1}^{m_2}\int\limits_{S_k} G(x,y)f(y)ds - \\
&-\frac{1}{2}\sum_{k=m_2+1}^{m_3}\int\limits_{S_k}\Big[\widetilde{P}\Big(\frac{\partial}{\partial y},n\Big)G^T(x,y)\Big]^T f(y)ds + \\
&+\frac{1}{2}\sum_{k=m_3+1}^{m_4}\int\limits_{S_k}\Big[Q\Big(\frac{\partial}{\partial y},n\Big)G^T(x,y)\Big]^T f(y)ds + \\
&+\sum_{k=m_4+1}^{m_5}\int\limits_{S_k}\Big[\widetilde{\Omega}\Big(\frac{\partial}{\partial y},n\Big)G^T(x,y)\Big]^T \varphi(y)ds + \\
&+\sum_{k=m_5+1}^{m_6}\int\limits_{S_k}\Big[N\Big(\frac{\partial}{\partial y},n\Big)G^T(x,y)\Big]^T \varphi(y)ds + \\
&+\sum_{k=m_6+1}^{m_7}\int\limits_{S_k}\Big[\widetilde{I}\Big(\frac{\partial}{\partial y},n\Big)G^T(x,y)\Big]^T \varphi(y)ds + \\
&+\sum_{k=m_7+1}^{m}\int\limits_{S_k}\Big[L\Big(\frac{\partial}{\partial y},n\Big)G^T(x,y)\Big]^T \varphi(y)ds,
\end{aligned} \tag{9}$$

where $f(y)=f^{(k)}(y)$, $y\in S_k$, $k=\overline{0,m_4}$, is a given vector; $\varphi(y)$, $y\in S_k$, $k=\overline{m_4+1,m}$ is an unknown six-dimensional vector. We can easily see that the differential equation and the boundary conditions on S_k, $k=\overline{0,m_4}$, are automatically satisfied for an arbitrary $\varphi(y)\in C^{1,\beta}(S_k)$, $k=\overline{m_4+1,m}$. To satisfy the rest of the boundary conditions, we obtain the following system of singular integral equations with respect to $\varphi(y)$:

$$a(z)\varphi(z)+\sum_{m_4+1}^{m}\int_{S_k}K(z,y)\varphi(y)ds+$$
$$+\sum_{k=m_4+1}^{m}\int_{S_k}K^{(1)}(z,y)\varphi(y)ds=l(z), \tag{10}$$

where $K(z,y)$ is the singular matrix,

$$K(z,y)=\begin{cases} N\Big(\dfrac{\partial}{\partial z},n\Big)\Big[\widetilde{\Omega}\Big(\dfrac{\partial}{\partial y},n\Big)\Gamma^T(z-y),i\tau)\Big]^T, & z,y\in\bigcup\limits_{k=m_4+1}^{m_5}S_k,\\ \Omega\Big(\dfrac{\partial}{\partial z},n\Big)\Big[N\Big(\dfrac{\partial}{\partial y},n\Big)\Gamma^T(z-y),i\tau)\Big]^T, & z,y\in\bigcup\limits_{k=m_5+1}^{m_6}S_k,\\ L\Big(\dfrac{\partial}{\partial z},n\Big)\Big[\widetilde{I}\Big(\dfrac{\partial}{\partial y},n\Big)\Gamma^T(z-y),i\tau)\Big]^T, & z,y\in\bigcup\limits_{k=m_6+1}^{m_7}S_k,\\ I\Big(\dfrac{\partial}{\partial z},n\Big)\Big[L\Big(\dfrac{\partial}{\partial y},n\Big)\Gamma^T(z-y),i\tau)\Big]^T, & z,y\in\bigcup\limits_{k=m_7+1}^{m}S_k,\\ 0, & \text{otherwise},\end{cases} \tag{11}$$

$$a(z)=\begin{cases}-1, & z\in\bigcup\limits_{k=m_4+1}^{m_5}S_k\cup\bigcup\limits_{k=m_6+1}^{m_7}S_k,\\ 1, & z\in\bigcup\limits_{k=m_5+1}^{m_6}S_k\cup\bigcup\limits_{k=m_7+1}^{m}S_k,\end{cases} \tag{12}$$

$K^{(1)}(z,y)$, $z,y\in\cup_{k=m_4+1}^{m}S_k$ is a completely definite continuous matrix, $l(z)$ is the known vector given in terms of the data of the problem; there is no need to write out this vector explicitly.

