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NONCONVEX DIFFERENTIAL INCLUSIONS WITH NONLINEAR MONOTONE BOUNDARY CONDITIONS

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ABSTRACT. Existence results for problems with monotone nonlinear boundary conditions obtained in the previous publications by the author for functional differential equations are transferred to the case of nonconvex differential inclusions with the help of the selection theorem due to A. Bressan and G. Colombo.

The existence of solutions of boundary value problems for differential inclusions with possibly nonconvex right-hand sides was studied in [1–6]. The technique of continuous selections of multifunctions with decomposable values is helpful in these investigations. In particular, this technique allows us to establish a connection between the considered differential inclusion and a functional differential equation. Below we use this connection to transfer some assertions, previously proved by the author for functional differential equations [7–9], to the case of differential inclusions. Thus we obtain solvability results for problems for lower semicontinuous differential inclusions with nonlinear monotone boundary conditions. We essentially employ the selection theorem for multifunctions with decomposable values due to Bressan and Colombo [10]. A systematic account of different aspects of the theory of differential inclusions and the corresponding bibliography can be found in [11–13]. For results on boundary value problems, see e.g., [14, 15] in the case of equations and [16] in the case of convex-valued inclusions, and the references therein.

The following notation is used below. We fix a norm in the n-dimensional space $\mathbb{R}^n$ and denote it by $|\cdot|_n$. Let C^k with an integer $k \geq 0$ denote the space of k times continuously differentiable functions (C^0 is the class of all continuous functions). The space L_1 consists of all measurable integrable functions. Here and everywhere below we use the Lebesgue measure. CL_1^m

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for integer $m \geq 1$ contains all $x(\cdot) \in C^{m-1}$ such that $x^{(m-1)}(\cdot)$ is absolutely continuous, and consequently $x^{(m)}(\cdot) \in L_1$. We put

$$\| x(\cdot) \|_{CL_1^m} = \| x(\cdot) \|_{C^{m-1}} + \| x^{(m)}(\cdot) \|_{L_1}.$$

Unless otherwise stated explicitly, we assume that all considered functional spaces consist of functions from $[a,b]$ to $\mathbb{R}^n$. The interval $[a,b]$ is fixed, $-\infty < a < b < +\infty$.

Let us recall some definitions. For a set $A \subset [a,b]$ denote by $\chi_A(\cdot)$ the characteristic function of A. Thus $\chi_A(t) = 1$ if $t \in A$, and $\chi_A(t) = 0$ otherwise. A set $K \subset L_1$ is called decomposable if for any $u(\cdot)$, $v(\cdot) \in K$ and any measurable $A \subset [a,b]$ the function $w(t) = \chi_A(t)u(t) + \chi_{[a,b]\setminus A}(t)v(t)$ belongs to K. A multifunction G from X to Y, where X, Y are metric spaces, is called lower semicontinuous (l.s.c.) if for any closed $E \subset Y$ the set $\{x \in X : \ G(x) \subset E\}$ is closed.

We begin with a description of the required selection technique in the form convenient to us. Let us fix an arbitrary mapping $h : \ CL_1^m \to L_1$, an arbitrary set $V \subset CL_1^m$, and a function $\Theta : \ [a,b] \times [0,\infty) \to [0,\infty)$ such that the function $\Theta(\cdot, N)$ is measurable for any real number $N \geq 0$. Consider some l.s.c. multifunction G from C^{m-1} to L_1 with nonempty closed decomposable values. Assume that if $y(\cdot) \in G(x(\cdot))$, then the inequality

$$| \, y(t) \, |_n \leq \Theta(t, \| x(\cdot) \|_{C^{m-1}}) \tag{1}$$

holds a.e. on $[a,b]$.

