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ON THE GLOBAL SOLVABILITY OF THE CAUCHY PROBLEM FOR SINGULAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. Sufficient conditions are found for the global solvability of the weighted Cauchy problem

$$\frac{dx(t)}{dt} = f(x)(t), \quad \lim_{t\to a} \frac{\|x(t) - c_0\|}{h(t)} = 0,$$

where $f : C([a,b];R^n) \to L_{loc}(]a,b];R^n)$ is a singular Volterra operator, $c_0 \in R^n$, $h : [a,b] \to [0,+\infty[$ is a function continuous and positive on $]a,b]$, and $\|\cdot\|$ is the norm in R^n.

Throughout the paper the following notation will be used:

R is the set of real numbers, $R_+ = [0,+\infty[$; if $u \in R$, then $[u]_+ = \frac{1}{2}(|u| + u)$;

R^n is the space of n-dimensional column vectors $x = (x_i)_{i=1}^n$ with elements $x_i \in R$ $(i = 1,\dots,n)$ and the norm $\|x\| = \sum_{i=1}^n |x_i|$;

$R^n_\rho = \{x \in R^n : \|x\| \le \rho\}$;

if $x = (x_i)_{i=1}^n$, then $\operatorname{sgn}(x) = (\operatorname{sgn} x_i)_{i=1}^n$;

$x \cdot y$ is the scalar product of the vectors x and $y \in R^n$;

$C([a,b];R^n)$ is the space of continuous vector functions $x : [a,b] \to R^n$ with the norm $\|x\|_C = \max\{\|x(t)\| : a \le t \le b\}$;

$$C_\rho([a,b];R^n) = \big\{x \in C([a,b];R^n) : \|x\|_C \le \rho\big\};$$
$$C([a,b];R_+) = \big\{x \in C([a,b];R) : x(t) \ge 0 \text{ for } a \le t \le b\big\};$$

if $x \in C([a,b];R^n)$ and $a \le s \le t \le b$, then

$$\nu(x)(s,t) = \max\big\{\|x(\xi)\| : s \le \xi \le t\big\};$$

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$L_{loc}(]a,b];R^n)$ is the space of vector functions $x:]a,b]\to R^n$ which are summable on each segment of $]a,b]$ with the topology of convergence in the mean on each segment from $]a,b]$;

$$L_{loc}(]a,b];R_+)=\big\{x\in L_{loc}(]a,b];R):\ x(t)\geq 0 \text{ for almost all } t\in[a,b]\big\}.$$

Definition 1. An operator $f:C([a,b];R^n)\to L_{loc}(]a,b];R^n)$ is called a *Volterra* one if the equality $f(x)(t)=f(y)(t)$ holds almost everywhere on $]a,t_0]$ for any $t_0\in]a,b]$ and any vector-functions x and $y\in C([a,b];R^n)$ satisfying the condition $x(t)=y(t)$ when $a<t\leq t_0$.

Definition 2. An operator $f:C([a,b];R^n)\to L_{loc}(]a,b];R^n)$ will be said to satisfy the *local Carathéodory conditions* if it is continuous and there exists a function $\gamma:]a,b]\times R_+\to R_+$ nondecreasing with respect to the second argument such that $\gamma(\cdot,\rho)\in L_{loc}(]a,b];R)$ for $\rho\in R_+$, and the inequality

$$\|f(x)(t)\|\leq\gamma(t,\|x\|_C)$$

is fulfilled for any $x\in C([a,b];R^n)$ almost everywhere on $]a,b[$.

If

$$\int_a^b\gamma(t,\rho)dt<+\infty \quad \text{for} \quad \rho\in R_+,$$

then the operator f is called *regular*, and, otherwise, *singular*.

Here we will consider the vector functional differential equation

$$\frac{dx(t)}{dt}=f(x)(t) \tag{1}$$

with the weighted initial condition

$$\lim_{t\to a}\frac{\|x(t)-c_0\|}{h(t)}=0. \tag{2}$$

It is assumed everywhere that $f:C([a,b];R^n)\to L_{loc}(]a,b];R^n)$ is a Volterra, generally speaking, singular operator satisfying the local Carathéodory conditions, $c_0\in R^n$, and $h:[a,b]\to[0,+\infty[$ is a continuous function nondecreasing and positive on $]a,b]$.

We will separately consider the case where $h(a)>0$ so that condition (2) takes the form

$$x(a)=c_0. \tag{2_1}$$

The vector differential equation with delay

$$\frac{dx(t)}{dt}=f_0\big(t,x(t),x(\tau_1(t)),\dots,x(\tau_m(t))\big) \tag{3}$$

is the important particular case of the functional differential equation (1).

Below, whenever equation (3) is discussed, it will be assumed that the vector function $f_0 :]a,b[\times R^{(m+1)n} \to R^n$ satisfies the local Carathéodory conditions, i.e., $f_0(t,\cdot,\dots,\cdot) : R^{(m+1)n} \to R^n$ is continuous for almost all $t \in]a,b[$, $f_0(\cdot,x_0,x_1,\dots,x_m) :]a,b[\to R^n$ is measurable for all $x_k \in R^n$ $(k=0,1,\dots,m)$, and on the set $]a,b[\times R^{(m+1)n}$ there holds the inequality

$$\big\|f_0(t,x_0,x_1,\dots,x_m)\big\| \le \gamma\Big(t,\sum_{k=0}^{m}\|x_k\|\Big),$$

where $\gamma :]a,b] \times R_+ \to R_+$ does not decrease with respect to the second argument and $\gamma(\cdot,\rho) \in L_{loc}(]a,b];R_+)$ for $\rho \in R_+$. As for $\tau_i : [a,b] \to [a,b]$ $(i=1,\dots,m)$, they are measurable and

$$\tau_i(t) \le t \quad \text{for} \quad a \le t \le b \ \ (i=1,\dots,m).$$

Definition 3. If $b_0 \in]a,b]$, then:

