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ON THE ABSOLUTE CONVERGENCE OF FOURIER SERIES

T. KARCHAVA

Abstract. The necessary and sufficient conditions of the absolute convergence of a trigonometric Fourier series are established for continuous 2π-periodic functions which in $[0,2\pi]$ have a finite number of intervals of convexity, and whose nth Fourier coefficients are $O(\omega(1/n;f)/n)$, where $\omega(\delta;f)$ is the continuity modulus of the function f.

Let ω be an arbitrary modulus of continuity, i.e., a nondecreasing function continuous on $[0,1]$, $\omega(0)=0$ and $\omega(\delta_1+\delta_2)\leq\omega(\delta_1)+\omega(\delta_2)$. As usual, denote by H^ω the class of all functions f continuous on $[0,2\pi]$ for which

$$\omega(\delta;f)=\sup_{|x_1-x_2|\leq\delta}|f(x_1)-f(x_2)|=O(\omega(\delta)),\quad 0\leq\delta\leq 1$$

(see, for instance, [5, Ch. 3, pp. 150, 157]).

Let M be the class of all continuous 2π-periodic functions f for which there exists a partitioning of the segment $[0,2\pi]$ by the points $0=x_1(f)<\cdots<x_{m+1}(f)=2\pi$ such that f is convex or concave on each segment $[x_k(f),x_{x+1}(f)]$, $k=1,\dots m$.

The Fourier coefficients of a function f with respect to the trigonometric system will be denoted by $a_n=a_n(f)$, $b_n=b_n(f)$.

Problems pertaining to the absolute convergence of Fourier series have been studied quite completely (see, for instance, the monographs of Bari [2, Ch. 9], Zygmund [3, Ch. 6], Kahane [4, Ch. 2], and the survey by Guter and Ulyanov [5, p. 391]).

This paper deals with some problems of the absolute convergence of trigonometric Fourier series of a function from the class M.

The following facts are well known:

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(1) The Fourier series of any 2π-periodic continuous even function, convex on $[0, 2\pi]$, converges absolutely ([4, Ch. 2]).

(2) Let f be an odd function convex on $[0, +\infty)$. Then $f \in A^{loc}$ if and only if $\int_0^1 f(t)\frac{dt}{t} < \infty$ [4, Ch. 2]. Here A^{loc} is the set of all functions f, continuous on $(-\infty, +\infty)$, for which every point can be encircled by an interval on which $f = g$, where g is a function, continuous on $[0, 2\pi]$, whose Fourier series converges absolutely.

We have obtained the following results:

Theorem 1. *If $f \in M$, then for the absolute convergence of the Fourier series of the function f it is necessary and sufficient that*

$$\sum_{n=1}^{\infty} \Big| f\Big(x_k(f) + \frac{1}{n}\Big) - f\Big(x_k(f) - \frac{1}{n}\Big)\Big| \frac{1}{n} < +\infty, \quad k = 1, \dots, m.$$

Theorem 2.

(*a*) *Let $f \in M$; then*

$$a_n(f) = O\Big(\omega\Big(\frac{1}{n}; f\Big)\frac{1}{n}\Big), \quad b_n(f) = O\Big(\omega\Big(\frac{1}{n}; f\Big)\frac{1}{n}\Big).$$

(*b*) *If $\sum_{n=1}^{\infty} \omega(\frac{1}{n})\frac{1}{n} = +\infty$, then in the class $H^\omega \cap M$ there exists a a function whose Fourier series does not converge absolutely.*

Theorem 3. *Let $f \in M$ and at least one of the following conditions be fulfilled:*

(1) *for any adjacent intervals $(x_k(f), x_{k+1}(f))$ and $(x_{k+1}(f), x_{k+2}(f))$, the function f is convex on one of them and concave on the other;*

(2) *each point $x_k(f)$ can be encircled by an interval where the function f is monotonous;*

(3) *for any $x_k(f)$, at least one of the two series*

$$\sum_{n=1}^{\infty} \Big| f\Big(x_k(f) + \frac{1}{n}\Big) - f(x_k(f))\Big| \frac{1}{n} \quad \text{and} \quad \sum_{n=1}^{\infty} \Big| f\Big(x_k(f) - \frac{1}{n}\Big) - f(x_k(f))\Big| \frac{1}{n}$$

converges.