Lemma. *Assume that for any continuous $g : \ C^{m-1} \to L_1$ which for all $x : \ [a,b] \to \mathbb{R}^n$, $x(\cdot) \in C^{m-1}$, satisfies the estimate*

$$| \, g(x(\cdot))(t) \, |_n \leq \Theta(t, \| x(\cdot) \|_{C^{m-1}}) \tag{2}$$

a.e. on $[a,b]$, there exists some $x(\cdot) \in V$ such that $h(x(\cdot)) = g(x(\cdot))$. Then there exists at least one $x(\cdot) \in V$ for which $h(x(\cdot)) \in G(x(\cdot))$.

Remark 1. Inequalities (1), (2) can be replaced by the relations $y(t) \in U(t, x(\cdot))$, $g(x(\cdot))(t) \in U(t, x(\cdot))$ respectively, where U is a fixed multifunction from $[a,b] \times C^{m-1}$ to $\mathbb{R}^n$ with nonempty values.

Proof of Lemma consists in a direct application of the selection theorem of A. Bressan and G. Colombo (Theorem 3 in [10]), due to which the assumptions on G ensure the existence of a continuous selection $g_0 : \ C^{m-1} \to L_1$, $g_0(x(\cdot)) \in G(x(\cdot))$. Because of (1) the mapping g_0 satisfies (2). According to the conditions of Lemma, one can find a function $x(\cdot) \in V$ such that $h(x(\cdot)) = g_0(x(\cdot)) \in G(x(\cdot))$, which completes the proof. $\square$

We can apply Lemma to a concrete problem in the following way. Take for V the set of all functions that satisfy the boundary conditions. Suppose that we know an existence result for a boundary value problem $h(x(\cdot)) =$

$g(x(\cdot))$, $x(\cdot) \in V$. Here we deal with a functional differential equation with an arbitrary continuous mapping g which satisfies the growth restrictions. Based on this result and the lemma, we can immediately obtain an existence theorem for the boundary value problem $h(x(\cdot)) \in G(x(\cdot))$, $x(\cdot) \in V$. That is, we can consider a differential inclusion with the possibly nonconvex right-hand side G satisfying the growth conditions.

In particular, the differential inclusion $h(x(\cdot)) \in G(x(\cdot))$ may have the form

$$x^{(m)}(t) + a_{m-1}(t)x^{(m-1)}(t) + \ldots + a_0(t)x(t) \in \\ \in F(t, x(t), \ldots, x^{(m-1)}(t)), \quad t \in [a, b], \tag{3}$$

where $a_0(t), \ldots, a_{m-1}(t)$ are $n \times n$ matrices, F is defined on $[a, b] \times \mathbb{R}^{mn}$ and its values are closed sets in $\mathbb{R}^n$. In that case $h(x(\cdot))(t)$ coincides with the left-hand side of (3), and $G(x(\cdot))$ is the set of all measurable selections of the multifunction $t \to F(t, x(t), \ldots, x^{(m-1)}(t))$. Conditions on F can be given that ensure the necessary properties of the corresponding multifunction G; see Proposition 2.1 in [17]. Namely, in addition to the upper estimate that guarantees (1), one should impose on F properties of measurability and of lower semicontinuity in the arguments $x(t), \ldots, x^{(m-1)}(t)$. In particular, these assumptions hold for Hausdorff continuous F.

Now, let us apply the above reasoning to the case of nonlinear monotone boundary conditions. Consider a boundary value problem

$$x^{(m)}(\cdot) \in G(x(\cdot)); \tag{4}$$

$$B_{ki}(x_i^{(k)}(\cdot), x(\cdot)) = 0, \quad k = 0, \ldots, m-1, \quad i = 1, \ldots, n. \tag{5}$$

Here $x : [a, b] \to \mathbb{R}^n$, $x(\cdot) \in CL_1^m$ is the unknown function. By $x_i^{(k)}(\cdot)$ we denote the coordinate number i of the derivative $x^{(k)}(\cdot)$. The multifunction G from C^{m-1} to L_1 is lower semicontinuous and has nonempty closed decomposable values. For some fixed integrable function $\xi(t) \geq 0$, for any $x(\cdot) \in C^{m-1}$ and $y(\cdot) \in G(x(\cdot))$, the inequality

$$| \, y(t) \, |_n \leq \xi(t) \tag{6}$$

holds a.e. on $[a, b]$. The mappings $B_{ki} : C^0 \times CL_1^m \to \mathbb{R}$ are continuous. (Here $C^0 = C^0([a, b], \mathbb{R})$ is the space of scalar functions, whereas the other employed functional spaces have values in $\mathbb{R}^n$.)