(i) For any $x \in C([a,b_0];R^n)$, by $f(x)$ is understood the vector function given by the equality $f(x)(t) = f(\overline{x})(t)$ for $a \le t \le b_0$, where

$$\overline{x}(t) = \begin{cases} x(t) & \text{for} \quad a \le t \le b_0 \\ x(b_0) & \text{for} \quad b_0 < t \le b_0 \end{cases};$$

(ii) a continuous vector function $x : [a,b_0] \to R^n$ *is called a solution of equation* (1) *on the segment* $[a,b_0]$ if x is absolutely continuous on each segment contained in $]a,b_0]$ and satisfies equation (1) almost everywhere on $]a,b_0[$;

(iii) a vector function $x : [a,b_0] \to R^n$ *is called a solution of equation* (1) *on the semi-open segment* $[a,b_0[$ if for each $b_1 \in]a,b_0[$ the restriction of x on $[a,b_1]$ is a solution of this equation on the segment $[a,b_1]$;

(iv) a solution x of equation (1) satisfying the initial condition (2) *is called a solution of problem* (1), (2).

Definition 4. Problem (1), (2) is said to be globally solvable if it has at least one solution on the segment $[a,b]$.

Definition 5. A solution x of equation (1) defined on the segment $[a,b_0] \subset [a,b[$ (on the semi-open segment $[a,b_0[\subset [a,b[$) is called *continuable* if for some $b_1 \in]b_0,b]$ $(b_1 \in [b_0,b])$ equation (1) has, on the segment $[a,b_1]$, a solution y satisfying the condition $x(t) = y(t)$ for $a \le t \le b_0$. Otherwise, the solution x is called *noncontinuable*.

Definition 6. An operator $\varphi : C([a,b];R_+) \to L_{loc}(]a,b];R)$ is called nondecreasing if the inequality $\varphi(u)(t) \le \varphi(v)(t)$ is fulfilled almost everywhere on $]a,b[$ for any u and $v \in C([a,b];R_+)$ satisfying the condition $u(t) \le v(t)$ when $a \le t \le b$.

Theorem 1. *Let there exist a positive number ρ, summable functions p and $q : [a,b] \to R_+$, and a continuous nondecreasing Volterra operator $\varphi : C([a,b];R_+) \to L_{loc}(]a,b];R_+)$ such that*

$$\limsup_{t\to a}\left(\frac{1}{h(t)}\int_a^t p(s)ds\right) < 1, \quad \lim_{t\to a}\left(\frac{1}{h(t)}\int_a^t q(s)ds\right) = 0 \tag{4}$$

and the inequalities

$$f(c_0 + hy)(t)\cdot \operatorname{sgn}(y(t)) \le \varphi(\|y\|)(t), \tag{5}$$

$$\varphi(u)(t) \le p(t)\nu(u)(a,t) + q(t) \tag{6}$$

are fulfilled for any $y \in C([a,b];R^n)$ and $u \in C_\rho([a,b];R_+)$ almost everywhere on $]a,b[$. Let furthermore the problem

$$\frac{dv(t)}{dt} = \varphi\Big(\frac{v}{h}\Big)(t); \quad \lim_{t\to a}\frac{v(t)}{h(t)} = 0 \tag{7}$$

have an upper solution on the segment $[a,b]$. Then problem (1), (2) *is globally solvable and each of its noncontinuable solutions is defined on $[a,b]$.*

Proof. By virtue of Theorem 3.2 from [1] conditions (4)–(6) imply the existence of a noncontinuable solution of problem (1), (2). Let x be an arbitrary noncontinuable solution of problem (1), (2) defined on the segment I_0. Our aim is to prove that $I_0 = [a,b]$.

By (4) there exist $b_0 \in]a,b]$ and $\alpha \in]0,1[$ such that

$$\int_a^t p(s)ds \le \alpha h(t), \quad h_0(t) \le \rho h(t) \quad \text{for } a \le t \le b_0, \tag{8}$$

where $h_0(a) = 0$, and

$$h_0(t) = \frac{h(t)}{1-\alpha}\sup\left\{\frac{1}{h(s)}\int_a^s q(\xi)d\xi : \ a < s \le t\right\} \quad \text{for } a < t \le b_0. \tag{9}$$

By Lemma 2.3 and Corollary 3.1 from [1], conditions (4)–(6) and (8) imply that $I_0 \supset [a,b_0]$ and

$$\|x(t) - c_0\| \le h_0(t) \quad \text{for} \quad a \le t \le b_0. \tag{10}$$

Let v^* be an upper solution of problem (7). Then by conditions (4), (6), (8) and Lemma 2.3 from [1]

$$v^*(t) \le h_0(t) \quad \text{for} \quad a \le t \le b_0. \tag{11}$$

By (9) the function

$$\varepsilon(t)=\begin{cases}\dfrac{h_0(t)}{h(t)}+\dfrac{t-a}{b_0-t} & \text{for} \quad a<t<b_0\\ 0 & \text{for} \quad t=a\end{cases} \tag{12}$$

is continuous on $[a,b_0[$.

For any $u\in C([a,b];R)$ we set

$$\chi(u)(t)=\begin{cases}\varepsilon(t) & \text{for} \quad u(t)>\varepsilon(t), \quad a\le t<b_0\\ [u(t)]_+ & \text{for} \quad u(t)\le\varepsilon(t), \quad a\le t<b_0\\ [u(t)]_+ & \text{for} \quad b_0\le t\le b\end{cases} \tag{13}$$

and

$$\overline{\varphi}(u)(t)=\varphi\Big(\chi\Big(\frac{u}{h}\Big)\Big)(t). \tag{14}$$

Obviously, $\chi: C([a,b];R)\to C([a,b];R_+)$ and $\overline{\varphi}: C([a,b];R)\to L_{loc}(]a,b];R_+)$ are continuous nondecreasing Volterra operators and the inequality

$$\overline{\varphi}(hu)(t)\cdot\operatorname{sgn}(u(t))\le p(t)\nu(u)(a,t)+q(t) \tag{15}$$

holds for any $u\in C_\rho([a,b];R)$ almost everywhere on $]a,b[$.

