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ON THE RIEMANN–HILBERT PROBLEM IN THE DOMAIN WITH A NONSMOOTH BOUNDARY

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Abstract. The following Riemann–Hilbert problem is solved: find an analytical function Φ from the Smirnov class $E^p(D)$, whose angular boundary values satisfy the condition

$$\mathrm{Re}[(a(t)+ib(t))\Phi^+(t)] = f(t).$$

The boundary Γ of the domain D is assumed to be a piecewise smooth curve whose nonintersecting Lyapunov arcs form, with respect to D, the inner angles with values $\nu_k\pi$, $0 < \nu_k \leq 2$.

Let Γ be a plane piecewise smooth closed Jordanian curve bounding the finite simply-connected domain D, $0 \in D$; $z = z(w)$ be the function conformally mapping the unit circle onto the domain D; $w = w(z)$ be its inverse function. Denote by $E^p(D)$ the Smirnov class, i.e., the set of analytic functions Φ in D for which

$$\sup_{r\in(0,1)} \int_{\Gamma_r} |\Phi(z)|^p |dz| < \infty,$$

where $p > 0$, and Γ_r is the image of the circumference $|w| = r$ for the conformal mapping of the unit circle U onto the domain D. As is well known, functions of the class $E^p(D)$ have, almost everywhere on Γ, angular boundary values belonging to the class $L^p(\Gamma)$, i.e., to the set of functions summable in power p on Γ with respect to the arc length measure.

Let the real measurable functions a, b and f be given on Γ, a and b being bounded and f being from the class $L^p(\Gamma)$.

We shall solve the following Riemann–Hilbert problem: define a function $\Phi \in E^p(D)$, $p > 1$, whose boundary values satisfy, almost everywhere on Γ,

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the boundary condition

$$\operatorname{Re}\big[(a(t)+ib(t))\Phi^+(t)\big]=f(t), \quad t\in\Gamma. \tag{1}$$

The Riemann–Hilbert problem and related singular integral equations under various assumptions for Γ and Φ were investigated by many authors (see, e.g., [1]–[20]); the case where Γ is a nonsmooth curve was also considered, (see, e.g., [10]–[20]). In the papers where the problem is treated in the above-given (or close to the above-given) formulation, certain restrictions are imposed on the angles under which there are arcs making up Γ. This is done with the aim of establishing the Fredholmian property of the problem or by the limited capacity of the investigation method used.

In this paper we investigate the problem for arbitrary piecewise Lyapunov curves without cusps and also for curves having cusps with the inner angle 2π. The solvability is discussed and when the problem is not solvable we give, as we think, an optimal and simple condition ensuring the existence of a solution. In all cases where solutions exist, they are constructed using Cauchy type integrals and the conformal mapping of a given domain onto the unit circle. In achieving this aim, we establish a two-weighted inequality for singular integrals with an optimal condition imposed on the pair of weights. Moreover, we give detailed consideration to the Dirichlet problem with respect to a harmonic function which is the real part of a function from $E^p(D)$. By virtue of these results, problem (1) is solved by Muskhelishvili's method, i.e., by reducing it to the linear conjugation problem [4], [5]. Though we have to solve the latter problem without making the traditional assumptions with respect to coefficients, we nevertheless succeed in creating the solvability situation and constructing solutions. In the conclusive part, the problem is considered in Smirnov weight classes.

We made essential use of the fact that when Γ is a piecewise Lyapunov curve and C is its angular point with an inner angle $\nu\pi$, $0<\nu\le 2$, then in the neighborhood of the point $c=w(C)$ we have

$$z'(w)=(w-c)^{\nu-1}z_0(w), \tag{2}$$

where $z_0(w)$ is continuous and different from zero [21], ([7], Ch. I).

1^0. A Two-Weighted Estimate for Singular Integrals. For a 2π-periodic summable function f on $(-\pi,\pi)$ we set

$$\widetilde{f}(x)=\int_{-\pi}^{\pi}\frac{f(y)}{e^{ix}-e^{iy}}\,dy.$$

It will be assumed that $1<p<\infty$ and the positive number α is so large that the function $\psi(x)=x^{p-1}\ln^p\frac{\alpha}{x}$ increases on $(0,\pi)$, $\alpha>e\pi$.

Theorem 1. *Let $1 < p < \infty$ and $x_0 \in (-\pi, \pi)$. Then there exists a constant $M(p) > 0$ such that the inequality*

$$\int_{-\pi}^{\pi} |\widetilde{f}(x)|^p |x - x_0|^{p-1} dx \leq$$
$$\leq M(p) \int_{-\pi}^{\pi} |f(x)^p |x - x_0|^{p-1} \ln^p \frac{\alpha}{|x - x_0|} \, dx \tag{3}$$

holds for arbitrary f for which the integral on the right-hand side is finite.

Moreover, the power p on the right-hand side with the logarithm is sharp, that is, it cannot be replaced by any $\gamma < p$.

The proof will be based on the following Hardy type two-weight inequality.

Theorem A ([22]). *Let $1 \leq p < q < \infty$ and the functions u, w defined on $(0, \pi)$ be positive. Then for the equality*

$$\int_0^{\pi} v(x) \Big| \int_0^x F(y) dy \Big|^p dx \leq N(p) \int_0^{\pi} w(x) |F(x)|^p dx \tag{4}$$

with the constant $N(p)$ not depending on F to hold, it is necessary and sufficient that the conditions

$$\sup_{x>0} \Big(\int_x^{\pi} v(y) dy \Big) \Big(\int_0^x w^{1-p'}(y) dy \Big)^{p-1} < \infty \tag{5}$$

($p' = \frac{p}{p-1}$) be fulfilled.

Proof of Theorem 1. It is assumed without loss of generality that $x_0 = 0$. Note that if the integral on the right-hand side of (3) is finite, then, as readily follows from the Hölder inequality, the function f is summable on $(-\pi, \pi)$ and therefore $\widetilde{f}(x)$ exists almost everywhere. Indeed,

$$\int_{-\pi}^{\pi} |f(x)| dx = \int_{-\pi}^{\pi} |f(x)| \, |x|^{1-\frac{1}{p}} \ln \frac{\alpha}{|x|} |x|^{\frac{1}{p}-1} \ln^{-1} \frac{\alpha}{|x|} \, dx \leq$$
$$\leq \Big(\int_{-\pi}^{\pi} |f(x)|^p |x|^{p-1} \ln^p \frac{\alpha}{|x|} \, dx \Big)^{\frac{1}{p}} \Big(\int_{-\pi}^{\pi} \frac{dx}{|x| \ln^{p'} \frac{\alpha}{|x|}} \Big)^{\frac{1}{p'}} < \infty.$$

Further we have

$$\int_{-\pi}^{\pi}|\widetilde{f}(x)|^p|x|^{p-1}dx=(p-1)\int_{-\pi}^{\pi}|\widetilde{f}(x)|^p\Big(\int_0^{|x|}\tau^{p-2}d\tau\Big)dx=$$
$$=(p-1)\int_0^{\pi}\tau^{p-2}\Big(\int_{\pi>|x|>\tau}|\widetilde{f}(x)|^pdx\Big)d\tau\leq$$
$$\leq 2^{p-1}(p-1)\Big[\int_0^{\pi}\tau^{p-2}\Big(\int_{\pi>|x|>\tau}\Big|\int_{\pi>|y|>\frac{\tau}{2}}\frac{f(y)}{e^{ix}-e^{iy}}\,dy\Big|^pdx\Big)d\tau+$$
$$+\int_0^{\pi}\tau^{p-2}\Big(\int_{\pi>|x|>\tau}\Big|\int_{0<|y|<\frac{\tau}{2}}\frac{f(y)}{e^{ix}-e^{iy}}\,dy\Big|^pdx\Big)d\tau\Big]=$$
$$=2^{p-1}(p-1)(I_1+I_2).$$

By Riesz' theorem we conclude, that

$$I_1=\int_0^{\pi}\tau^{p-2}\Big(\int_{\pi>|x|>\tau}\Big(\Big|\int_{-\pi}^{\pi}\frac{f(y)\chi\{y:\,\pi>|y|>\frac{\tau}{2}\}}{e^{ix}-e^{iy}}\,dy\Big|^pdx\Big)d\tau\leq$$
$$\leq R_p\int_0^{\pi}\tau^{p-2}\Big[\int_{-\pi}^{\pi}\Big(|f(y)|\chi\Big\{y:\,\pi>|y|>\frac{\tau}{2}\Big\}\Big)^pdy\Big]d\tau\leq$$
$$\leq R_p\int_0^{\pi}\tau^{p-2}\Big(\int_{\pi>|y|>\frac{\tau}{2}}|f(y)|^pd\tau\Big).$$