Then the convergence of the series $\sum_{n=1}^{\infty} \omega(\frac{1}{n}; f)\frac{1}{n}$ is the necessary and sufficient condition for the Fourier series of the function f to converge absolutely.

Proof of Theorem 1. Let f_1, f_2, f be continuous 2π-periodic functions defined as follows: f_1 is convex or concave on a segment $[0, \pi]$, $f_1(0) = f_1(\pi) = 0$, linear on $[1, \pi]$, $f_1(x) = 0$, for $x \in [-\pi, 0]$; f_2 is convex or concave on $[-\pi, 0]$, $f_2(-\pi) = f_2(0) = 0$, linear on $[-\pi, -1]$, $f_1(x) = 0$ for $x \in [0, \pi]$; $f = f_1 + f_2$.

The theorem will be proved by showing that for the Fourier series of f to converge it is necessary and sufficient that

$$\sum_{n=1}^{\infty}\left|f\Big(\frac{1}{n}\Big)-f\Big(-\frac{1}{n}\Big)\right|\frac{1}{n}<+\infty.$$

This follows from Wiener's theorem and from the following facts: If the function f is convex or concave on a segment $[a,b]$, then $f\in \operatorname{Lip}1$ on any segment $[c,d]$ entirely lying inside $[a,b]$, and the Fourier series of the functions $f(x)$ and $f(x+c)$ simultaneously converge or diverge absolutely.

The function f_1 is convex on $[0,\pi]$ and continuous, which means that it is absolutely continuous so that one can apply integration by parts and the Newton–Leibniz formulas to obtain $a_n(f)=a_n(f_1)+a_n(f_2)$.

$$a_n(f_1)=\frac{1}{\pi}\int\limits_0^{2\pi} f_1(t)\cos nt\,dt=\frac{1}{\pi}\int\limits_0^{2\pi} f_1(t)d\frac{\sin nt}{n}=\frac{1}{\pi}\Big(f_1(t)\frac{\sin nt}{n}\Big)_0^{2\pi}-$$

$$-\frac{1}{\pi n}\int\limits_0^{2\pi} f_1'(t)\sin nt\,dt=\frac{-1}{\pi n}\int\limits_0^{2\pi} f_1'(t)\sin nt\,dt=\frac{-1}{\pi n}\int\limits_0^{\pi} f_1'(t)\sin nt\,dt=$$

$$=-\frac{1}{\pi n}\int\limits_0^{1/n} f_1'(t)\sin nt\,dt-\frac{1}{\pi n}\int\limits_{1/n}^{1} f_1'(t)\sin nt\,dt-\frac{1}{\pi n}\int\limits_1^{\pi} f_1'(t)\sin nt\,dt.$$

The derivative f' of the convex or concave function f is monotonous and therefore, applying the second theorem of the mean value, we obtain

$$\Big|\frac{1}{\pi n}\int\limits_{1/n}^{1} f_1'(t)\sin nt\,dt\Big|=\Big|\frac{1}{\pi n}f_1'\Big(\frac{1}{n}+0\Big)\int\limits_{1/n}^{\xi}\sin nt+$$

$$+\frac{1}{\pi n}f_1'(1-0)\int\limits_{\xi}^{1}\sin nt\,dt\Big|\leq\frac{1}{\pi n^2}\Big|f_1'\Big(\frac{1}{n}+0\Big)\Big|+\frac{1}{\pi n^2}\big|f_1'(1-0)\big|$$

with $1/n<\xi<1$.