By Corollary 1.3 from [2] the problem

$$\frac{dv(t)}{dt}=\overline{\varphi}(v)(t); \quad \lim_{t\to a}\frac{v(t)}{h(t)}=0 \tag{16}$$

has a noncontinuable upper solution $\overline{v}$ defined on some interval I. On the other hand, by conditions (11)–(14),

$$\varphi\Big(\frac{v^*}{h}\Big)(t)\equiv\overline{\varphi}(v^*)(t)$$

and therefore v^* is a solution of problem (16). Thus

$$v^*(t)\le\overline{v}(t) \quad \text{for} \quad t\in I. \tag{17}$$

By Lemma 2.3 and Corollary 3.1 from [1], conditions (4), (6), and (8) imply the estimate

$$0\le\overline{v}(t)\le h_0(t) \quad \text{for} \quad a\le t\le b_0$$

and hence by (12)—(14) we obtain

$$\overline{\varphi}(\overline{v})(t)\equiv\varphi\Big(\frac{\overline{v}}{h}\Big)(t).$$

Therefore $\overline{v}$ is a solution of problem (7) so that

$$\overline{v}(t)\le v^*(t) \quad \text{for} \quad t\in I. \tag{18}$$

By (17) and (18) it is obvious that $I = [a,b]$ and $\overline{v}(t) \equiv v^*(t)$. Thus v^* is an upper solution of problem (16).

Due to conditions (5), (10) and (12)–(14) we have

$$\frac{d}{dt}\big(\|x(t)-c_0\|\big) \le \overline{\varphi}\big(\|x-c_0\|\big)(t) \tag{19}$$

almost everywhere on I_0.

By Theorem 1.3 from [2], (19) and (2) imply the estimate

$$\|x(t)-c_0\| \le v^*(t) \quad \text{for} \quad t \in I_0.$$

Hence by Corollary 3.1 from [1], we conclude that $I_0 = [a,b]$. □

For our further discussion we need

Definition 7. Let $D \subset C([a,b];R^k)$ and $M \subset L_{loc}(]a,b];R^l)$. An operator $g : D \to M$ is called a strictly Volterra operator if there exists a continuous nondecreasing function $\tau : [a,b] \to [a,b]$ such that

$$\tau(t) < t \quad \text{for} \quad a < t \le b$$

and the equality

$$g(x)(t) = g(y)(t)$$

holds almost everywhere on $[a,t_0]$ for any $t_0 \in \,]a,b]$ and any vector functions x and $y \in D$ satisfying the condition

$$x(t) = y(t) \quad \text{for} \quad a \le t \le \tau(t_0).$$

Corollary 1. *Let, for any $y \in C([a,b];R^n)$, the inequality*

$$f(c_0+hy)\cdot \operatorname{sgn}(y(t)) \le p(\|y\|)(t)\varphi_0(\nu(y)(a,t)) + q(\|y\|)(t)$$

be fulfilled almost everywhere on $]a,b[$, where p and $q : C([a,b];R_+) \to L_{loc}([a,b];R_+)$ are continuous, nondecreasing, strictly Volterra operators satisfying, for some $\rho > 0$, the conditions

$$\limsup_{t\to a}\left(\frac{1}{h(t)}\int_a^t p(\rho)(s)ds\right) < 1, \;\; \lim_{t\to a}\left(\frac{1}{h(t)}\int_a^t q(\rho)(s)ds\right) = 0,$$

and $\varphi_0 : R_+ \to R_+$ is a continuous, nondecreasing function such that

$$\limsup_{s\to 0}\frac{\varphi_0(s)}{s} < 1, \tag{20}$$

$$\varphi_0(s) > 0, \quad \int_s^{+\infty}\frac{d\xi}{\varphi_0(\xi)} = +\infty \quad \textit{for} \quad s > 0. \tag{21}$$

Then the conclusion of Theorem 1 is valid.

Proof. By (20) the number $\rho > 0$ can be assumed to be so small that

$$\frac{\varphi_0(s)}{s} < \rho \quad \text{for} \quad 0 < s \le \rho.$$

We set

$$\varphi(u)(t) = p(u)(t)\varphi_0(\nu(u)(a,t)) + q(u)(t). \tag{22}$$

Then inequalities (5) and (6), where

$$p(t) \equiv p(\rho)(t), \quad q(t) \equiv q(\rho)(t),$$

are fulfilled almost everywhere on $]a,b[$ for any $y \in C([a,b];R^n)$ and $u \in C_\rho([a,b];R_+)$.

By virtue of Theorem 1, to prove the corollary it is sufficient to show that problem (7) has an upper solution on the segment $[a,b]$.

Choose $b_0 \in]a,b]$ and $\alpha \in]0,1[$ such that inequalities (8) are fulfilled.

Let h_0, ε, χ and $\overline{\varphi}$ be the functions and operators given by equalities (9) and (12)–(14). As shown while proving Theorem 1, problem (7) is equivalent to problem (16).

Following Corollary 1.3 from [2], problem (16) has an upper solution v^* defined on some interval I. Since problems (7) and (16) are equivalent it remains for us only to show that $I = [a,b]$.

By virtue of Corollary 3.1 and Lemma 2.3 from [1], $I \supset [a,b_0]$ and inequality (11) is fulfilled.

Now assume that $I \ne [a,b]$. Then, by Corollary 3.1 from [1] and the nonnegativeness of the operator $\overline{\varphi}$, there exists $b_1 \in]b_0,b]$ such that $I = [a,b_1[$ and

$$\lim_{t\to b_1} v^*(t) = +\infty. \tag{23}$$

By (8) and (11)–(13)

$$\frac{v^*(t)}{h(t)} < \rho \quad \text{for} \quad 0 \le t \le b_0$$

and

$$\nu\Big(\chi\Big(\frac{v^*}{h}\Big)\Big)(a,t) \le \rho + \frac{v^*(t)}{h(b_0)} \quad \text{for} \quad b_0 \le t < b_1,$$

due to which (14) and (22) imply

$$\overline{\varphi}(v^*)(t) \le p_0(t)\varphi_0\Big(\rho + \frac{v^*(t)}{h(b_0)}\Big) \quad \text{for} \quad b_0 \le t < b_1,$$

where

$$p_0(t) = p\Big(\chi\Big(\frac{v^*}{h}\Big)\Big)(t) + \frac{q(\chi(\frac{v^*}{h}))(t)}{\varphi_0(\rho)}.$$

On the other hand, since the operators p and q are the strictly Volterra ones, the function $p_0 : [b_0, b_1] \to R_+$ is summable.