The system of integral equations (10) is singular; singular integrals involved in (10) are understood in the sense of the Cauchy principal value. These systems have been studied in detail in [1]. The only difference is in the completely continuous operators which do not affect the general theory. The system (10) is of normal type with zero index. We outline the proof, for example, in the case where $z,y\in\cup_{k=m_4+1}^{m_5}S_k$. For (10) we have

$$-\varphi(z)+\sum_{k=m_4+1}^{m_5}\int_{S_k}N\Big(\frac{\partial}{\partial z},n\Big)\Big[\widetilde{\Omega}\Big(\frac{\partial}{\partial y},n\Big)\Gamma^T(z-y,i\tau)\Big]^T\varphi(y)ds+$$
$$+\mathbf{T}(\varphi)=l(z). \tag{13}$$

(Here $\mathbf{T}$ is the completely continuous operator.) Direct calculation shows that the 6×6 symbolic matrix has the form (see [1,2,4,5,7])

$$\sigma^+(z,\theta)=\|\sigma^+_{jk}\|_{6\times 6},\quad \sigma_{jj}=-1,\ \ j=\overline{1,6},\quad \sigma_{j4}=\sigma_j,\ \ j=\overline{1,3},$$
$$\sigma_{jk}=0 \text{ in all the rest cases},$$

$$\sigma_j = 2\pi i\Big[K(z)(\alpha_{j2}\sin\theta - \alpha_{j1}\cos\theta) + \\ + \sum_{k=1}^{3}(\alpha_{k2}\sin\theta - \alpha_{k1}\cos\theta)D^j\Big(\frac{\partial}{\partial z}, n\Big)n_k\Big], \quad j = 1, 3,$$

where $K(z) = Dn(z) = \sum_{j=1}^{3} D^j(z)n_j(z)$ is the mean curvature of the surface S_k at the point $z \in S_k$. Hence, $\det\sigma^+(z,\theta) = 1$, and the singular operator of the system (10) is also of normal type. The system (13) has zero index. This follows from the fact that for the principal minors Δ_j, $j = \overline{1,6}$, of the symbolic matrix $\sigma^+(z,\theta)$ we have $\Delta_j = (-1)^j$. Thus, the lower bounds of modules of these minors are positive (see Mikhlin [8], p. 192). Moreover, the classical theory of Fredholm integral equations extends entirely to such systems. Consequently, the system (13) is regular-solvable.

Let us now prove that the system (10) is uniquely solvable. Indeed, let $\varphi^{(0)}(y) \in C^{1,\beta}(S_k)$, $k = \overline{m_4+1, m}$, be a solution of the homogeneous system corresponding to (10). Let us consider the potential

$$\overset{\circ}{U}(x) = \sum_{k=m_4+1}^{m_5} \int_{S_k} \Big[\widetilde{\Omega}\Big(\frac{\partial}{\partial y}, n\Big)G^T(x,y)\Big]^T \varphi^{(0)}(y)ds + \\ + \sum_{k=m_5+1}^{m_6} \int_{S_k} \Big[N\Big(\frac{\partial}{\partial y}, n\Big)G^T(x,y)\Big]^T \varphi^{(0)}(y)ds + \\ + \sum_{k=m_6+1}^{m_7} \int_{S_k} \Big[\widetilde{I}\Big(\frac{\partial}{\partial y}, n\Big)G^T(x,y)\Big]^T \varphi^{(0)}(y)ds + \\ + \sum_{k=m_7+1}^{m} \int_{S_k} \Big[L\Big(\frac{\partial}{\partial y}, n\Big)G^T(x,y)\Big]^T \varphi^{(0)}(y)ds.$$

It is clear that $\overset{\circ}{U}(x)$ is the regular solution of the homogeneous problem $\widehat{M_0^+}(i\tau)$. By Theorem 1 we have

$$\overset{\circ}{U}(x) \equiv 0, \quad x \in D^+. \tag{14}$$

Furthermore,

$$B\Big(\frac{\partial}{\partial x}, i\tau\Big)\overset{\circ}{U}(x) = 0, \quad x \in \bigcup_{k=m_4+1}^{m} D_k. \tag{15}$$