By changing the integration order in the latter expression we obtain

$$I_1\leq M_1\int_{-\pi}^{\pi}|f(y)|^p\Big(\int_0^{2|y|}\tau^{p-2}d\tau\Big)dy\leq$$
$$\leq M_2\int_{-\pi}^{\pi}|f(y)|^p|y|^{p-1}dy\leq M_2\int_{-\pi}^{\pi}|f(y)|^p|y|^{p-1}\ln^p\frac{\alpha}{|y|}\,dy. \tag{6}$$

Let us now estimate I_2. For $0<\tau<\pi$, $\pi>|x|>\tau$, $0<|y|<\frac{\tau}{2}$ we have $|x-y|\leq|x|+|y|\leq\pi+\frac{\pi}{2}=\frac{3\pi}{2}$. Moreover, $|x|\leq|x-y|+|y|\leq|x-y|+\frac{\tau}{2}\leq$

$|x-y|+\frac{1}{2}|x|$ and hence $|x-y|\geq\frac{1}{2}|x|>\frac{1}{2}\tau$. Also,

$$|e^{ix}-e^{iy}|=2\Big|\sin\frac{x-y}{2}\Big|\geq\frac{2}{\pi}\,|x-y|$$

for $\frac{1}{2}\tau\leq|x-y|\leq\pi$, and $|e^{ix}-e^{iy}|\geq 2\sin\frac{3\pi}{4}$ for $\pi\leq|x-y|\leq\frac{3\pi}{2}$.

By virtue of all of the above inequalities we obtain

$$I_2\leq\int\limits_0^\pi\tau^{p-2}\bigg(\int\limits_{\pi>|x|>\tau}\bigg(\int\limits_{\{y:\,|y|<\frac{\tau}{2}\}\cap\{\frac{\tau}{2}<|x-y|<\pi\}}|f(y)|\,\frac{1}{|e^{ix}-e^{iy}|}\,dy\bigg)^p dx\bigg)d\tau+$$
$$+\int\limits_0^\pi\tau^{p-2}\bigg(\int\limits_{\pi>|x|>\tau}\bigg(\int\limits_{\{y:\,|y|<\frac{\pi}{2}\}\cap\{\pi<|x-y|<\frac{3\pi}{2}\}}|f(y)|\,\frac{1}{|e^{ix}-e^{iy}|}\,dy\bigg)^p dx\bigg)d\tau\leq$$
$$\leq M_3\bigg[\int\limits_0^\pi\tau^{p-2}\bigg(\int\limits_{\pi>|x|>\tau}\frac{dx}{|x|^p}\bigg(\int\limits_{|y|<\frac{\tau}{2}}|f(y)|dy\bigg)^p\bigg)d\tau+$$
$$+\int\limits_0^\pi\tau^{p-2}\bigg(\int\limits_{\pi>|x|>\tau}dx\int\limits_{|y|<\frac{\pi}{2}}|f(y)|dy\bigg)^p d\tau\bigg].$$

Furthermore,

$$I_2\leq M_4\int\limits_0^\pi\frac{1}{\tau}\bigg(\int\limits_{|y|<\frac{\tau}{2}}|f(y)|dy\bigg)^p d\tau+$$
$$+M_3\int\limits_0^\pi\tau^{p-2}\bigg(\int\limits_{\pi>|x|>\tau}dx\int\limits_{|y|<\frac{\pi}{2}}|f(y)|dy\bigg)^p\bigg)d\tau=I_{21}+I_{22}. \tag{7}$$

Let us verify whether condition (5) is fulfilled for the pair of weights $v(\tau)=\frac{1}{\tau}$, $w(\tau)=\tau^{p-1}\ln^p\frac{\alpha}{\tau}$. We have

$$\int\limits_x^\pi\frac{d\tau}{\tau}\bigg(\int\limits_0^x\frac{1}{\tau}\ln^{p(1-p')}\frac{\alpha}{\tau}\,d\tau\bigg)^{p-1}=M_5\int\limits_x^\pi\frac{d\tau}{\tau}\bigg(\int\limits_0^x\frac{d\ln\frac{\alpha}{\tau}}{\ln^{p'}\frac{\alpha}{\tau}}\bigg)^{p-1}=$$
$$=c\ln\frac{\pi}{x}\,\frac{1}{\ln\frac{\alpha}{x}}\leq M_6.$$

Therefore we can use Theorem A to estimate I_{21}. We obtain

$$I_{21}\leq M_7\int\limits_{-\pi}^{\pi}|f(x)|^p|x|^{p-1}\ln^p\frac{\alpha}{|x|}\,dx. \tag{8}$$

Now we shall estimate I_{22} as

$$
\begin{aligned}
I_{22} &\le M_8 \int\limits_0^\pi \tau^{p-2} \int\limits_{\pi>|x|>\tau} dx \Big(\int\limits_{|y|<\frac{|x|}{2}} |f(y)|dy \Big)^p d\tau \le \\
&\le M_9 \int\limits_{-\pi}^{\pi} |x|^{p-1} \Big(\int\limits_{|y|<\frac{|x|}{2}} |f(y)|dy \Big)^p dx \le \\
&\le M_{10} \int\limits_{-\pi}^{\pi} |x|^{-1} \Big(\int\limits_{|y|<\frac{|x|}{2}} |f(y)|dy \Big)^p dx \le \\
&\le M_{11} \int\limits_{-\pi}^{\pi} |f(x)|^p |x|^{p-1} \ln^p \frac{\alpha}{|x|}\, dx. \qquad (9)
\end{aligned}
$$

By (6), (8), and (9) we conclude that

$$
\int\limits_{-\pi}^{\pi} |\widetilde{f}(x)|^p |x|^{p-1} dx \le M(p) \int\limits_{-\pi}^{\pi} |f(x)|^p |x|^{p-1} \ln^p \frac{\alpha}{|x|}\, dx. \qquad (10)
$$

It remains to show that in inequality (10) the power index p of the logarithm cannot be replaced by a smaller number. Let us assume the contrary. Let $\varepsilon \in (0,1)$. Fix the number $t > 0$ and set

$$
f_t(y) = \begin{cases} \dfrac{\alpha}{y} \ln^{(p-\varepsilon)(1-p')} \dfrac{1}{y}, & \text{for} \quad 0 < y < \dfrac{t}{2} \\ 0, & \text{for} \quad y \overline{\in} \Big(0, \dfrac{t}{2}\Big) \end{cases}.
$$

By substituting the function f_t into inequality (10), where the power index p of the logarithm is replaced by $p-\varepsilon$, we obtain

$$
\begin{aligned}
\int\limits_{-\pi}^{\pi} \Big| \int\limits_0^{\frac{t}{2}} \frac{f_t(y)}{x-y}\, dy \Big|^p |x|^{p-1} dx &\le M \int\limits_0^{\frac{t}{2}} \frac{1}{y} \ln^{(p-\varepsilon)p(1-p')} \frac{\alpha}{y} \ln^{p-\varepsilon} \frac{\alpha}{y}\, dy = \\
&= M \int\limits_0^{\frac{t}{2}} \frac{1}{y} \ln^{(p-\varepsilon)(1-p')} \ln \frac{\alpha}{y}\, dy.
\end{aligned}
$$

Hence

$$\int_t^\pi \Big(\int_0^{\frac{t}{2}} \frac{f_t(y)}{x-y}\,dy\Big)^p |x|^{p-1}dx \le M \int_0^{\frac{t}{2}} \frac{1}{y} \ln^{(p-\varepsilon)(1-p')} \frac{\alpha}{y}\,dy. \tag{11}$$

On the other hand, it is obvious that

$$\int_t^\pi \Big(\int_0^{\frac{t}{2}} \frac{f_t(y)}{x-y}\,dy\Big)^p |x|^{p-1}dx \ge \int_t^\pi \frac{1}{x}\Big(\int_0^{\frac{t}{2}} f_t(y)dy\Big)^p dx. \tag{12}$$

By virtue of (11) and (12) we must have

$$\int_t^\pi \frac{dx}{x}\Big(\int_0^{\frac{t}{2}} f_t(y)dy\Big)^p \le M \int_0^{\frac{t}{2}} \frac{1}{y} \ln^{(p-\varepsilon)(1-p')} \frac{\alpha}{y}\,dy,$$

i.e., the inequality

$$\ln\frac{\pi}{t}\Big(\int_0^t \frac{1}{y} \ln^{(p-\varepsilon)(1-p')} \frac{\alpha}{y}\,dy\Big)^{p-1} \le M \tag{13}$$

must be fulfilled for $0 < t < \pi$. But this is impossible, since

$$\Big(\int_0^t \frac{1}{y} \ln^{(p-\varepsilon)(1-p')} \frac{\alpha}{y}\,dy\Big)^{p-1} \sim \ln^{\varepsilon-1}\frac{\alpha}{y}\,. \tag{14}$$

We have therefore proved the validity of the last part of the theorem.