From the above discussion it follows that the inequality

$$\frac{dv^*(t)}{dt} / \varphi_0\Big(\rho + \frac{v^*(t)}{h(b_0)}\Big) \leq p_0(t)$$

is fulfilled almost everywhere on $]b_0, b_1[$.

If we integrate both parts of this inequality from b_0 to b_1, by (23) we will obtain

$$h(b_0) \int\limits_{s_0}^{+\infty} \frac{ds}{\varphi_0(s)} \leq \int\limits_{b_0}^{b_1} p_0(s)ds < +\infty,$$

which contradicts condition (21). The obtained contradiction proves the corollary. □

The next corollary is proved similarly.

Corollary 2. *Let, for any* $y \in C([a,b]; R^n)$*, the inequality*

$$f(c_0 + hy) \cdot \operatorname{sgn}(y(t)) \leq p(\|y\|)(t)\varphi_0(\nu(y)(a,t))$$

be fulfilled almost everywhere on $]a,b[$*, where* $p : C([a,b]; R_+) \to L([a,b]; R_+)$ *is a continuous, nondecreasing, strictly Volterra operator satisfying, for some* $\rho > 0$*, the condition*

$$\lim_{t \to a} \left(\frac{1}{h(t)} \int\limits_a^t p(\rho)(s)ds \right) = 0,$$

and $\varphi_0 : R_+ \to]0, +\infty[$ *is a continuous, nondecreasing function such that*

$$\int\limits_0^{+\infty} \frac{ds}{\varphi_0(s)} = +\infty. \tag{24}$$

Then the conclusion of Theorem 1 *is valid.*

Corollary 3. *Let, for any* $y \in C([a,b]; R^n)$*, the inequality*

$$f(c_0 + y)(t) \cdot \operatorname{sgn}(y(t)) \leq \frac{\alpha}{b-a}\Big(\frac{b-t}{b-a}\Big)^{\beta} \exp\big[\nu(y)(a, \tau(t))\big] \tag{25}$$

be fulfilled almost everywhere on $]a,b[$*, where*

$$\tau(t) = b - (b-a)\Big(1 + \ln\frac{b-a}{b-t}\Big)^{-1}, \tag{26}$$

$\alpha > 0$ *and* β *are constants such that*

$$\beta > [\alpha - 1]_+ - 1. \tag{27}$$

Then problem (1), (2) *is globally solvable and each of its noncontinuable solutions is defined on the entire* $[a,b]$.

Proof. By Theorem 1 it is sufficient to show that the problem

$$\frac{du(t)}{dt}=\frac{\alpha}{b-a}\Big(\frac{b-t}{b-a}\Big)^{\beta}\exp\big[\nu(u)(a,\tau(t))\big],\quad u(a)=0$$

has an upper solution on the segment $[a,b]$. But this problem is equivalent to the problem

$$\frac{du(t)}{dt}=\frac{\alpha}{b-a}\Big(\frac{b-t}{b-a}\Big)^{\beta}\exp\big[u(\tau(t))\big],\quad u(a)=0. \tag{28}$$

By Theorem 3.2 from [1] problem (28) has a noncontinuable solution. It is not difficult to verify that this solution is unique. We denote it by u. Let I be the definition interval of u. Our aim is to show that $I=[a,b]$.

Consider the function

$$v(t)=\alpha\Big(\frac{b-a}{b-t}-1\Big)\quad\text{for}\quad a\le t<b.$$

By virtue of (26) and (27)

$$a\le\tau(t)<t\quad\text{for}\quad a<t<b$$

and

$$\frac{dv(t)}{dt}=\frac{\alpha}{b-a}\Big(\frac{b-t}{b-a}\Big)^{\alpha-2}\exp\big[v(\tau(t))\big]\ge$$
$$\ge\frac{\alpha}{b-a}\Big(\frac{b-t}{b-a}\Big)^{\beta}\exp\big[v(\tau(t))\big]\quad\text{for}\quad a\le t<b.$$

Hence, by Corollary 1.8 from [2], we have

$$v(t)\ge u(t)\quad\text{for}\quad t\in I.$$

However, by Corollary 3.1 from [2] this estimate implies that $I\supset[a,b[$ and

$$u(t)\le\alpha\Big(\frac{b-a}{b-t}-1\Big)\quad\text{for}\quad a\le t<b. \tag{29}$$

On account of (27) there is $\varepsilon\in]0,1[$ such that $\beta>[\alpha-1]_+-\varepsilon$. If, along with this inequality, we use equality (26) and estimate (29), then (28) will yield

$$u(t)\le\frac{\alpha}{b-a}\int\limits_a^t\Big(\frac{b-s}{b-a}\Big)^{[\alpha-1]_+-\varepsilon}\exp\big[u(\tau(s))\big]ds\quad\text{for}\quad a\le t<b, \tag{30}$$

$$u(t) \le \frac{\alpha}{b-a} \int\limits_a^t \Big(\frac{b-s}{b-a}\Big)^{\alpha-1-\varepsilon} \Big(\frac{b-s}{b-a}\Big)^{-\alpha} ds =$$

$$= \alpha(b-a)^\varepsilon \int\limits_a^t (b-s)^{-1-\varepsilon} ds < \frac{\alpha}{\varepsilon}\Big(\frac{b-a}{b-t}\Big)^\varepsilon \quad \text{for} \quad a \le t < b,$$

and hence

$$u(t) \le \alpha_1 + \frac{1-\varepsilon}{2}\Big(\frac{b-a}{b-t}\Big) \quad \text{for} \quad a \le t < b, \tag{31}$$

where

$$\alpha_1 = \frac{1}{\varepsilon}\Big(\frac{2\alpha}{1-\varepsilon}\Big)^{\frac{1}{1-\varepsilon}}.$$