Wherever we come across expressions of the form $f'(x\pm 0)$, the left and right limits are considered with respect to the set at whose points the derivative f' exists.

For the convex (concave) function f we have the relation

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}\geq f'(x_2\pm 0)\geq\frac{f(x_3)-f(x_2)}{x_3-x_2}$$

$$\Big(\frac{f(x_2)-f(x_1)}{x_2-x_1} \le f'(x_2 \pm 0) \le \frac{f(x_3)-f(x_2)}{x_3-x_2}\Big)$$

where $x_1 < x_2 < x_3$. Therefore

$$\Big|f_1'\Big(\frac{1}{n}+0\Big)\Big| \le \frac{f_1\big(\frac{1}{n}\big)-f_1\big(\frac{1}{n+1}\big)}{1/n-1/(n+1)} \le (n+1)^2\Big(f_1\Big(\frac{1}{n}\Big)-f_1\Big(\frac{1}{n+1}\Big)\Big).$$

Hence

$$\sum_{n=1}^{\infty}\Big|f_1'\Big(\frac{1}{n}+0\Big)\Big|\frac{1}{n^2} \le 2\sum_{n=1}^{\infty}\Big(f_1\Big(\frac{1}{n}\Big)-f_1\Big(\frac{1}{n+1}\Big)\Big) < +\infty.$$

Since $f_1 \in \operatorname{Lip} 1$ on the segment $[\varepsilon,\pi]$, we have $|f_1'(1-0)| \le M$ and $\sum_{n=1}^{\infty}|f_1'(1-0)|/n^2 \le \sum_{n=1}^{\infty} M/n^2 < +\infty$.

The function f_1 is linear on the segment $[1,\pi]$, i.e., $f_1'(t)=\cos nt = c$, so that $1/n\big|\int_1^{\pi} f_1'(t)\sin nt\big| \le c/n^2$.

Finally, $a_n(f_1) = -\frac{1}{\pi n}\int_0^{1/n} f_1'(t)\sin nt\,dt + \gamma_n$, where $\sum_{n=1}^{\infty}|\gamma_n| < +\infty$.

If we introduce the notation $I_n = \frac{-1}{\pi n}\int_0^{1/n} f_1'(t)\sin nt\,dt$, then $a_n(f_1) = I_n+\gamma_n$, $I_n = a_n(f_1)-\gamma_n$.

Since the function f_1 has a bounded variation, we have

$$f_1(x) = \frac{a_n(f_1)}{2} + \sum_{n=1}^{\infty} a_n(f_1)\cos nx + b_n(f_1)\sin nx.$$

By substituting here $x=0$ we obtain $\sum_{n=1}^{\infty} a_n(f_1) < \infty$. Therefore $\sum_{n=1}^{\infty} I_n = \sum_{n=1}^{\infty}(a_n(f_1)-\gamma_n) < \infty$.

One can easily verify that the values I_n do not change their sign for sufficiently large n. Thus $\sum_{n=1}^{\infty}|I_n| < +\infty$. Since $|a_n(f_1)| \le |I_n|+|\gamma_n|$, we obtain $\sum_{n=1}^{\infty}|a_n(f_1)| < +\infty$.

In a similar manner we shall show that $\sum_{n=1}^{\infty}|a_n(f_2)| < \infty$. We have $|a_n(f)| = |a_n(f_1)+a_n(f_2)| \le |a_n(f_1)|+|a_n(f_2)|$ and $\sum_{n=1}^{\infty}|a_n(f)| < +\infty$.