We shall now formulate the theorem as needed for our further discussion:

Theorem 1′. *Let $1 < p < \infty$, γ be the unit circumference, $c \in \gamma$. Then the operator*

$$T:\, f \to Tf, \quad (Tf)(\zeta_0) = (\zeta_0 - c)^{\frac{1}{p'}} \int_\gamma \frac{f(\zeta)}{(\zeta - c)^{\frac{1}{p'}} \ln|\zeta - c|(\zeta - \zeta_0)}\,d\zeta$$

is continuous in $L^p(\gamma)$.

2^0. Let Γ be a simple piecewise Lyapunov closed curve bounding the finite domain D. It is assumed that $\Gamma = \cup_{k=1}^n \Gamma_k$, where the non-intersecting Lyapunov arcs Γ_k meet at the points C so as to form, with respect to D, the inner angles

$$\varphi_k = \nu_k\pi, \quad 0 < \nu_k \le 2, \quad k = \overline{1,n}. \tag{15}$$

We shall investigate the following Dirichlet problem: In the domain D define a function u by the conditions

$$\left.\begin{array}{c}\Delta u=0, \quad u=\operatorname{Re}\Phi, \quad \Phi\in E^p(D)\\ u^+(t)=f(t), \quad f\in L^p(\Gamma)\end{array}\right\}. \tag{16}$$

It will be sufficient for us to define a function $\Phi \in E^p(D)$ whose angular boundary values $\Phi^*(t)$ satisfy the condition

$$\operatorname{Re}[\Phi^+(t)]=f(t). \tag{17}$$

The latter condition is the particular case of problem (1).

The function $\Phi(z(w))$ is regular in the unit circle. In this case, $\Phi(z)\in E^p(D)$ iff the function $\Psi(w)=\sqrt[p]{z'(w)}\Phi(z(w))$ belongs to the Hardy class H^p (see, e.g., [23], Ch. IX). Hence the solution of problem (17) in the class $E^p(D)$ is equivalent to the solution of the boundary value problem

$$\operatorname{Re}\left[\frac{1}{\sqrt[p]{z'(\zeta)}}\,\Psi^+(\zeta)\right]=f(z(\zeta)), \quad |\zeta|=1 \tag{18}$$

in the class H^p.

Following [4], [5], we introduce the function

$$\Omega(w)=\begin{cases}\Psi(w), & |w|<1\\ \overline{\Psi\Big(\dfrac{1}{\overline{w}}\Big)}, & |w|>1\end{cases}. \tag{19}$$

It is not difficult to show that (see, e.g., [8], [19]) in a domain $|w|>1$ the function $\Omega(w)-\Omega(\infty)$ is representable by a Cauchy integral. Therefore in a complex plane cut along the circumference $\gamma=\{\zeta:\ |\zeta|=1\}$ the function $\Omega(w)$ is representable by a Cauchy type integral with the constant principal part.

Let $p\geq 1$ and n be an integer ≥ 0. We set

$$\mathcal{K}_{p,n}=\Big\{\Phi:\ \exists q,\ q(w)=a_0+a_1e+\cdots+a_nw^n,\ \varphi\in L^p(\gamma),$$
$$\Phi(w)=\frac{1}{2\pi i}\int\limits_{\gamma}\frac{\varphi(\zeta)}{\zeta-w}\,d\zeta+q(w)\Big\}. \tag{20}$$

Denote $\widetilde{\mathcal{K}}_p=\mathcal{K}_{p,0}$ and

$$\mathcal{K}_p=\Big\{\Phi:\ \exists\varphi\in L^p(\gamma),\ \Phi(w)=\frac{1}{2\pi i}\int\limits_{\gamma}\frac{\varphi(\zeta)}{\zeta-w}\,d\zeta\Big\}. \tag{21}$$

Thus, if $\Psi \in H^p$ is a solution of problem (18), then the function Ω defined by (19) belongs to $\widetilde{\mathcal{K}}_p$. It is not difficult to find that (see, e.g., [5], §41)

$$\Omega^+(\zeta) = -\frac{\sqrt[p]{z'(\zeta)}}{\overline{\sqrt[p]{z'(\zeta)}}}\,\Omega^-(\zeta) + g(\zeta), \tag{22}$$

where $g(\zeta) = 2f(z(\zeta)\sqrt[p]{z'(\zeta)}$. We therefore conclude that if $\Psi \in H^p$ is a solution of problem (18), then $\Omega \in \widetilde{\mathcal{K}}_p$ is a solution of the boundary value problem (22). The converse statement is not true. For the restriction of $\Omega(w)$ onto the unit circle U to give a solution of problem (18), it is necessary and sufficient that the equality

$$\Omega_*(w) = \Omega(w), \tag{23}$$

where $\Omega_*(w) = \overline{\Omega(\frac{1}{\overline{w}})}$ (see [5], Ch. 5), be fulfilled for any $|w| \neq 1$.

We have come to the problem: Define solutions $\Omega \in \widetilde{\mathcal{K}}_p$ of problem (22) satisfying the additional condition (23). If Ω is such a solution, then its restriction onto U will give the desired solution of problem (18), while the function

$$\Phi(z) = \big(\sqrt[p]{w'(z)}\big)^{-1}\Omega(w(z)), \quad z \in D, \tag{24}$$

will be a solution of the class $E^p(D)$ of our main problem (17). Also, if Ω is a solution of the boundary value problem (22), then the function

$$\frac{1}{2}\big(\Omega(w) + \Omega_*(w)\big) \tag{25}$$

is a solution of problem (22) satisfying condition (23).

We introduce the notation $G(\zeta) = -\sqrt[p]{z'(\zeta)}[\overline{\sqrt[p]{z'(\zeta)}}]^{-1}$. The boundary value problem

$$\Omega^+(\zeta) = G((\zeta)\Omega_0^-(\zeta) \tag{22_0}$$

will be called the homogeneous problem corresponding to the nonhomogeneous problem (22).

A function X defined on the set $|w| \neq 1$ will be called a factor-function for G in the class $\mathcal{K}_p$ if it satisfies the conditions: (i) $X(z) \in \cup_n \mathcal{K}_{p,n}$, $[X(z)]^{-1} \in \cup_n \mathcal{K}_{p',n}$; (ii) $X^+(\zeta)[X^-(\zeta)]^{-1} = G(\zeta)$; (iii) the operator

$$g \to Tg, \quad (Tg)(\zeta_0) = \frac{X^+(\zeta_0)}{2\pi i}\int\limits_{\gamma} \frac{g(\zeta)}{X^+(\zeta)(\zeta-\zeta_0)}\,d\zeta \tag{26}$$

is continuous in $L^p(\gamma)$.

3^0. First, it is assumed that Γ has an angular point C with the inner angle $\nu\pi$, $0<\nu\le 2\pi$. We shall consider the following cases: (1) $0<\nu<p$; (2) $p<\nu<2$; (3) $\nu=p$; (4) $\nu=2$.