By (26) and (31), from (30) we obtain

$$u(t) \le \frac{\alpha}{b-a} \exp(\alpha_1) \int\limits_a^t \Big(\frac{b-s}{b-a}\Big)^{[\alpha-1]_+ -\varepsilon} \Big(\frac{b-s}{b-a}\Big)^{\frac{\varepsilon-1}{2}} ds \le$$

$$\le \frac{\alpha}{b-a} \exp(\alpha_1) \int\limits_a^t \Big(\frac{b-s}{b-a}\Big)^{-\frac{1+\varepsilon}{2}} ds \le \alpha_2 \quad \text{for} \quad a \le t < b,$$

where

$$\alpha_2 = \frac{2\alpha}{1-\varepsilon}.$$

Hence by Corollary 3.1 from [1] it follows that $I = [a, b]$. $\square$

Corollaries 1–3 imply the following propositions for equation (3).

Corollary 4. *Let there exist $m_0 \in \{1, \dots, m\}$ and a continuous nondecreasing function $\tau : [a, b] \to [a, b]$ such that*

$$\tau_k(t) \le \tau(t) < t \quad \textit{for} \quad a < t \le b \quad (k = m_0, \dots, m). \tag{32}$$

Let furthermore on the set $]a, b[\, \times R^{(m+1)n}$ there hold the inequality

$$f_0\big(t, c_0 + h(t)y_0, c_0 + h(\tau_1(t))y_1, \dots, c_0 + h(\tau_m(t))y_m\big) \cdot \operatorname{sgn}(y_0) \le$$

$$\le p\Big(t, \sum_{k=m_0}^m \|y_k\|\Big)\varphi_0\Big(\sum_{k=0}^{m_0-1} \eta_k \|y_k\|\Big) + q\Big(t, \sum_{k=m_0}^m \|y_k\|\Big),$$

where p and $q : [a,b] \times R_+ \to R_+$ are functions summable with respect to the first argument and continuously nondecreasing with respect to the second argument, and satisfying, for some $\rho > 0$, the conditions

$$\limsup_{t\to a} \left(\frac{1}{h(t)} \int_a^t p(s,\rho)ds \right) < 1, \ \lim_{t\to a} \left(\frac{1}{h(t)} \int_a^t q(s,\rho)ds \right) = 0,$$

$\varphi_0 : R_+ \to R_+$ is a continuous nondecreasing function satisfying conditions (20) *and* (21), *η_k $(k = 0, \dots, m_0)$ are non-negative constants such that*

$$\sum_{k=0}^{m_0} \eta_k = 1.$$

Then problem (3), (2) *is globally solvable and each of its noncontinuable solutions is defined on $[a,b]$.*

Corollary 5. *Let the function h be nondecreasing and inequalities* (32) *be fulfilled, where $m_0 \in \{1, \dots, m\}$ and $\tau : [a,b] \to [a,b]$ is a continuous nondecreasing function. Let, furthermore, on the set $]a,b[\times R^{(m+1)n}$ there hold the inequality*

$$f_0\big(t, c_0 + h(t)y_0, c_0 + h(\tau_1(t))y_1, \dots, c_0 + h(\tau_m(t))y_m\big) \cdot \operatorname{sgn}(y_0) \le$$
$$\le p\Big(t, \sum_{k=m_0}^{m} \|y_k\|\Big) \varphi_0\Big(\sum_{k=0}^{m_0-1} \|y_k\|\Big),$$

where $p : [a,b] \times R_+ \to R_+$ is a function summable with respect to the first argument and continuously nondecreasing with respect to the second argument, and satisfying, for some $\rho > 0$, the condition

$$\lim_{t\to a} \left(\frac{1}{h(t)} \int_a^t p(s,\rho)ds \right) = 0,$$

$\varphi_0 : R_+ \to]0, +\infty[$ is a continuous nondecreasing function satisfying condition (21). *Then the conclusion of Corollary* 4 *is valid.*

Corollary 6. *Let on the set $]a,b[\times R^{(m+1)n}$ and the segment $]a,b[$ there hold respectively the inequalities*

$$f_0(t, c_0 + y_0, c_0 + y_1, \dots, c_0 + y_m) \cdot \operatorname{sgn}(y_0) \le$$
$$\le \frac{\alpha}{b-a} \Big(\frac{b-t}{b-a} \Big)^{\beta} \exp\Big(\sum_{k=1}^{m} \eta_k \|y_k\| \Big)$$

and

$$\tau_k(t) \le b - (b-a)\Big(1 + \ln \frac{b-a}{b-t}\Big)^{-1} \quad (k = 1, \dots, m),$$

where $\alpha > 0$, β *and* $\eta_k \geq 0$ $(k = 1, \dots, m)$ *are constants satisfying condition* (27) *and*

$$\sum_{k=1}^{m} \eta_k = 1.$$

Then problem (3),(2_1) *is globally solvable and each of its noncontinuable solutions is defined on the entire* $[a, b]$.

Example 1. Consider problem (28) where $\alpha > 1$, $\beta = \alpha - 2$, τ is the function given by equality (26). All conditions of Corollary 6 except (27) are fulfilled. Nevertheless it has the noncontinuable solution

$$u(t) = \alpha\Big(\frac{b-a}{b-t} - 1\Big),$$

which is defined not on the segment $[a, b]$ but on $[a, b[$. This example shows that in Corollaries 3 and 6 the strict inequality (27) cannot be replaced by the nonstrict inequality $\beta \geq [\alpha - 1]_+ - 1$.