Theorem 1. *Let the nonlinear functionals B_{ki} be monotone with respect to the first argument in the following sense. For any $u(\cdot)$, $v(\cdot)$, $z(\cdot)$, if $u(t) \leq v(t)$ for all $t \in [a, b]$, then $B_{ki}(u(\cdot), z(\cdot)) \leq B_{ki}(v(\cdot), z(\cdot))$. Fix* *nondecreasing functions* $\Omega_{ki}: [0,\infty) \to [0,\infty)$. *For any indices* k, i *and function* $z(\cdot)$, *let there exist some* $u(\cdot)$ *such that* $B_{ki}(u(\cdot), z(\cdot)) = 0$ *and*

$$\| u(\cdot) \|_{C^0} \leq \Omega_{ki}(\| z^{(k+1)}(\cdot) \|_{CL_1^{m-k-1}}) \tag{7}$$

(*for* $k = m-1$ *we take* $CL_1^0 = L_1$). *Then problem* (4), (5) *has at least one solution.*

Remark 2. In fact, a more general assertion is valid. For any fixed $u(\cdot)$, $z(\cdot)$, let the function $B_{ki}(u(\cdot)+c, z(\cdot))$ be nondecreasing with respect to the real argument c. Fix mappings $\Upsilon_{ki}: [0,\infty)\times[0,\infty) \to [0,\infty)$ which are nondecreasing in each of their two arguments. For any k, i, $u(\cdot)$, $z(\cdot)$, let there exist a number c such that the equality $B_{ki}(u(\cdot)+c, z(\cdot)) = 0$ and the estimate

$$| c | \leq \Upsilon_{ki}(\| u(\cdot) \|_{C^0}, \| z^{(k+1)}(\cdot) \|_{CL_1^{m-k-1}})$$

hold. Then problem (4), (5) has at least one solution.

It is easy to check that the requirement for monotonicity of $B_{ki}(u(\cdot) + c, z(\cdot))$ with respect to the number c is weaker than the assumption of monotonicity of $B_{ki}(u(\cdot), z(\cdot))$ with respect to the function $u(\cdot)$.

Remark 3. In the assumptions of Theorem 1, let the functions $\Omega_{ki}(M) = p_{ki}M + q_{ki}$ be linear. Then estimate (6) can be replaced by the following weaker requirement. For some fixed integrable $\zeta(t) \geq 0$ and nondecreasing $\Phi: [0,\infty) \to [0,\infty)$ such that

$$\liminf_{M\to\infty} M^{-1}\Phi(M) = 0 \tag{8}$$

for any $x(\cdot) \in C^{m-1}$ and $y(\cdot) \in G(x(\cdot))$, the inequality

$$| y(t) |_n \leq \zeta(t) + \Phi(\| x(\cdot) \|_{C^{m-1}}) \tag{9}$$

holds a.e. on $[a,b]$.

In connection with Remark 3, let us specify a simple case in which functions Ω_{ki} can be chosen not only linear, but even constant. This is the case where the mappings $B_{ki}(u(\cdot), z(\cdot)) = B_{ki}(u(\cdot))$ do not depend on the second argument.

Proof of Theorem 1. Put

$$h(x(\cdot))(t) = x^{(m)}(t), \quad \Theta(t,N) = \xi(t)$$

and take for V the set of all $x(\cdot) \in CL_1^m$ that satisfy the boundary conditions (5). The application of Lemma reduces Theorem 1 to an existence result for the case of functional differential equations, which can be found in [7, 8]. $\square$

In the same way, Remarks 2 and 3 are reduced to existence results of [7, 8]. The only difference is that to prove Remark 3 we put $\Theta(t,N)=\zeta(t)+\Phi(N)$.