4^0. The Case $0<\nu<p$. Consider the function

$$X(w)=\begin{cases} -\sqrt[p]{z'(w)}, & |w|<1\\ \overline{\sqrt[p]{z'\Big(\dfrac{1}{\overline{w}}\Big)}}, & |w|>1\end{cases}. \tag{27}$$

In the case under consideration we have

$$-\frac{1}{p}<\frac{\nu-1}{p}<\frac{1}{p'}\,. \tag{28}$$

By virtue of (2) it is easy to establish that (see, e.g., [8]) X is a factor-function for G in $\mathcal{K}_p$ and all solutions of problem (22) of the class $\widetilde{\mathcal{K}}_p$ are given by the formula

$$\Omega(w)=\frac{X(w)}{2\pi i}\int\limits_{\gamma}\frac{g(\zeta)}{X^+(\zeta)(\zeta-w)}\,d\zeta+\alpha X(w), \tag{29}$$

where α is an arbitrary constant.

Let us turn to condition (23). A general solution of problem (22_0) is given by the equality $\Omega_0(w)=\alpha X(w)$. But

$$(\alpha X)_*(w)=\begin{cases} \overline{\alpha}\sqrt[p]{z'(w)}, & |w|<1\\ -\overline{\alpha}\,\overline{\sqrt[p]{z'\Big(\dfrac{1}{\overline{w}}\Big)}}, & |w|>1\end{cases}, \tag{30}$$

while the equality $(\alpha X)_*=\alpha X$ implies $\overline{\alpha}=-\alpha$. Thus we have $\operatorname{Re}\alpha=0$. Hence it follows that all solutions of problem (22_0) have the form $\alpha\sqrt[p]{z'(w)}$, $\operatorname{Re}\alpha=0$ and by equality (24) we conclude that for $f=0$ problem (16) has only a trivial solution. Moreover, assuming additionally that $\alpha=0$ and using (25), from (29) we obtain the solution of (22) satisfying condition (23)

$$\begin{aligned}\Omega(w)=\frac{1}{2}\Bigg[\frac{X(w)}{2\pi i}\int\limits_{\gamma}\frac{g(\zeta)}{X^+(\zeta)(\zeta-w)}\,d\zeta\,+\\ +\Bigg(\overline{\frac{X(\frac{1}{\overline{w}})}{2\pi i}}\Bigg)\overline{\int\limits_{\gamma}\frac{g(\zeta)}{X^+(\zeta)(\zeta-\frac{1}{\overline{w}})}\,d\zeta}\Bigg].\end{aligned} \tag{31}$$

Since $X(w(z)) = -\frac{1}{\sqrt[p]{w'(z)}}$, $X(1/\overline{w'(z)}) = 1/\overline{\sqrt[p]{w'(z)}}$, from (24) and (31) we have the equality

$$u(z) = \mathrm{Re}\left[\frac{1}{2\pi i}\int\limits_{\gamma}\frac{f(z(\zeta))}{\zeta - w(z)}\,d\zeta + \frac{w(z)}{2\pi i}\int\limits_{\gamma}\frac{\overline{f(z(\zeta))}}{\zeta}\,\frac{d\zeta}{\zeta - w(z)}\right]. \tag{32}$$

Eventually, we find that for $0 < \nu < p$ problem (16) is uniquely solvable for any $f \in L^p(\Gamma)$ and its solution is given by (32).

5^0. The Case $p < \nu < 2$. This time we shall investigate the function

$$\widetilde{X}(w) = \begin{cases} -\dfrac{\sqrt[p]{z'(w)}}{w - c}, & |w| < 1 \\ \dfrac{\overline{\sqrt[p]{z'(\frac{1}{\overline{w}})}}}{w - c}, & |w| > 1 \end{cases} . \tag{$\widetilde{27}$}$$

By virtue of (2) we have $\widetilde{X} = O((w-c)^{\frac{\nu - p - 1}{p}})$ in the neighborhood of the point c. Since $-\frac{1}{p} < \frac{\nu-p-1}{p} < \frac{1}{p'}$ (this is equivalent to the inequality $p < \nu < 2p$ which we have in the case under consideration), only the function

$$\Omega(w) = \frac{\widetilde{X}(w)}{2\pi i}\int\limits_{\gamma}\frac{g(\zeta)}{\zeta - w}\,d\zeta + (\alpha w + \beta)\widetilde{X}(\beta)$$

can be a solution of problem (22), and the function $\Omega_0(w) = (\alpha w + \beta)\widetilde{X}(w)$ a solution of problem (22_0). Let us see for which α and β condition (23) is fulfilled. We must have

$$-\Big(\overline{\alpha}\frac{1}{w} + \overline{\beta}\Big)\frac{\sqrt[p]{z'(w)}}{\frac{1}{w} - \overline{c}} = \frac{(\alpha w + \beta)\sqrt[p]{z'(w)}}{w - c} .$$

Since $\overline{c} = c^{-1}$, we obtain

$$-\Big(\frac{\overline{\alpha}}{w} + \overline{\beta}\Big)\frac{wc}{c - w} = \frac{\alpha w + \beta}{w - c} .$$

Hence $\overline{\beta}c - \alpha = 0$, $\overline{\alpha}c - \beta = 0$. This gives us $\beta = \overline{\alpha}c$. For arbitrarily chosen β we obtain $\alpha = \overline{\beta}c$. Thus the function

$$u_0(z(w)) = \mathrm{Re}\left[\frac{1}{\sqrt[p]{z'(w)}}\,\frac{w\overline{\beta}c + \beta}{w - c}\,\sqrt[p]{z'(w)}\right] = \mathrm{Re}\left[\frac{w\overline{\beta}c + \beta}{w - c}\right]$$

will be a solution of the homogeneous Dirichlet problem in the circle.

But if $w = re^{i\theta}$, $c = c_1 + ic_2 = e^{i\theta_c}$, $\beta = \lambda + i\mu$, then

$$\frac{\overline{\beta}cw + \beta}{w - c} = \frac{(\overline{\beta}cw + \beta)(\overline{w} - \overline{c})}{|w - c|^2} = \frac{(\overline{\beta}cr^2 - \beta\overline{c}) - (\overline{\beta}w - \beta\overline{w})}{|w - c|^2}$$

and, keeping in mind that $\operatorname{Re}[\overline{\beta}w - \beta\overline{w}] = 0$ and assuming $\overline{\beta}c = d + ie$, we obtain

$$\operatorname{Re}\frac{\overline{\beta}cw+\beta}{w-c} = \operatorname{Re}\frac{\overline{\beta}cr^2-\overline{\overline{\beta}c}}{|w-c|^2} = \operatorname{Re}\frac{(d+ie)r^2-(d-ie)}{|w-c|^2} = \frac{d(r^2-1)}{|w-c|}$$

with $d = \operatorname{Re}\overline{\beta}c = \operatorname{Re}[(\lambda - i\mu)(c_1 + ic_2)] = \lambda c_1 + \mu c_2$.

Thus

$$u_0(z(re^{i\theta})) = \frac{(\lambda c_1+\mu c_2)(1-r^2)}{1+r^2-2r\cos(\theta-\theta_c)}\,,$$

where λ, μ are arbitrary real constants. Clearly, $\lambda c_1 + \mu c_2$ runs through the set of all real numbers so that

$$u_0(z(re^{i\theta})) = M\operatorname{Re}\frac{c+w}{c-w}\,,$$

where M is an arbitrary real constant. For $f = 0$ a general solution of problem (16) is given by the equality

$$u_0(z) = M\operatorname{Re}\frac{c+w(z)}{c-w(z)}\,. \tag{33}$$

A particular solution of the nonhomogeneous problem is obtained after replacing in it $X(w)$ by $\widetilde{X}(w)$. Finally, we obtain a particular solution of the form

$$\widetilde{u}(z) = \operatorname{Re}\bigg[\frac{1}{w(z)-c}\bigg(\frac{1}{2\pi i}\int\limits_\gamma \frac{f(z(\zeta))(\zeta-c)}{\zeta-w}\,d\zeta - \\ -\frac{cw^2}{2\pi i}\int\limits_\gamma \frac{\overline{f(z(\zeta))}(\overline{\zeta}-\overline{c})}{\zeta(\zeta-w)}\,d\zeta\bigg)\bigg]. \tag{34}$$

Thus for $p < \nu < 2$ the Dirichlet problem (16) is solvable for any $f \in L^p(\Gamma)$ and has an infinite number of solutions given by the equality $u(z) = u_0(z) + \widetilde{u}(z)$, where u_0 and $\widetilde{u}$ are defined by (33) and (34), respectively.