Theorem 2. *Let there exist* $\delta :]a, b] \to]0, \infty[$ *and* $c :]a, b] \to R^n$ *such that* $\delta(s) < s - a$ *for* $a < s \leq b$, *and for any numbers* $s \in]a, b]$, $\rho > 0$ *and any vector function* $y \in C([a, s]; R^n)$ *satisfying the condition*

$$\|y(t)\| \leq \rho \quad \text{for} \quad a \leq t \leq s - \delta(s), \tag{33}$$

let the inequality

$$f(c(s) + y)(t) \cdot \operatorname{sgn}(y(t)) \leq \varphi_{s,\rho}(\|y\|)(t) \tag{34}$$

hold almost everywhere on $]s - \delta(s), s[$, *where* $\varphi_{s,\rho} : C([s - \delta(s), s]; R_+) \to L([s - \delta(s), s]; R_+)$ *is a continuous nondecreasing Volterra operator. Let furthermore the problem*

$$\frac{du(t)}{dt} = \varphi_{s,\rho}(u)(t), \quad u(s - \delta(s)) = \rho \tag{35}$$

have an upper solution on the interval $[s - \delta(s), s]$ *for any* $s \in]a, b]$ *and* $\rho > 0$. *Then each noncontinuable solution of equation* (1) *is defined on* $[a, b]$.

Proof. Assume that the theorem is not valid. Then by virtue of Corollary 3.1 from [1] there exist $s \in]a, b]$ and a noncontinuable solution $x : [a, s[\to R^n$ of equation (1) such that

$$\limsup_{t \to s} \|x(t)\| = +\infty. \tag{36}$$

We set $y(t) = x(t) - c(s)$ and choose $\rho > 0$ such that inequality (33) is fulfilled. Then by (34) the inequality

$$\frac{d\|y(t)\|}{dt} \le \varphi_{s,\rho}(\|y\|)(t)$$

is fulfilled almost everywhere on $]s - \delta(s), s[$.

Moreover,

$$\|y(s - \delta(s))\| \le \rho.$$

By Corollary 1.7 from [2] the latter two inequalities yield the estimate

$$\|y(t)\| \le u(t) \quad \text{for} \quad s - \delta(s) \le t < s.$$

Therefore

$$\limsup_{t \to s} \|x(t)\| \le \|c(s)\| + \limsup_{t \to s} \|y(t)\| \le \|c(s)\| + u(s) < +\infty,$$

which contradicts equality (36). The obtained contradiction proves the theorem. □

Remark. It is obvious that if the conditions of Theorem 2 are fulfilled, then the local solvability of problem (1), (2) guarantees its global solvability. Therefore if the conditions of Theorem 2, as well as of Theorem 2.1 from [1], are fulfilled, then problem (1), (2) is globally solvable and each of its noncontinuable solutions is defined on $[a, b]$.

Corollary 7. *Let there exist functions* $\delta :]a, b] \to]0, 1[$, $c :]a, b] \to R^n$, $\alpha :]a, b] \times R_+ \to R_+$, $\beta :]a, b] \times R_+ \to]-1, 0]$, *and* $\lambda_k :]a, b] \times R_+ \to [1, +\infty[$ $(k = 1, \dots, m)$ *such that* $\delta(s) < s - a$ *for* $a < s \le b$, *and for any numbers* $s \in]a, b]$, $\rho > 0$ *and any vector function* $y \in C([a, b]; R^n)$ *satisfying condition* (33), *there hold, almost everywhere on* $]s - \delta(s), s[$, *the inequality*

$$\begin{aligned} f(c(s) + y)(t) \cdot \operatorname{sgn}(y(t)) \le \\ \le \alpha(s, \rho)(s - t)^{\beta(s,\rho)} \left(1 + \sum_{k=1}^{m} \left[\nu(y)(s - \delta(s), \tau_{ks\rho}(t))\right]^{\lambda_k(s,\rho)}\right) \times \\ \times \ln\left(2 + \sum_{k=1}^{m} \nu(y)(s - \delta(s), \tau_{ks\rho}(t))\right), \end{aligned} \tag{37}$$

where

$$\tau_{ks\rho}(t) = \begin{cases} s - \delta(s) & \text{for } s - \delta(s) \le t \le s - [\delta(s)]^{\lambda_k(s,\rho)} \\ s - (s - t)^{\frac{1}{\lambda_k(s,\rho)}} & \text{for } s - [\delta(s)]^{\lambda_k(s,\rho)} < t \le s \end{cases}. \tag{38}$$

Then each noncontinuable solution of equation (1) *is defined on* $[a, b]$.

Proof. Let $s \in]a, b]$ and $\rho > 0$ be arbitrarily fixed. We introduce $\varepsilon \in]0, \frac{1}{2}]$ such that

$$\beta(s, \rho) \geq 2\varepsilon - 1.$$

By virtue of Theorem 2 and inequality (37), to prove the corollary it is sufficient to establish that problem (35), where

$$\varphi_{s\rho}(u)(t) = \alpha(s, \rho)(s - t)^{2\varepsilon - 1} \left(1 + \sum_{k=1}^{m} \big[\nu([u]_+)(s - \delta(s), \tau_{ks\rho}(t))\big]^{\lambda_k(s,\rho)}\right) \times$$
$$\times \ln\left(2 + \sum_{k=1}^{m} \nu([u]_+)(s - \delta(s), \tau_{ks\rho}(t))\right), \tag{39}$$

has an upper solution on the interval $[s - \delta(s), s]$.

By Corollary 1.3 from [2] and equalities (38) and (39), problem (35) has a noncontinuable upper solution u^* on some interval $I \subset [s - \delta(s), s]$. On the other hand, from Corollary 2 it immediately follows that $I \supset [s - \delta(s), s[$. It remains for us to show that $s \in I$.