Now consider the coefficients $b_n(f)$. We have $b_n(f) = b_n(f_1)+b_n(f_2)$,

$$b_n(f_1) = \frac{1}{\pi}\int_0^{2\pi} f_1(t)\sin nt\,dt = \frac{1}{\pi}\int_0^{2\pi} f_1(t)d\frac{\cos nt}{n} = \frac{1}{\pi}\Big(f_1(t)\frac{\cos nt}{n}\Big)\Big|_0^{2\pi} -$$

$$-\frac{1}{\pi n}\int_0^{2\pi} f_1'(t)\cos nt\,dt = -\frac{1}{\pi n}\int_0^{2\pi} f_1'(t)\cos nt\,dt = -\frac{1}{\pi n}\int_0^{\pi} f_1'(t)\cos nt\,dt =$$

$$= -\frac{1}{\pi n}\int_0^{1/n} f_1'(t)\cos nt\,dt - \frac{1}{\pi n}\int_{1/n}^{1} f_1'(t)\cos nt\,dt - \frac{1}{\pi n}\int_1^{\pi} f_1'(t)\cos nt\,dt.$$

The function f_1 is linear on the segment $[1,\pi]$, i.e., $f_1'(t) = const = C$, so that

$$\frac{1}{n}\Big|\int\limits_1^{\pi} f_1'(t)\cos nt\,dt\Big| \le \frac{C}{n^2}.$$

Again applying the theorem of the mean, we obtain (with $1/n < \xi < 1$)

$$\Big|\frac{1}{n}\int\limits_{1/n}^{1} f_1'(t)\cos nt\,dt\Big| = \frac{1}{n}\Big|f_1'\Big(\frac{1}{n}+0\Big)\int\limits_{1/n}^{\xi}\cos nt\,dt +$$

$$+f_1'(1-0)\int\limits_{\xi}^{1}\cos nt\,dt\Big| \le \frac{1}{n^2}\Big|f_1'\Big(\frac{1}{n}+0\Big)\Big| + \frac{1}{n^2}|f_1'(1-0)| < +\infty.$$

Therefore $b_n(f_1) = -\frac{1}{\pi n}\int_0^{1/n} f_1'(t)\cos nt\,dt + \gamma_n$, $\sum_{n=1}^{\infty}|\gamma_n| < +\infty$.

$$-\frac{1}{\pi n}\int\limits_0^{1/n} f_1'(t)\cos nt\,dt = \frac{1}{\pi n}\int\limits_0^{1/n} f_1'(t)(1-\cos nt - 1)dt =$$

$$= \frac{-1}{\pi n}\int\limits_0^{1/n} f_1'(t)dt + \frac{1}{\pi n}\int\limits_0^{1/n} f_1'(t)(1-\cos nt)dt =$$

$$= \frac{-1}{\pi n} f_1\Big(\frac{1}{n}\Big) + \frac{1}{\pi n}\int\limits_0^{1/n} f_1'(t)2\sin^2\frac{nt}{2}dt,$$

$$\Big|\frac{1}{\pi n}\int\limits_0^{1/n} f_1'(t)2\sin^2\frac{nt}{2}dt\Big| \le \frac{2}{\pi n}\int\limits_0^{1/n}|f_1'(t)|\,|\sin nt|dt = 2|I_n|.$$

As we have seen above, $\sum_{n=1}^{\infty}|I_n| < +\infty$ and therefore

$$b_n(f_1) = -\frac{1}{\pi n} f_1\Big(\frac{1}{n}\Big) + C_n = \frac{-1}{\pi n} f\Big(\frac{1}{n}\Big) + C_n,$$

where $\sum_{n=1}^{\infty}|C_n| < +\infty$. In a similar manner it will be shown that

$$b_n(f_2) = \frac{1}{\pi n} f\Big(-\frac{1}{n}\Big) + P_n, \ \text{ where } \sum_{n=1}^{\infty}|P_n| < +\infty.$$

Since $b_n(f) = b_n(f_1) + b_n(f_2)$, we have

$$b_n(f) = \frac{-1}{\pi n}\Big\{f\Big(\frac{1}{n}\Big) - f\Big(\frac{-1}{n}\Big)\Big\} + \gamma_n, \quad \sum_{n=1}^{\infty}|\gamma_n| < +\infty. \qquad \square$$

Proof of Theorem 2.