6^0. The Case $\nu = p$. Let X be the function defined by (27). Then by (2) we have $X(w) = O((w-c)^{\frac{1}{p'}})$ (including the case $\nu = 2 = p$). We shall first consider the homogeneous problem. The function $F(w) = \Omega(w)[X(w)]^{-1}$ satisfies the condition $F^+ = F^-$ and, in the domains $|w| < 1$ and $|w| > 1$, belongs to the set $\cap_{\delta<1}H^\delta$, where H^δ is the Hardy class (it is assumed that $F \in H^\delta$ in the domain $|w| > 1$ if $F(\frac{1}{w}) \in H^\delta$ in the domain $|w| < 1$). We shall show that $F(w)$ is regular in the whole plane except the point c. Let ζ be an arbitrary point on γ, different from c. Let us take a pair of points ζ_1 and ζ_2 from the different sides of ζ and consider the domain S_ζ^+ which is a sector of the circle U bounded by the radii passing through ζ_1 and ζ_2

and by the circumference arc $\gamma(\zeta_1, \zeta_2)$ containing ζ. Since for the function $\Omega \in H^p$ we have by the Feyer–Riesz theorem (see, e.g., [24], p. 46)

$$\int_0^1 |\Omega(re^{i\theta})|^p dr \le M \int_0^{2\pi} |\Omega(e^{i\psi})|^p d\psi,$$

it follows that $\Omega \in E^p(S_\zeta^+)$. Hence we easily find that $\Omega \in E^p(S_\zeta^-)$, where S_ζ^- is the domain bounded by the arc $\gamma(\zeta_1, \zeta_2)$, by the continuation of the radii passing through ζ_1 and ζ_2, and by the circumference arc $|w| = 1 + \eta$, $\eta > 0$. For $\zeta \ne c$ points ζ_1 and ζ_2 can be chosen so that c would not lie on $\gamma_1(\zeta_1, \zeta_2)$. Therefore the function $[X(w)]^{-1}$ is bounded in the domains $S_\zeta^\pm$ and hence in these domains the function $F(w)$ belongs to the Smirnov class and is represented by the Cauchy integral

$$\begin{aligned} F(w) &= \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_{\gamma_1} \dfrac{F(\zeta)}{\zeta - w}\, d\zeta, & w \in S_\zeta^+ \\ 0, & w \in S_\zeta^- \end{cases}, \\ F(w) &= \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_{\gamma_2} \dfrac{F(\zeta)}{\zeta - w}\, d\zeta, & w \in S_\zeta^- \\ 0, & w \in S_\zeta^+ \end{cases}, \end{aligned} \tag{35}$$

where γ_1 and γ_2 are the boundaries of the domains S_ζ^+ and S_ζ^-, respectively. Within these boundaries there lies the arc $\gamma(\zeta_1, \zeta_2)$ on which the integration on γ_1 and γ_2 is performed in the opposite directions and on which $F^+(\zeta) = F^-(\zeta)$. Hence the function

$$\widetilde{F}(w) = \frac{1}{2\pi i} \int_{\gamma_1 \cup \gamma_2} \frac{F(\zeta)}{\zeta - w}\, d\zeta = \frac{1}{2\pi i} \int_{\gamma_3} \frac{F(\zeta)}{\zeta - w}\, d\zeta,$$
$$\gamma_3 = (\gamma_1 \cup \gamma_2) \backslash \gamma(\zeta_1, \zeta_2),$$

on the one hand, coincides in $S_\zeta^+ \cup S_\zeta^-$ with the function $F(w)$ (by virtue of (35)) and, on the other hand, is analytic inside γ_3 and therefore in the neighborhood of the point ζ. Thus F is regular almost everywhere except the point c at which it may have only a first order pole, since otherwise it could not belong to the class H^δ, $\delta \in (0, 1)$. Therefore $F(w) = \alpha + \frac{\beta}{w-c}$ and only the function $\Omega_0(w) = \alpha X(w) + \beta[w - c]^{-1} X(w)$ can be a solution of problem (22_0). But since in the neighborhood of c we have $\Omega_0(w) = O((w-c)^{-\frac{1}{p}})$, Ω_0 will belong to $\widetilde{\mathcal{K}}_p$ iff $\beta = 0$, i.e., for $\nu = p$ the homogeneous problem has only a solution of the form $\Omega_0(w) = \alpha X(w)$. For this solution to satisfy condition (23) we shall show, like we did in Subsection 4^0, that $\alpha = 0$.

Returning to the nonhomogeneous problem, note that there are functions in $L^p(\gamma)$ for which it is not solvable (see Subsection 8^0 below),

We shall show now how using Theorem $1'$ one can construct a solution of problem (22)–(23) for a wide subclass of functions $f \in L^p(\Gamma)$.

Assuming that

$$g(\zeta)\ln(\zeta - c) \in L^p(\gamma), \tag{36}$$

we obtain $g(\zeta) = \frac{g_1(\zeta)}{\ln(\zeta-c)}$, where $g_1 \in L^p(\gamma)$ and therefore by Theorem $1'$ the function

$$\frac{(\zeta_0 - c)^{\frac{1}{p'}}}{2\pi i}\int\limits_{\gamma} \frac{1}{(\zeta - c)^{\frac{1}{p'}}\ln(\zeta - c)}\,\frac{g_1(\zeta)}{\zeta - \zeta_0}\,d\zeta = (\zeta_0 - c)^{\frac{1}{p'}}F(\zeta)$$

belongs to $L^p(\gamma)$. Hence due to the inclusion $F \in H^\delta$ and the relation $X(w) = O((w - c)^{\frac{1}{p'}})$ it follows that the function

$$\Omega(w) = \frac{X(w)}{2\pi i}\int\limits_{\gamma} \frac{g(\zeta)}{X^+(\zeta)(\zeta - w)}\,d\zeta$$

belongs to $\widetilde{\mathcal{K}}_p$ (we have used here Smirnov's theorem: if $\Phi \in H^{p_1}$ has boundary values $\Phi^+ \in L^{p_2}$, $p_2 > p_1$, then $\Phi \in H^{p_2}$). Condition (36) is equivalent to the condition $f(t)\ln|\zeta(t)-\zeta(c)| \in L^p(\Gamma)$, since $\arg(\zeta(t)-\zeta(c))$ is the bounded function. But in the case under consideration

$$\zeta(t) - \zeta(c) = (t - C)^{\frac{1}{p'}}\zeta_0(t), \quad 0 < m < |\zeta_0(t)| < M$$

(see [21] or [7]). Therefore $\ln|\zeta(t) - \zeta(c)| < \operatorname{const}\ln|t - C|$ and (36) is equivalent to the condition

$$f(t)\ln|t - C| \in L^p(\Gamma). \tag{37}$$

If this condition is fulfilled, then the nonhomogeneous problem (16) is solvable and its solution is given by equality (32).

7^0. The Case $\nu = 2$. Equality (2) implies $z'(w) = (w - c)z_0(w)$, $0 < m < |z_0(w)| < M$. For $p < 2$ we shall establish, like we did in Subsection 5^0, that $\Omega_0(w) = (\alpha w + \beta)\widetilde{X}(w)$, where $\widetilde{X}$ is given by equality $(\widetilde{27})$. Using corresponding arguments we again find that the homogeneous Dirichlet problem has an infinite number of solutions given by (33), while the function $\widetilde{u}$ given by (34) is a particular solution of the nonhomogeneous problem.

For $p > 2$ we have $2 = \nu < p$ (see Subsection 4^0). If $p = 2$, then $\nu = p = 2$ (see Subsection 6^0).

To summarize, we have: for $\nu = 2$ the homogeneous Dirichlet problem has only a trivial solution if $p \geq 2$. If however $p < 2$, then it has an infinite set of solutions given by equality (33). The nonhomogeneous problem is solvable for any $f \in L^p(\Gamma)$ if $p \neq 2$. When $p = 2$, for this problem to be solvable it is sufficient that condition (37) be fulfilled. In that case, a particular solution is constructed by (32) if $p \geq 2$, and by (34) if $p < 2$.

8^0. An Example of the Function $f_0 \in L^p(\Gamma)$ for Which the Dirichlet Problem (16) Has No Solution if Γ Has an Angular Point with an Angle $p\pi$.