To demonstrate the application of Theorem 1 and Remark 3, let us employ the following boundary value problem with nonlinear integral boundary conditions:

$$x^{(m)}(\cdot)\in G(x(\cdot)); \tag{10}$$

$$\int\limits_a^b \varphi_k(x^{(k)}(\tau))d\tau=0,\quad k=0,\dots,m-1. \tag{11}$$

The unknown function $x(\cdot)\in CL_1^m([a,b],\mathbb{R})$ is scalar, G is a lower semicontinuous multifunction from $C^{m-1}([a,b],\mathbb{R})$ to $L_1([a,b],\mathbb{R})$ with nonempty closed decomposable values. We fix an integrable function $\zeta(t)\geq 0$ and numbers $\varepsilon\in(0,1)$, $A\geq 0$. Assume that the estimate

$$|\, y(t)\,|\leq \zeta(t)+A\|\, x(\cdot)\,\|_{C^{m-1}}^{1-\varepsilon}$$

holds for any $x(\cdot)\in C^{m-1}([a,b],\mathbb{R})$, $y(\cdot)\in G(x(\cdot))$ and almost all $t\in[a,b]$. For any index k, the function $\varphi_k:\ \mathbb{R}\to\mathbb{R}$ is continuous, nondecreasing, and takes the value zero at least at one point.

Corollary. *Problem* (10), (11) *admits a solution.*

Proof of Corollary is based on Theorem 1 and Remark 3. We take

$$B_k(u(\cdot),z(\cdot))=\int\limits_a^b \varphi_k(u(\tau))d\tau$$

omitting the index i in B_{ki} as $n=1$. The required continuity and monotonicity in $u(\cdot)$ hold. To ensure that $B_k(u(\cdot),z(\cdot))=0$, it suffices to put $u(t)\equiv c_k$, where c_k is a constant such that $\varphi_k(c_k)=0$. Thus in (7) the functions $\Omega_k\equiv|\, c_k\,|$ can be taken constant, and Remark 3 is applicable. We assume in (9) $\Phi(M)=AM^{1-\varepsilon}$, which makes (8) valid. Problem (10), (11) coincides with (4), (5) and is solvable. $\square$

In [7], [8] one can find another example of nonlinear integral boundary conditions of form (5), which contain some odd-degree moments of the graph of the solution and are related to the center of gravity of this graph. Now, based on [9], we will give an example with a minimum and a maximum in the boundary conditions.

Consider the problem

$$\ddot{x}(\cdot) \in G(x(\cdot)), \tag{12}$$

$$\min_t x(t) = \alpha, \tag{13}$$

$$\max_t x(t) = \beta. \tag{14}$$

The solution x is a scalar function belonging to the space $CL_1^2([a,b],\mathbb{R})$. The numbers α, β are fixed. Conditions (13), (14) specify the minimal and maximal values of the unknown function on the interval $[a,b]$. The lower semicontinuous multifunction G from $C^1([a,b],\mathbb{R})$ to $L_1([a,b],\mathbb{R})$ has nonempty closed decomposable values. Assume also that for fixed $M \geq 0$ and any $x(\cdot)$, $y(\cdot)$, if $y(\cdot) \in G(x(\cdot))$, then $| y(t) | \leq M$ a.e. on $[a,b]$.

Theorem 2. *If $\beta - \alpha > \frac{M}{8}(b-a)^2$, then problem* (12)–(14) *is solvable.*

The constant $\frac{M}{8}(b-a)^2$ in Theorem 2 is the best possible one and cannot be replaced by a smaller one, see Remark 1 in [9].

One can derive Theorem 2 from Theorem 1 after some transformation of conditions (13), (14). The transformation of (13), (14) was described in detail in [9], where it was applied to the case of ordinary differential equations. The reasoning in the case of differential inclusion (12) is quite similar. The paper [9] also gives some properties of solutions concerned with the points of a minimum and a maximum. Actually, in the proof in [9], two solutions with different properties appear.

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