Choose numbers $\delta_0 \in]0, \delta(s)[$ and $\rho_0 \in]\rho, +\infty[$ such that

$$\frac{1}{\varepsilon} < \ln \frac{1}{\delta_0} < \big[\alpha(s, \rho)(m + 1)(1 + \ln(2 + m))\big]^{-1} \delta_0^{-\varepsilon}, \tag{40}$$

$$u^*(s - \delta_0) < \rho_0, \tag{41}$$

and for any $k \in \{1, \dots, m\}$ and $u \in C([s - \delta_0, s]; R)$ put

$$\tau_k^*(t) = \begin{cases} s - \delta_0 & \text{for} \quad s - \delta_0 \leq t \leq s - \delta_0^{\lambda_k(s,\rho)} \\ s - (s - t)^{\frac{1}{\lambda_k(s,\rho)}} & \text{for} \quad s - \delta_0^{\lambda_k(s,\rho)} < t \leq s \end{cases}, \tag{42}$$

$$\nu_k^*(u)(t) = \begin{cases} \rho_0 & \text{for} \quad s - \delta_0 \leq t \leq s - \delta_0^{\lambda_k(s,\rho)} \\ [u(\tau_k^*(t)]_+ & \text{for} \quad s - \delta_0^{\lambda_k(s,\rho)} < t \leq s \end{cases}, \tag{43}$$

$$\varphi^*(u)(t) = \alpha(s, \rho)(s - t)^{2\varepsilon - 1} \left(1 + \sum_{k=1}^{m} \big[\nu_k^*(u)(t)\big]^{\lambda_k(s,\rho)}\right) \times$$
$$\times \ln\left(2 + \sum_{k=1}^{m} \nu_k^*(u)(t)\right). \tag{44}$$

Then by virtue of (38) and (39) the inequality $\varphi_{s,\rho}(u^*)(t) \leq \varphi^*(u^*)(t)$ holds almost everywhere on $]s - \delta_0, s[$ and therefore

$$0 < \frac{du^*(t)}{dt} \leq \varphi^*(u^*)(t). \tag{45}$$

Let l be a natural number so large that

$$\delta_0^{-\frac{(l + \frac{1}{2})\varepsilon}{\lambda_k(s,\rho)}} > \rho_0 \quad (k = 1, \dots, m). \tag{46}$$

Setting $v(t) = (s-t)^{-(l+\frac{1}{2})\varepsilon}$, we obtain

$$v(s-\delta_0) > \rho_0. \tag{47}$$

Moreover, with (42), (43), and (46) taken into account we find

$$\nu_k^*(v)(t) = \begin{cases} \rho_0 & \text{for} \quad s-\delta_0 \le t \le s - \delta_0^{\lambda_k(s,\rho)} \\ (s-t)^{-\frac{(l+\frac{1}{2})\varepsilon}{\lambda_k(s,\rho)}} & \text{for} \quad s-\delta_0^{\lambda_k(s,\rho)} < t \le s \end{cases},$$

and

$$\nu_k^*(v)(t) \le (s-t)^{-\frac{(l+\frac{1}{2})\varepsilon}{\lambda_k(s,\rho)}} \ \text{ for } \ s_0-\delta_0 \le t < s \ \ (k=1,\dots,m). \tag{48}$$

By (44) and (48) the inequality

$$\begin{aligned} \varphi^*(v)(t) \le \alpha(s,\rho)(s-t)^{2\varepsilon-1}\big(1+m(s-t)^{-(l+\frac{1}{2})\varepsilon}\big) \times \\ \times \ln\big(2+m(s-t)^{-(l+\frac{1}{2})\varepsilon}\big) \le \alpha(s,\rho)(m+1) \times \\ \times (s-t)^{2\varepsilon-1-(l+\frac{1}{2})\varepsilon} \ln\big[(2+m)(s-t)^{-(l+\frac{1}{2})\varepsilon}\big] \end{aligned}$$

holds almost everywhere on $]s-\delta_0, s[$.

On the other hand, using (40) we have

$$\begin{gathered} (s-t)^{\varepsilon} \ln\frac{1}{s-t} \le \delta_0^{\varepsilon} \ln\frac{1}{\delta_0} \le \\ \le \big[\alpha(s,\rho)(m+1)(1+\ln(2+m))\big]^{-1} \quad \text{for} \quad s-\delta_0 \le t < s \end{gathered}$$

and

$$\begin{gathered} \ln\big[(2+m)(s-t)^{-(l+\frac{1}{2})\varepsilon}\big] = \ln(2+m) + \Big(l+\frac{1}{2}\Big)\varepsilon \ln\frac{1}{s-t} \le \\ \le \big(1+\ln(2+m)\big)\Big(l+\frac{1}{2}\Big)\varepsilon \ln\frac{1}{s-t} \le \\ \le \frac{(l+\frac{1}{2})\varepsilon}{\alpha(s,\rho)(m+1)}(s-t)^{-\varepsilon} \ \text{ for } \ s-\delta_0 \le t < s. \end{gathered}$$

Therefore

$$\varphi^*(v)(t) \le \Big(l+\frac{1}{2}\Big)\varepsilon(s-t)^{\varepsilon-1-\left(l+\frac{1}{2}\right)\varepsilon} \tag{49}$$

and

$$\frac{dv(t)}{dt} > \varphi^*(v)(t). \tag{50}$$

By Theorem 1.4 from [2] inequalities (41), (45), (50) imply the estimate

$$0 < u^*(t) < (s-t)^{-(l+\frac{1}{2})\varepsilon} \quad \text{for} \quad s-\delta_0 \le t < s,$$

by means of which we find from (45) and (49) that

$$0 \le \frac{du^*(t)}{dt} \le \rho_1 (s-t)^{-(l-1+\frac{1}{2})\varepsilon - 1},$$

where $\rho_1 = (l + \frac{1}{2})\varepsilon$. Assume now that the inequality

$$0 \le \frac{du^*(t)}{dt} \le \rho_k (s-t)^{-(l-k+\frac{1}{2})\varepsilon - 1}, \tag{51}$$

where ρ_k is a positive constant, holds almost everywhere on $]s-\delta_0, s[$ for some $k \in \{1, \dots, l\}$. Then