To construct such a function we need to show first that a solution of the Dirichlet problem, if it exists, has the definite form. We rewrite condition (22) as follows:

$$(\zeta - c)\Omega^+(\zeta) = (\zeta - c)G(\zeta)\Omega^-(\zeta) + g(\zeta)(\zeta - c).$$

It can be easily verified that if X is defined by equality (27), then the function $F(w) = (w - c)[X(w)]^{-1}$ belongs to the class $\mathcal{K}_{p,1}$. Therefore possible solutions of the class $\widetilde{\mathcal{K}}_p$ of problem (22) lie in the set of functions

$$\widetilde{\Omega}(w) = \frac{X(w)}{w - c}\,\frac{1}{2\pi i}\int\limits_{\gamma} \frac{g(\zeta)(\zeta - c)}{X^+(\zeta)(\zeta - w)}\,d\zeta + \frac{\alpha w + \beta}{w - c}\,X(w), \tag{38}$$

where α and β are arbitrary constants. Since $\frac{\alpha w+\beta}{w-c}\,X(w) = \alpha + \frac{\beta+\alpha c}{w-c}\,X(w)$ and the function $\alpha X(w)$ belongs to $\widetilde{\mathcal{K}}_p$, we conclude that $\widetilde{\Omega}(w)$ will be a solution of the class $\widetilde{\mathcal{K}}_p$ only if the function

$$\Omega(w) = \frac{X(w)}{w - c}\,\frac{1}{2\pi i}\int\limits_{\gamma} \frac{g(\zeta)(\zeta - c)}{X^+(\zeta)(\zeta - w)}\,d\zeta + \frac{DX(w)}{w - c}\,, \tag{39}$$

where D is some constant, belongs to this class. But $\Omega \in H^\delta$, $\delta < 1$, and therefore Ω will belong to the class $\widetilde{\mathcal{K}}_p$ if the function $\Omega^+(\zeta)$ belongs to $L^p(\Gamma)$. For this it is necessary and sufficient that the function

$$\widetilde{g}(\zeta_0) = \frac{X^+(\zeta_0)}{\zeta_0 - c}\int\limits_{\gamma} \frac{g(\zeta)(\zeta - c)}{X^+(\zeta)(\zeta - \zeta_0)}\,d\zeta + \frac{DX^+(\zeta_0)}{\zeta_0 - c} \tag{40}$$

belong to the class $L^p(\gamma)$.

Let us now find a function $g_0 \in L^p(\gamma)$ for which $\widetilde{g}_0 \notin L^p(\gamma)$ no matter what the constant D is. We set $c = 1$ and assume that

$$g_0(\zeta) = g_0(e^{i\theta}) = \begin{cases} \dfrac{m_n X^+(\zeta)}{\zeta - 1}\,, & \theta_n \leq \theta \leq \theta_{n+1} \\ 0, & \theta \in (1, 2\pi) \end{cases}\,, \tag{41}$$

where $\theta_n = \frac{1}{n}$, $m_n = \frac{1}{\ln(n+1)}$.

Since in the case under consideration $|\frac{X^+}{\zeta-1}| \le \frac{M}{|\zeta-1|^{\frac{1}{p}}}$, we have

$$\int\limits_{\gamma} |g_0|^p |d\zeta| \le \sum_{n=1}^{\infty} m_n^p \int\limits_{\theta_n}^{\theta_{n+1}} \frac{d\theta}{|\zeta-1|} \le$$
$$\le \sum_{n=1}^{\infty} \frac{m_n^p}{\sqrt{(1-\cos\frac{1}{n})^2+\sin^2\frac{1}{n}}}\Big(\frac{1}{n}-\frac{1}{n+1}\Big) \le$$
$$\le \operatorname{const} \sum_{n=1}^{\infty} \frac{1}{n\ln^p(n+1)} < \infty.$$

Thus $g_0 \in L^p(\gamma)$. We shall show that $\widetilde{g}_0 \notin L^p(\gamma)$ no matter what the constant D is. Since

$$\widetilde{g}_0(\zeta) = \frac{X(\zeta_0)}{(\zeta_0-c)}\Big[\int\limits_{\gamma} \frac{g_0(\zeta)(\zeta-c)}{X^+(\zeta)(\zeta-\zeta_0)}\,d\zeta + D\Big]$$

and, by (2) $\frac{X(\zeta)}{\zeta_0-c} = (\zeta_0-c)^{-\frac{1}{p}} z_0(\zeta_0)$, $0<m<|z_0|<M$, it is sufficient to prove the equality

$$\lim_{\zeta_0\to 1} \int\limits_{\gamma} \frac{g_0(\zeta)(\zeta-c)}{X^+(\zeta)(\zeta-\zeta_0)}\,d\zeta = \infty. \tag{42}$$

Given a number $K>0$, we choose N such that $\sum_{n=1}^{N} \frac{1}{n\ln(n+1)} > K$ and assume that $\zeta_0 = e^{i\theta}$, $\theta \in (2\pi - \frac{1}{N}, 2\pi)$. We have

$$\Big|\int\limits_{\gamma} \frac{g_0(\zeta)(\zeta-c)}{X^+(\zeta)(\zeta-\zeta_0)}\,d\zeta\Big| \ge \Big|\sum_{n=1}^{\infty} m_n \ln\Big|\frac{\zeta_n-\zeta_0}{\zeta_{n+1}-\zeta_0}\Big|\Big|. \tag{43}$$

But

$$\ln\Big|\frac{\zeta_n-\zeta_0}{\zeta_{n+1}-\zeta_0}\Big| = \ln\Big|\frac{\sin\frac{1}{2}(\frac{1}{n}-\theta)}{\sin\frac{1}{2}(\frac{1}{n+1}-\theta)}\Big|$$

so that assuming that $\theta = 2\pi - \alpha$ we obtain

$$\ln\Big|\frac{\zeta_n-\zeta_0}{\zeta_{n+1}-\zeta_0}\Big| = \ln\Big|\frac{\sin(\frac{\alpha}{2}+\frac{1}{2n})}{\sin(\frac{\alpha}{2}+\frac{1}{2(n+1)})}\Big| > 0, \quad \alpha \in \Big(0, \frac{1}{N}\Big).$$

Now (43) implies

$$|\widetilde{g}_0(\zeta)| \ge \sum_{n=1}^{\infty} m_n \ln\Big|\frac{\sin(\frac{\alpha}{2}+\frac{1}{2n})}{\sin(\frac{\alpha}{2}+\frac{1}{2(n+1)})}\Big| =$$

$$= \sum_{n=1}^{\infty} m_n \ln\left(1 + \frac{2\cos(\frac{\alpha}{2} + \frac{1}{4n(n+1)})\sin\frac{1}{4n(n+1)}}{\sin(\frac{\alpha}{2} + \frac{1}{2(n+1)})}\right) \geq$$

$$\geq \sum_{n=1}^{N} m_n \frac{2\cos(\frac{\alpha}{2} + \frac{1}{4n(n+1)})\sin\frac{1}{4n(n+1)}}{\sin(\frac{\alpha}{2} + \frac{1}{2(n+1)})} \geq m_0 K,$$

where $m_0 = \frac{2}{\pi}\cos\frac{2N+3}{4N(N+1)}$. Thus relation (42) is proved.

Clearly, the function $f_0(t) = g_0(w(t))[\sqrt[p]{w'(t)}]^{-1}$ belongs to $L^p(\Gamma)$ and the corresponding Dirichlet problem is not solvable for it.

9^0. Formulation of the Result on Problem (16) in the Case of One Angular Point. Based on the arguments of Subsections 4^0–8^0, we come to a conclusion that the following theorem is valid.

Theorem 2$'$. *Let D be a finite singly connected domain with a piecewise Lyapunov boundary, one angular point C with an inner angle $\nu\pi$, $0 < \nu \leq 2$. Then for the Dirichlet problem the following statements are true:*

If $0 < \nu < p$, then the problem is uniquely solvable and its solution is given by equality (32). *If $p < \nu \leq 2$, then it has an infinite number of solutions of the form*

$$u(z) = \widetilde{u}(z) + M\,\mathrm{Re}\,\frac{c + w(z)}{c - w(z)}\,, \quad c = w(C),$$

where $w(z)$ is the function mapping conformally the domain D onto the unit circle, M is an arbitrary constant, and $\widetilde{u}$ is defined by equality (34). *If $\nu = p$, then the fulfillment of the condition $f \in L^p(\Gamma)$ is not enough for the problem to be solvable. However, if the condition*

$$f(t)\ln|t - C| \in L^p(\Gamma) \tag{37}$$

is satisfied, it has a (*unique*) *solution given by equality* (34).