Moreover, taking into account certain Liapounov–Tauber type theorems proven in [1,2], we can show that

$$\begin{aligned}
&\lim_{x\to z}\Omega\Big(\frac{\partial}{\partial x},n\Big)\overset{\circ}{U}(x)=0, \quad x\in\bigcup_{k=m_4+1}^{m_5}D_k, \quad z\in\bigcup_{k=m_4+1}^{m_5}S_k,\\
&\lim_{x\to z}N\Big(\frac{\partial}{\partial x},n\Big)\overset{\circ}{U}(x)=0, \quad x\in\bigcup_{k=m_5+1}^{m_6}D_k, \quad z\in\bigcup_{k=m_5+1}^{m_6}S_k,\\
&\lim_{x\to z}I\Big(\frac{\partial}{\partial x},n\Big)\overset{\circ}{U}(x)=0, \quad x\in\bigcup_{k=m_6+1}^{m_7}D_k, \quad z\in\bigcup_{k=m_6+1}^{m_7}S_k,\\
&\lim_{x\to z}L\Big(\frac{\partial}{\partial x},n\Big)\overset{\circ}{U}(x)=0, \quad x\in\bigcup_{k=m_7+1}^{m}D_k, \quad z\in\bigcup_{k=m_7+1}^{m}S_k.
\end{aligned}\tag{16}$$

On account of (15), (16) and Theorem 5 we obtain

$$\overset{\circ}{U}(x)\equiv 0, \quad x\in x\in\bigcup_{k=m_4+1}^{m_5}D_k. \tag{17}$$

Using the jump formulas of thermodiffusion potential, from (14) and (17) we have

$$\varphi^{(0)}(z)\equiv 0, \quad z\in\bigcup_{k=m_4+1}^{m_5}S_k.$$

Thus the following theorem is valid.

Theorem 6. *The solution of the inhomogeneous problem $\widehat{M}^+(i\tau)$, $\sigma>0$ does exist; it is unique and can be represented in the form of* (9)*, where $\varphi(y)$ is the solution of the integral equation* (10) *which is solvable for an arbitrary right-hand side.*

Consider now the problem $\widehat{M}^-(i\tau)$.

Let $D_\infty^{(m_4)}=\mathbb{R}^3\setminus\cup_{k=1}^{m_4}D_k$, $G(x,y)=G(x,y,D_\infty^{(m_4)},i\tau)$ be the Green's tensor of the boundary value problem $\widehat{M}^-(i\tau)$ for $1\le m_1\le m_2\le m_3\le m$. The existence of such a tensor follows from the results of [2, 4].

We seek the solution in the form

$$\begin{aligned}
U(x)=&-\frac{1}{2}\int_{D^-}G(x,y)\Phi(y)dy-\\
&-\frac{1}{2}\sum_{k=1}^{m_1}\int_{S_k}\Big[\widetilde{R}\Big(\frac{\partial}{\partial y},n\Big)G^T(x,y;D_\infty^{(m_4)},i\tau)\Big]^T f(y)ds+\\
&+\frac{1}{2}\sum_{k=m_1+1}^{m_2}\int_{S_k}G(x,y)f(y)ds-\\
&-\frac{1}{2}\sum_{k=m_2+1}^{m_3}\int_{S_k}\Big[\widetilde{P}\Big(\frac{\partial}{\partial y},n\Big)G^T(x,y)\Big]^T f(y)ds+
\end{aligned}$$

$$+\frac{1}{2}\sum_{k=m_3+1}^{m_4}\int\limits_{S_k}\Big[Q\Big(\frac{\partial}{\partial y},n\Big)G^T(x,y)\Big]^T f(y)ds+$$

$$+\sum_{k=m_4+1}^{m_5}\int\limits_{S_k}\Big[\widetilde{\Omega}\Big(\frac{\partial}{\partial y},n\Big)G^T(x,y)\Big]^T \psi(y)ds+$$

$$+\sum_{k=m_5+1}^{m_6}\int\limits_{S_k}\Big[N\Big(\frac{\partial}{\partial y},n\Big)G^T(x,y)\Big]^T \psi(y)ds+$$

$$+\sum_{k=m_6+1}^{m_7}\int\limits_{S_k}\Big[\widetilde{I}\Big(\frac{\partial}{\partial y},n\Big)G^T(x,y)\Big]^T \psi(y)ds+$$

$$+\sum_{k=m_7+1}^{m}\int\limits_{S_k}\Big[L\Big(\frac{\partial}{\partial y},n\Big)G^T(x,y)\Big]^T \psi(y)ds, \tag{18}$$