$$0 < u^*(t) \le \rho_0 + \Big(l - k + \frac{1}{2}\Big)^{-1} \varepsilon^{-1} (s-t)^{-(l-k+\frac{1}{2})\varepsilon} \le$$
$$\le \rho_{1k} (s-t)^{-(l-k+\frac{1}{2})\varepsilon} \quad \text{for} \quad s - \delta_0 \le t < s,$$

where $\rho_{1k} = \rho_0 + \big(l - k + \frac{1}{2}\big)^{-1} \varepsilon^{-1} \rho_k$. On account of this, estimates (42)–(46) yield

$$\nu_k^*(u^*)(t) \le \rho_0 + \rho_{1k} (s-t)^{-\frac{(l-k+\frac{1}{2})\varepsilon}{\lambda_k(s,\rho)}} \le$$
$$\le (\rho_0 + \rho_{1k}) (s-t)^{-\frac{(l-k+\frac{1}{2})\varepsilon}{\lambda_k(s,\rho)}},$$

$$1 + \sum_{k=1}^m \big[\nu_k^*(u^*)(t)\big]^{\lambda_k(s,\rho)} \le 1 + \sum_{k=1}^m (\rho_0 + \rho_{1k})^{\lambda_k(s,\rho)} (s-t)^{-(l-k+\frac{1}{2})\varepsilon} \le$$
$$\le \Big[1 + \sum_{k=1}^m (\rho_0 + \rho_{1k})^{\lambda_k(s,\rho)}\Big] (s-t)^{-(l-k+\frac{1}{2})\varepsilon},$$

$$\ln\Big(2 + \sum_{k=1}^m \nu_k^*(u)(t)\Big) \le \ln\Big[\Big(2 + \sum_{k=1}^m (\rho_0 + \rho_{1k})\Big)(s-t)^{-l}\Big] \le$$
$$\le \ln\Big(2 + \sum_{k=1}^m (\rho_0 + \rho_{1k})\Big) + l \ln\Big(\frac{1}{s-t}\Big) \le$$
$$\le \Big[\ln\Big(2 + \sum_{k=1}^m (\rho_0 + \rho_{1k})\Big) + l\Big] \ln\frac{1}{s-t}$$

and

$$0 < \frac{du^*(t)}{dt} \le \rho_{k+1} (s-t)^{-(l-k-1+\frac{1}{2})\varepsilon} (s-t)^{\varepsilon} \ln\frac{1}{s-t} \le$$
$$\le \rho_{k+1} (s-t)^{-(l-k-1+\frac{1}{2})\varepsilon} \delta_0^{\varepsilon} \ln\frac{1}{\delta_0} < \rho_{k+1} (s-t)^{-(l-k-1+\frac{1}{2})\varepsilon},$$

where

$$\rho_{k+1} = \alpha(s,\rho)\Big[1+\sum_{k=1}^{m}(\rho_0+\rho_{1k})^{\lambda_k(s,\rho)}\Big]\Big[\ln\Big(2+\sum_{k=1}^{m}(\rho_0+\rho_{1k})\Big)+l\Big].$$

We have thus shown by induction that inequality (51) holds almost everywhere on $]s-\delta_0, s[$ for each $k \in \{1,\dots,l+1\}$. Therefore

$$0 < \frac{du^*(t)}{dt} \le \rho_{l+1}(s-t)^{\frac{\varepsilon}{2}-1}$$

and $0 < u^*(t) < \rho^*$ for $s-\delta_0 \le t < s$, where $\rho^* = \rho_0 + \frac{2}{\varepsilon}\delta_0^{\frac{\varepsilon}{2}}$. By Corollary 3.3, the latter estimate implies $s \in I$. $\square$

The proved proposition immediately implies

Corollary 8. *Let there exist functions* $\delta :]a,b] \to]0,1[$, $c :]a,b] \to R^n$, $\alpha :]a,b] \times R_+ \to R_+$, $\beta :]a,b] \to]-1,0]$, $\lambda_k :]a,b] \to [1,+\infty[$ $(k=1,\dots,m)$ *and a number* $m_0 \in \{1,\dots,m\}$ *such that the inequalities*

$$\tau_k(t) \le s-(s-t)^{\frac{1}{\lambda_k(s)}} \quad (k=1,\dots,m_0),$$
$$\tau_k(t) \le s-\delta(s) \quad (k=m_0+1,\dots,m),$$

and

$$\begin{aligned} &f_0(t,c(s)+y_0,\dots,c(s)+y_m)\cdot \operatorname{sgn}(y_0) \le \\ &\le \alpha\Big(s,\sum_{k=m_0+1}^{m}\|y_k\|\Big)(s-t)^{\beta(s)}\Big(1+\sum_{k=1}^{m_0}\|y_k\|^{\lambda_k(s)}\Big)\times \\ &\qquad\times \ln\Big(2+\sum_{k=1}^{m_0}\|y_k\|\Big) \end{aligned} \tag{52}$$

hold respectively on $]s-\delta(s),s[$ *and* $]s-\delta(s),s[\,\times R^{(m+1)n}$. *Then each noncontinuable solution of equation* (3) *is defined on* $[a,b]$.

Example 2. [1] Let $b-a \le 1$, $\lambda \ge 1$ and $\varepsilon > 0$. Then the differential equation

$$\frac{dx(t)}{dt} = \frac{\lambda}{\varepsilon}\big|x(b-(b-t)^{\frac{1}{\lambda}})\big|^{\lambda+\varepsilon}$$

has the noncontinuable solution $x(t) = (b-t)^{-\frac{\lambda}{2}}$ defined on the interval $[a,b[$.

[1] See [3].

This example shows that the index $\lambda_k(s)$ on the right-hand side of (52) cannot be replaced by $\lambda_k(s)+\varepsilon$ for any $k \in \{1,\dots,m\}$ no matter how small $\varepsilon > 0$ is.

To conclude, note that Corollaries 1, 2, 4, and 5 are analogues of the well-known theorem of A. Wintner ([4], Ch. III, §3.5) for problems (1), (2) and (3), (2), and Corollaries 7 and 8 are generalizations of the theorem of A. Myshkis and Z. Tsalyuk [3] (see also [5]).

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