10^0. The Dirichlet Problem in a General Case. We shall formulate the result for a more general case which follows from Theorem 2$'$.

Theorem 2. *Let D be a domain bounded by a simple piecewise Lyapunov closed curve with angular points C_k, $k = \overline{1,n}$, at which the inner angle values are equal to $\nu_k\pi$, $0 < \nu_k \leq 2$. Note that there are n_1 angular points with values ν_k from the interval $(p, 2]$ (it is assumed that $(2, 2] = \{2\}$). In that case, all solutions of the homogeneous problem* (16) *are given by the equality*

$$u_0(z) = \sum_{\substack{k \\ \nu_k \in (p,2]}} M_k\,\mathrm{Re}\,\frac{c_k + w(z)}{c_k - w(z)}\,, \quad c_k = w(C_k), \tag{44}$$

where M_k are arbitrary real constants. Generally speaking, the fulfillment of the condition $f \in L^p(\Gamma)$ is not enough for a homogeneous problem to be solvable. However, if the condition

$$f(t)\ln \prod_{\substack{k \\ \nu_k=p}} (t-C_k) \in L^p(\Gamma) \tag{45}$$

is fulfilled, then the problem is solvable. In all cases in which a solution exists, it is given by the equality $u(z)=u_0(z)+\widetilde{u}(z)$, where $u_0(z)$ is defined by equality (44), *and*

$$\begin{aligned}\widetilde{u}(z)=\mathrm{Re}\Bigg[\bigg(\frac{1}{2\pi i}\int_\gamma \frac{f(z(\zeta))\rho(\zeta)}{\zeta-w(z)}\,d\zeta+{}\\ +(-1)^{n_1}\frac{w^{n_1+1}(z)\prod\limits_{\substack{k \\ \nu_k\in(p,2]}} c_k}{2\pi i}\int_\gamma \frac{\overline{f(z(\zeta))\rho(\zeta)}}{\zeta(\zeta-w(z))}\,d\zeta\bigg)\frac{1}{\rho(w(z))}\Bigg],\end{aligned} \tag{46}$$

where $\rho(w)=\prod\limits_{\substack{k \\ \nu_k\in(p,2]}}(w-c_k)$ and $\rho(w)\equiv 1$, when $\{\nu_k:\nu_k\in(p,2]\}=\varnothing$.

11^0. The Riemann–Hilbert Problem in a Domain whose Boundary Has One Angular Point. We make the same assumptions for Γ as in Subsection 3^0. It is assumed that the coefficients $a(t)$ and $b(t)$ are measurable and the function $G(t)=(a(t)-ib(t))(a(t)+ib(t))^{-1}$ belongs to the class $\widetilde{A}_p(\Gamma)$, i.e., (1) $\inf|G|>0$, $\sup|G|<\infty$; (2) for each point $t\in\Gamma$, except perhaps the points t_k, $k=\overline{1,m}$, there exists a neighborhood in which the values G lie in some sector with the vertex at the origin, whose angle is less than $2\pi[\max(p,p')]^{-1}$; (3) in unilateral neighborhoods of points t_k, G satisfies the Hölder condition, at points t_k there exist limits $G(t_k,\pm)$, and values of the angles δ_k between the vectors $G(t_k-)$ and $G(t_k+)$ are such that $\frac{2\pi}{p}<\delta_k\le\frac{2\pi}{p'}$ for $p>2$, $\frac{2\pi}{p'}\le\delta_k<\frac{2\pi}{p}$ for $1<p<2$ and $\delta_k\ne\pi$ for $p=2$.

For such a function we define, depending on p, the function $\Theta_p(t)=\arg_p G(t)$ and the integer number $\varkappa=\varkappa(p;G)=\frac{1}{2\pi}[\Theta_p]_\Gamma$ [13] ($[\varphi]_\Gamma$ denotes an increment of the function when t passes over the curve Γ).

Passing to the circle, we come to a problem of defining the function $\Omega\in\widetilde{\mathcal{K}}_p(\gamma)$ by the set of conditions

$$\left.\begin{aligned}\Omega^+(\zeta)=-\frac{\sqrt[p]{z'(\zeta)}}{\sqrt[p]{\overline{z'(\zeta)}}}\,\frac{A(\zeta)-iB(\zeta)}{A(\zeta)+iB(\zeta)}\,\Omega^-(\zeta)+g(\zeta),\ \ \zeta\in\Gamma,\\ \Omega_*(w)=\Omega(w),\quad |w|\ne 1,\end{aligned}\right\} \tag{47}$$

where $A(\zeta) = a(z(\zeta))$, $B(\zeta) = b(z(\zeta))$, $g(\zeta) = 2f(z(\zeta))\sqrt[p]{z'(\zeta)}[a(\zeta) + iB(\zeta)]^{-1}$.

The function $G_\gamma(\zeta) = G(z(\zeta))$ belongs to $\widetilde{A}_p(\gamma)$. We need only to check whether the Hölder property takes place in the unilateral neighborhoods of points $\tau_k = w(t_k)$. To this end, we recall that in our case the function $z(\zeta)$ satisfies the Hölder condition with the index $\min(1, \nu)$ [21]. By [13] $G_\gamma(\zeta)$ is factorized in $\mathcal{K}_p(\gamma)$ and its factor-function has the form

$$Y(w) = \begin{cases} \exp\left(\dfrac{1}{2\pi i}\displaystyle\int_\gamma \dfrac{\Theta_p(\zeta)}{\zeta - w}\, d\zeta\right), & |w| < 1 \\ w^{-\varkappa}\exp\left(\dfrac{1}{2\pi i}\displaystyle\int_\gamma \dfrac{\Theta_p(\zeta)}{\zeta - w}\, d\zeta\right), & |w| > 1 \end{cases}. \tag{48}$$

Assuming that Γ has one angular point $C \neq t_k$, $k = \overline{1, m}$, with an angle $\nu\pi$, $0 < \nu \leq 2$, we shall consider the cases: (i) $0 < \nu < p$; (ii) $p < \nu < 2$; (iii) $\nu = p$; (iv) $\nu = 2$.

(i) If $0 < \nu < p$ and the function X is given by (27), then the function

$$T(w) = AY(w)X(w), \tag{49}$$

where A is an arbitrary constant, will be the factor-function for $\widetilde{G}_\gamma(\zeta) = -\sqrt[p]{z'(\zeta)}[\overline{\sqrt[p]{z'(\zeta)}}]^{-1}G_\gamma(\zeta)$. By an appropriate choice of A [5] we can fulfill the equality $T_*(w) = w^\varkappa T(w)$ and hence make the following conclusion: if $\varkappa \geq 0$, then the homogeneous problem corresponding to problem (1) has an infinite number of solutions given by the equality

$$\Phi(z) = T(w(z))P_\varkappa(w(z))[\sqrt[p]{w'(\zeta)}]^{-1},$$

where $P_\varkappa(w) = a_0 + a_1 w + \cdots + a_\varkappa w^\varkappa$ is an arbitrary polynomial whose coefficients satisfy the conditions $c_i = \overline{c}_{\varkappa - i}$, $i = \overline{1, \varkappa}$. The nonhomogeneous problem is solved unconditionally. If, however, $\varkappa < 0$, then the homogeneous problem has only a trivial solution, while for the nonhomogeneous problem to be solvable it is necessary and sufficient that the conditions

$$\int_\Gamma w^k(t)\frac{f(t)w'(t)}{T^+(w(t))}\, dt = 0, \quad k = 0, 1, \ldots, |\varkappa| - 1, \tag{50}$$

be fulfilled.