(a) It is the well-known fact that estimates of Fourier coefficients can be derived using the integral modulus of continuity (see, for instance [3, Ch. 2])

$$|a_n| \leq \sup_{|h|\leq\frac{1}{n}} \frac{1}{\pi} \int_0^{2\pi} \big|f(x+h)+f(x-h)-2f(x)\big|dx.$$

Applying the above inequality, we obtain

$$|a_n| \leq \sup_{|h|\leq\frac{1}{n}} \frac{1}{\pi} \int_0^{2\pi} \big|f(x+h)+f(x-h)-2f(x)\big|dx =$$

$$= \sup_{|h|\leq\frac{1}{n}} \frac{1}{\pi} \int_0^{2\pi} \big|f(x+kh)+f(x+(k-2)h)-2f(x+(k-1)h)\big|dx =$$

$$= \frac{1}{\pi n} \sup_{|h|\leq\frac{1}{n}} \int_0^{2\pi} \sum_{k=1}^{n} \big|f(x+kh)+f(x+(k-2)h)-$$

$$-2f(x+(k-1)h)\big|dx = \frac{1}{\pi n} \sup_{|h|\leq\frac{1}{n}} \int_0^{2\pi} \Big(\sum_{k=1}^{n} |u_k-u_{k-1}|\Big)dx,$$

where $u_k = f(x+kh)-f(x+(k-1)h)$.

Convexity (concavity) of a function f on some segment $[a,b]$ implies that

$$f(x+kh)+f(x-(k-2)h)-2f(x+(k-1)h) = u_k-u_{k-1} \leq 0 \quad (u_k-u_{k-1} \geq 0)$$

for $x+(k-2)h$, $x+(k-1)h$, $x+kh \in [a,b]$. Therefore on the segment $[0,2\pi]$ the values u_k-u_{k-1} change their sign a finite number of times. Thus

$$|a_n(f)| \leq \frac{1}{\pi n} \sup_{|h|\leq\frac{1}{n}} \int_0^{2\pi} \Big(\sum_{k=1}^{n} |u_k-u_{k-1}|\Big)dx \leq$$

$$\leq \frac{1}{\pi n} \sup_{|h|\leq\frac{1}{n}} \int_0^{2\pi} \Big|\sum_{k=1}^{n} u_k-u_{k-1}\Big|dx + \frac{C(f)}{n}\omega\Big(\frac{1}{n},f\Big) \leq \frac{C_1(f)}{n}\omega\Big(\frac{1}{n},f\Big).$$

The proof of the estimate for $b_n(f)$ is similar.

(b) Let $\sum_{n=1}^{\infty} \omega(\frac{1}{n})\frac{1}{n} = +\infty$. By Stechkin's lemma (see [6]) there exists a convex modulus of continuity $\omega'(\delta)$ such that $H^{\omega} = H^{\omega'}$. Hence $\omega(\delta)$ can be regarded as convex function.

Consider a continuous function

$$f_0(t) = \begin{cases} \omega(t), & t \in [0,1], \\ \text{linear for} & [t \in [1, 2\pi], \quad f_0(t+2\pi) = f_0(t). \end{cases}$$

Clearly, $f_0 \in H^\omega \cap M$.