With respect to $\psi(y)$ we obtain the following system of singular integral equations:

$$a(z)\psi(z)+\sum_{k=m_4+1}^{m}\int\limits_{S_k}K(z,y)\psi(y)ds+$$

$$+\sum_{k=m_4+1}^{m}\int\limits_{S_k}K^{(2)}(z,y)\psi(y)ds=l(z), \tag{19}$$

where $z\in\cup_{k=m_4+1}^{m}S_k$, $K(z,y)$, $a(z)$ are determined by the formulas (11) and (12), respectively. $K^{(2)}(z,y)$ is a completely definite continuous matrix. In the same way as for the problem $\widehat{M}^+(i\tau)$ we can show that the system (19) is solvable for an arbitrary right-hand side (here use is made of Theorems 2 and 5).

Thus the theorem below is valid.

Theorem 7. *The solution of the inhomogeneous problem $\widehat{M}^-(i\tau)$ does exist; it is unique and can be represented in the form of* (18), *where $\psi(y)$ is the solution of the integral equation* (19) *which is solvable for an arbitrary right-hand side.*

Consider now the problem $\widehat{M}^-(p)$ in the particular case where $m_5=m_6=m_7=m$, $1\le m_1\le m_2\le m_3\le m_4\le m$.

We seek the solution in the form

$$U(x)=-\frac{1}{2}\int\limits_{D^-}G(x,y)\Phi(y)dy-$$

$$
\begin{aligned}
&-\frac{1}{2}\sum_{k=1}^{m_1}\int\limits_{S_k}\Big[\widetilde{R}\Big(\frac{\partial}{\partial y},n\Big)G^T(x,y)\Big]^T f(y)ds+\\
&+\frac{1}{2}\sum_{k=m_1+1}^{m_2}\int\limits_{S_k}G(x,y)f(y)ds-\\
&-\frac{1}{2}\sum_{k=m_2+1}^{m_3}\int\limits_{S_k}\Big[\widetilde{P}\Big(\frac{\partial}{\partial y},n\Big)G^T(x,y)\Big]^T f(y)ds+\\
&+\frac{1}{2}\sum_{k=m_3+1}^{m_4}\int\limits_{S_k}\Big[Q\Big(\frac{\partial}{\partial y},n\Big)G^T(x,y)\Big]^T f(y)ds+\\
&+\sum_{k=m_4+1}^{m}\int\limits_{S_k}\Big[\widetilde{\Omega}\Big(\frac{\partial}{\partial y},n\Big)G^T(x,y)\Big]^T \varphi(y)ds+\\
&+\sum_{k=m_4+1}^{m}\int\limits_{S_k}\Big[N\Big(\frac{\partial}{\partial y},n\Big)G^T(x,y)\Big]^T \chi(y)ds,
\end{aligned}\tag{20}
$$

where $G(x,y)=G(x,y;D_\infty^{(m_4)},p)$, $f(y)=f^{(k)}(y)$, $y\in S_k$, $k=\overline{1,m_4}$, is the given vector, and $\varphi(y)$, $\chi(y)$ are the unknown vectors. For the first vector we obtain the following system of singular integral equations:

$$
\begin{aligned}
&\varphi(z)+\sum_{k=m_4+1}^{m}\int\limits_{S_k}N\Big(\frac{\partial}{\partial z},n\Big)\Big[\widetilde{\Omega}\Big(\frac{\partial}{\partial y},n\Big)G^T(z,y,D_\infty^{(m_4)},p)\Big]^T\varphi(y)ds=\\
&=l(z)-\sum_{k=m_4+1}^{m}N\Big(\frac{\partial}{\partial z},n\Big)\int\limits_{S_k}\Big[N\Big(\frac{\partial}{\partial y},n\Big)G^T(z,y,D_\infty^{(m_4)},p)\Big]^T\chi(y)ds,
\end{aligned}\tag{21}
$$

where $l(z)$ is the known vector given in terms of the problem's data.