(ii) $p < \nu < 2$. We set $\widetilde{T}(w) = AY(w)\widetilde{X}(w)$, where Y and $\widetilde{X}$ are given by equalities (48) and $(\widetilde{27})$, respectively. By an appropriate choice of A we can fulfill the equality

$$\widetilde{T}_*(w) = w^{\varkappa+1}\widetilde{T}(w), \quad |w| \neq 1.$$

The result of (i) therefore remains in force if we replace $\varkappa$ and T by $\varkappa+1$ and $\widetilde{T}$, respectively. We write the solvability conditions as

$$\int_\Gamma w^k(t)\frac{f(t)w'(t)}{\widetilde{T}^+(w(t))}\,dt=0,\quad k=0,1,\dots,|\varkappa|-2. \tag{51}$$

(iii) $\nu=p$. Applying a reasoning similar to that of Subsection 6^0 we find that only the function

$$\Omega_0(w)=Y(w)\Big[P_\varkappa(w)+\frac{D}{w-c}\Big]X(w)$$

can be a solution of the homogeneous problem. But for $\nu=p$ we have $X(w)=(w-c)^{\frac{1}{p'}}X_0(w)$, while the assumptions made for G_γ imply that $|Y(w)|\geq m_0>0$ in the neighborhood of the point c. Thus for the function Ω_0 to belong to the class $\widetilde{\mathcal{K}}_p(\gamma)$ we must set $D=0$. So we obtain the same result for the homogeneous problem as in the case (i). The nonhomogeneous problem is not always solvable. Let the condition (37) be fulfilled for f. Then the function

$$\Omega(w)=\frac{T(w)}{2\pi i}\int_\gamma\frac{g(\zeta)}{T^+(\zeta)}\,\frac{d\zeta}{\zeta-w} \tag{52}$$

will be a solution of problem (47) from the class $\widetilde{\mathcal{K}}_p(\gamma)$ iff conditions (50) are fulfilled.

(iv) $\nu=2$. We have a different situation for $p\neq2$ and $p=2$. When $p<2$, we can show, as above, that the function $\Omega(w)[Y(w)X(w)]^{-1}$ is analytically continuable onto the entire plane except the point c and thus come to the conclusion that the general solution of the homogeneous problem has the form

$$\Omega_0(w)=Y(w)X(w)(w-c)^{-1}P_{\varkappa+1}(w) \tag{53}$$

($P_{\varkappa+1}(w)\equiv0$ for $\varkappa+1<0$). The nonhomogeneous problem is solved unconditionally for $\varkappa\geq-1$. If, however, $\varkappa\leq-2$, then for this problem to be uniquely solvable it is necessary and sufficient that conditions (51) be fulfilled. In all cases one can easily write all solutions.

If $p>2$, then $2=\nu<p$ and this case is considered in (i); if $p=2$, the $\nu=2=p$ and this case is treated in (iii).

Now we are able to summarize the results for the general case.

12^0. The Riemann–Hilbert Problem in the General Case. We introduce the notation

$$\rho(w)=\prod_{\substack{k\\ \nu_k\in(p,2]}}(w-c_k) \tag{54}$$

and assume that $(2,2]=\{2\}$. If there are no points C_k for which $\nu_k \in (p,2]$, then it is assumed that $\rho(w)=1$.

The results of Subsection 11^0 give rise to

Theorem 3. *Let the Riemann–Hilbert problem* (1) *from the class* $E^p(D)$ *be considered in a finite simply connected domain* D *bounded by the curve* Γ. *It is assumed that :*

(i) Γ *is a simple piecewise Lyapunov curve with angular points* C_k, $k=\overline{1,n}$, *with the inner angles* $\nu_k\pi$, $0<\nu_k\le 2$; n_1 *is the number of points at which* $\nu_k\in(p,2]$;

(ii) $G(t)=(a(t)-ib(t))(a(t)+ib(t))^{-1}$ *belongs to* $\widetilde{A}_p$, $C_k\ne t_k$, *and* $\varkappa(p,G)=\frac{1}{2\pi}[\arg_p G(t)]_\Gamma$ *(see Subsection* 11^0*);*

(iii)

$$\varkappa=\varkappa(p,G)+n_1,$$

$$T(w)=\begin{cases}-\dfrac{Y(w)\sqrt[p]{z'(w)}}{\rho(w)}\,, & |w|<1\\ \dfrac{\overline{Y(\frac{1}{\overline{w}})\sqrt[p]{z'(\frac{1}{\overline{w}})}}}{\rho(w)}\,, & |w|>1\end{cases}. \tag{55}$$

Then:

(1) *All solutions of the homogeneous problem are given by the equality*

$$\Phi_0(z)=AT(w(z))P_\varkappa(w(z))\big[\sqrt[p]{w'(z)}\big]^{-1}, \tag{56}$$

where, for $\varkappa\ge 0$, $P_\varkappa(w)=a_0+a_1w+\dots+a_\varkappa w^\varkappa$ *is an arbitrary polynomial whose coefficients are related by the conditions* $c_i=\overline{c}_{\varkappa-i}$, $i=\overline{1,n}$, $P_\varkappa(w)\equiv 0$ *for* $\varkappa<0$, *and the constant* A *is uniquely defined by the equality*

$$(AT)_*(w)=w^\varkappa AT(w).$$

(2) *For the nonhomogeneous problem we have:*

If $\varkappa\ge 0$ *and*

$$f(t)\ln\Big|\prod_{\substack{k\\ \nu_k=p}}(t-C_k)\Big|\in L^p(\Gamma), \tag{57}$$

then the problem is solvable.

If $\varkappa<0$ *and condition* (57) *is fulfilled, then for problem* (1) *to be solvable it is necessary and sufficient that*

$$\int_\Gamma w^k(t)\frac{f(t)w'(t)}{T^+(w(t))}\,dt=0,\quad k=0,1,\dots,|\varkappa|-1. \tag{58}$$

In all these cases the solution is representable by the formula

$$\Phi(z)=\widetilde{\Phi}(z)+\Phi_0(z),$$

where Φ_0 is given by (56), and

$$\widetilde{\Phi}(z)=\frac{T(w(z))}{2\pi i}\int\limits_{\gamma}\frac{f(z(\zeta))}{T(z(\zeta))}\,\frac{\sqrt[p]{z'(\zeta)}}{\zeta-w(\zeta)}\,d\zeta+ \\ +\left(\frac{\overline{T(1/\overline{w(\zeta)})}}{2\pi i}\right)\overline{\int\limits_{\gamma}\frac{f(z(\zeta))\sqrt[p]{z'(\zeta)}(\overline{w(z)}-\overline{\zeta})}{(\overline{w(z)}-\overline{\zeta})T(z(\zeta))\overline{\zeta}\,\overline{w(\zeta)}}\,d\zeta}. \tag{59}$$

13^0. The Riemann–Hilbert Problem in the Smirnov Weighted Class. Let β be a real number, $\tau\in\Gamma$ and $r(z)=(z-\tau)^{\beta}$, $z\in D$. An analytical function $\Phi(z)$ in the domain D will be said to belong to the class $E^p(D;r)$ if $\Phi(z)r(z)\in E^p(D)$. When one considers problem (1) in this class, it is of special interest to investigate the case where τ coincides with the angular point C (it is assumed that there is only one such point). On rewriting the condition $\mathrm{Re}[(A(\zeta)+iB(\zeta))\Phi^+(z(\zeta))]=f(z(\zeta))$ as

$$\mathrm{Re}\Big[\frac{(A(\zeta)+iB(\zeta))\sqrt[p]{z'(\zeta)r(z(\zeta))}\,\Phi^+(z(\zeta))}{\sqrt[p]{z'(\zeta)r(z(\zeta))}}\Big]=f(z(\zeta)), \tag{60}$$

we reduce problem (1) to a problem of form (22) in which $\sqrt[p]{z'(\zeta)}$ is replaced by $\sqrt[p]{z'(\zeta)r(z(\zeta))}$. But $r(z(\zeta))=(r(\zeta)-z(c))^{\beta}$ and, since $z(\zeta)$ satisfies the Hölder condition with the index $\widetilde{\nu}=\min(1,\nu)$, we have $\sqrt[p]{z'(\zeta)r(z(\zeta))}=(\zeta-c)^{\frac{\nu-1}{p}+\frac{\widetilde{\nu}\beta}{p}}r_0(\zeta)$, $r_0\neq 0$, in the neighborhood of c. Now, by setting $\nu_\beta=\nu+\widetilde{\nu}\beta$ and assuming that $\nu_\beta\in(0,2]$ we obtain analogues of Theorems 2 and 3 in which ν is replaced by ν_β. We shall not go into details and write only admissible values for β

$$\max(-1,-\nu)<\beta\le\min\Big(2-\nu,\frac{2-\nu}{\nu}\Big). \tag{61}$$

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