On the interval $[1, 2\pi]$ the function $f_0(t)$ is linear, $f(2\pi) = f(0) = 0$. Therefore $f_0\Big(-\frac{1}{n}\Big) = f\Big(2\pi - \frac{1}{n}\Big) = \frac{c_0}{n}$, where c_0 is some number . We have

$$\sum_{n=1}^{\infty} \Big| f_0\Big(\frac{1}{n}\Big) - f_0\Big(\frac{-1}{n}\Big)\Big| \frac{1}{n} \geq \sum_{n=1}^{\infty} \Big| f_0\Big(\frac{1}{n}\Big)\Big| - \Big| f_0\Big(\frac{-1}{n}\Big)\Big| \frac{1}{n} =$$

$$= \sum_{n=1}^{\infty} \Big| f_0\Big(\frac{1}{n}\Big)\Big| \frac{1}{n} - \sum_{n=1}^{\infty} \Big| f_0\Big(-\frac{1}{n}\Big)\Big| \frac{1}{n} = \sum_{n=1}^{\infty} \omega\Big(\frac{1}{n}\Big)\frac{1}{n} - \sum_{n=1}^{\infty} \frac{C_0}{n^2} = +\infty.$$

By virtue of Theorem 1 we see that the necessary condition for Fourier series to be absolutely convergent is not fulfilled. The Fourier series of f_0 does not converge absolutely. □

Proof of Theorem 3. The sufficiency follows from part (a) of the proof of Theorem 2.

First note that if the function is convex (concave) on $[a, b]$, then its modulus of continuity on $[a, b]$ equals

$$\max\big\{|f(a+\delta) - f(a)|, |f(b-\delta) - f(b)|\big\}.$$

Therefore

$$\omega(1/n, f) \leq \max_{1 \leq k \leq m} \big\{|f(x_k + 1/n) - f(x_k)| + |f(x_k - 1/n) - f(x_k)|\big\},$$

where $x_k \equiv x_k(f)$.

Hence it follows that if $\sum_{n=1}^{\infty} \omega(\frac{1}{n}, f)\frac{1}{n} = +\infty$, then for some x_k we have

$$\sum_{n=1}^{\infty} \Big| f\Big(x_k + \frac{1}{n}\Big) - f(x_k)\Big| \frac{1}{n} = +\infty \quad \text{or} \quad \sum_{n=1}^{\infty} \Big| f\Big(x_k - \frac{1}{n}\Big) - f(x_k)\Big| \frac{1}{n} = +\infty.$$

For convenience we assume that

$$\sum_{n=1}^{\infty} \Big| f\Big(x_k + \frac{1}{n}\Big) - f(x_k)\Big| \frac{1}{n} = +\infty.$$

If the conditions (2) are fulfilled, then

$$\sum_{n=1}^{\infty} \Big| f\Big(x_k + \frac{1}{n}\Big) - f\Big(x_k - \frac{1}{n}\Big)\Big| \frac{1}{n} \geq \sum_{n=1}^{\infty} \Big| f\Big(x_k + \frac{1}{n}\Big) - f(x_k)\Big| \frac{1}{n} = +\infty,$$

i.e., by Theorem 1 the Fourier series of the function f is not absolutely convergent.

If the conditions (3) are fulfilled, then

$$\sum_{n=1}^{\infty}\Big|f\Big(x_k+\frac{1}{n}\Big)-f\Big(x_k-\frac{1}{n}\Big)\Big|\frac{1}{n}=$$
$$=\sum_{n=1}^{\infty}\Big|f\Big(x_k+\frac{1}{n}\Big)-f(x_k)+f(x_k)-f\Big(x_k-\frac{1}{n}\Big)\Big|\frac{1}{n}\geq$$
$$\geq\sum_{n=1}^{\infty}\Big|f\Big(x_k+\frac{1}{n}\Big)-f(x_k)\Big|\frac{1}{n}-\sum_{n=1}^{\infty}\Big|f\Big(x_k-\frac{1}{n}\Big)-f(x_k)\Big|\frac{1}{n}=+\infty.$$

which proves the theorem under conditions (3).

If , however, the conditions of (1) are fulfilled, then, as one can easily verify, the conditions of (2) or (3) are fulfilled too. □

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Authors' address:
Faculty of Physics and Mathematics
Sukhumi Branch of I. Javakhishvili Tbilisi State University
1, I. Chavchavadze ave., Tbilisi 380028
Georgia