Consider the homogeneous integral equation corresponding to (21),

$$
\begin{aligned}
&\varphi(z)+\sum_{k=m_4+1}^{m}\int\limits_{S_k}N\Big(\frac{\partial}{\partial z},n\Big)\times\\
&\times\Big[\widetilde{\Omega}\Big(\frac{\partial}{\partial y},n\Big)G^T(z,y,D_\infty^{(m_4)},p)\Big]^T\varphi(y)ds=0.
\end{aligned}\tag{22}
$$

The theorem below is proved in just the same way as in [1, 2].

Theorem 8. *The equality of the parameter p^2 to one of the eigenfre-*

quencies of the problem

$$\begin{gathered}
\Big(\Delta+\frac{\rho p^2}{\mu}\Big)u(x)=0, \quad \operatorname{div} u(x)=0, \quad x\in\bigcup_{k=m_4+1}^{m} D_k, \\
\big(u(x)-n(u(z)\cdot n)\big)^+=0, \quad \Big(n\cdot\frac{\partial u(z)}{\partial n}\Big)^+=0, \quad z\in\bigcup_{k=m_4+1}^{m} S_k
\end{gathered} \tag{23}$$

is the necessary and sufficient condition for equation (22) *to have the nontrivial solution.*

If p^2 *is a* ν*-multiple eigenfrequency of the problem, then the integral equation* (22) *has* ν *linearly independent solutions, and they coincide with the boundary values* $N(\frac{\partial}{\partial x}, n)$ *of the operator over the solutions of the problem* (23).

It follows from Theorem 8 that if p^2 is not an eigenfrequency of the problem (23), then the integral equation (21) is solvable due to Fredholm's first theorem, and the solution is representable by the formula (20), where $\chi(y)\equiv 0$. If, however, p^2 coincides with one of the eigenfrequencies of the problem (23), then the necessary and sufficient condition for solvability of equation (21) has the form

$$\begin{gathered}
\sum_{i=m_4+1}^{m}\int_{S_i}\bigg\{\sum_{k=m_4+1}^{m} N\Big(\frac{\partial}{\partial z},n\Big)\times \\
\times\int_{S_k}\Big[N\Big(\frac{\partial}{\partial y},n\Big)G^T(z,y)\Big]^T\chi(y)d_yS\bigg\}\overset{i}{\psi}(z)d_zS=B_i, \quad i=\overline{1,\nu},
\end{gathered} \tag{24}$$

where $\{\overset{i}{\psi}(z)\}_{i=1}^{\nu}$ is a complete system of solutions of the associated homogeneous equation corresponding to (22), B_i are the known constants, and $\chi(y)\in C^{1,\beta}(S_k)$, $y\in S_k$, $k=\overline{m_4+1,m}$ is still an unknown vector.

Choosing $\chi(y)$ appropriately, we can show [5, 6, 7] that the conditions (24) are fulfilled.

Thus the following theorem holds.

Theorem 9. *The solution of the problem* $\widehat{M}^-(p)$ *does exist for* $1\le m_1<m_2<m_3<m_4<m$*; it is unique and can be represented in the form of* (20)*, where* $\varphi(y)$ *is the solution of the integral equation* (21)*. If* p^2 *is not an eigenvalue of the problem* (23)*, then the integral equation* (21) *is solvable due to Fredholm's first theorem, and* $\chi(y)\equiv 0$*. If, however,* p^2 *is the eigenvalue of the problem* (23)*, then* (21) *is solvable due to Fredholm's third theorem.*

Now we can construct the Green's tensor of this mixed problem for the region $\mathbb{R}^3\setminus\cup_{k=1}^{m_5}\overline{D_k}$. By means of that tensor we can solve the problem

$\widehat{M}^{-}(p)$ in the case $1 \leq m_1 < \cdots < m_5 < m$ $(m_6 = m_7 = m)$. Continuing this process, we can solve the problem $\widehat{M}^{-}(p)$ in the general case,

$$1 \leq m_1 < \cdots < m_7 < m.$$

The problem $\widehat{M}^{+}(\rho)$, $\rho > 0$, can be investigated in a way similar to [4,5].

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