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ON A SINGULAR TWO-POINT BOUNDARY VALUE PROBLEM FOR THE NONLINEAR mTH-ORDER DIFFERENTIAL EQUATION WITH DEVIATING ARGUMENTS

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Abstract. Sufficient conditions of solvability and unique solvability of the boundary value problem

$$u^{(m)}(t) = f\big(t, u(\tau_{11}(t)), \dots, u(\tau_{1k}(t)), \dots, u^{(m-1)}(\tau_{m1}(t)), \dots$$
$$\dots, u^{(m-1)}(\tau_{mk}(t))\big), \quad u(t) = 0, \ \text{ for } \ t \notin [a,b],$$
$$u^{(i-1)}(a) = 0 \quad (i = 1, \dots, m-1), \ \ u^{(m-1)}(b) = 0,$$

are established, where $\tau_{ij} : [a,b] \to R$ $(i = 1, \dots, m;\ j = 1, \dots, k)$ are measurable functions and the vector function $f :]a,b[\times R^{kmn} \to R^n$ is measurable in the first and continuous in the last kmn arguments; moreover, this function may have nonintegrable singularities with respect to the first argument.

In the present paper we consider on a segment $I = [a,b]$ the n-dimensional mth-order vector differential equation

$$u^{(m)}(t) = f\big(t, u(\tau_{11}(t)), \dots, u(\tau_{1k}(t)), \dots$$
$$\dots, u^{(m-1)}(\tau_{m1}(t)), \dots, u^{(m-1)}(\tau_{mk}(t))\big) \tag{1}$$

with the conditions

$$u(t) = 0, \quad \text{for} \quad t \notin I, \tag{2}$$
$$u^{(i-1)}(a) = 0 \ \ (i = 1, \dots, m-1), \ \ u^{(m-1)}(b) = 0, \tag{3}$$

where $k \geq 1$, $m \geq 2$, $n \geq 1$, the functions $\tau_{ij} : I \to R$ $(i = 1, \dots, m;\ j = 1, \dots, k)$ are measurable, the vector function $f :]a,b[\times R^{kmn} \to R^n$ is such that $f(\cdot, x_{11}, \dots, x_{1k}, \dots, x_{m1}, \dots, x_{mk}) : [a,b[\to R^n$ is measurable for all

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$x_{ij} \in R$ $(i = 1,\dots,m;\ j = 1,\dots,k)$, and $f(t,.,\dots,.) : R^{kmn} \to R$ is continuous for almost all $t \in]a,b[$.

Let χ_I be the characteristic function of the interval I,

$$\begin{aligned} f_0(t,x_{11},\dots,x_{1k},\dots,x_{m1},\dots,x_{mk}) = \\ = f\big(t,\chi_I(\tau_{11}(t))x_{11},\dots,\chi_I(\tau_{1k}(t))x_{1k},\dots \\ \dots,\chi_I(\tau_{m1}(t))x_{m1},\dots,\chi_I(\tau_{mk}(t))x_{mk}\big) \end{aligned} \tag{4}$$

and

$$\tau_{0ij}(t) = \begin{cases} a & \text{for } \ \tau_{ij}(t) < a \\ \tau_{ij}(t) & \text{for } \ a \le \tau_{ij}(t) \le b \\ b & \text{for } \ \tau_{ij}(t) > b \end{cases} \quad (i = 1,\dots,m;\ j = 1,\dots,k). \tag{5}$$

A vector function $u : I \to R^n$ is called a solution of problem (1), (2), (3) if: (i) it is absolutely continuous along with its derivatives up to the order $m-1$ inclusive and satisfies conditions (3), (ii) the equality

$$\begin{aligned} u^{(m)}(t) = f_0\big(t,u(\tau_{011}(t)),\dots,u(\tau_{01k}(t)),\dots,u^{(m-1)}(\tau_{0m1}(t)),\dots \\ \dots,u^{(m-1)}(\tau_{0mk}(t))\big) \end{aligned} \tag{6}$$

is fulfilled almost everywhere on I.

It should be noted that if $\tau_{ij}(t) \in I$ for $t \in I$ $(i = 1,\dots,m;\ j = 1,\dots,k)$, then equations (1) and (6) coincide, and hence condition (2) drops out.

We are principally interested in the case where the vector-function $f(\cdot,x_{11},\dots,x_{1k},\dots,x_{m1},\dots,x_{mk})$ is not summable on I for some $x_{ij} \in R^n$ $(i = 1,\dots,m;\ j = 1,\dots,k)$. In this sense, problem (1), (2), (3) is singular. The existence and uniqueness theorems proven below generalize the results of paper [1] dealing with the case where equation (1) is linear. In the case with $k = 1$ and $\tau_{i1}(t) \equiv t$ $(i = 1,\dots,m)$, problems of type (1), (3) were investigated earlier in [2–5].

Before passing to the formulation of the existence theorem, we introduce the notation which will be used throughout this paper.

R is the set of real numbers; $R_+ = [0,+\infty[$;

R^l is the space of column vectors $x = (x_i)_{i=1}^l$ with the components $x_i \in R$ $(i = 1,\dots,l)$ and the norm

$$\|x\| = \sum_{i=1}^{l} |x_i|;$$

$R^{l\times l}$ is the space of $l\times l$ matrices $X=(x_{ij})_{i,j=1}^{l}$ with the components $x_{ij}\in R$ $(i,j=1,\dots,l)$ and the norm $\|X\|=\sum_{i,j=1}^{l}|x_{ij}|$;

$$R_{+}^{l}=\big\{x=(x_i)_{i=1}^{l}\in R^{l}:x_i\geq 0\ (i=1,\dots,l)\big\};$$
$$R_{+}^{l\times l}=\big\{X=(x_{ij})_{i,j=1}^{l}\in R^{l\times l}:x_{i,j}\geq 0\ (i,j=1,\dots,l)\big\};$$

$r(X)$ is the spectral radius of the matrix $X\in R^{l\times l}$; if $x=(x_i)_{i=1}^{l}$ and $X=(x_{ij})_{i,j=1}^{l}$, then

$$|x|=(|x_i|)_{i=1}^{l},\quad |X|=(|x_{ij}|)_{i,j=1}^{l};$$

if x and $y\in R^{l}$, then

$$x\leq y\Longleftrightarrow y-x\in R_{+}^{l};$$

if X and $Y\in R^{l\times l}$, then

$$X\leq Y\Longleftrightarrow Y-X\in R_{+}^{l\times l}.$$

A matrix or a vector function is called continuous, summable, etc., if its components are such.

In what follows, by f_0 will be meant the vector function given by equality (4), and by τ_{0ij} $(i=1,\dots,m;\ j=1,\dots,k)$ the functions given by equalities (5).

Theorem 1. *Let on the set* $]a,b[\times R^{kmn}$ *the inequality*

$$\big|f_0(t,x_{11},\dots,x_{1k},\dots,x_{m1},\dots,x_{mk})\big|\leq$$
$$\leq\sum_{i=1}^{m}\sum_{l=1}^{k}\mathcal{P}_{ij}(t)|x_{ij}|+q(t)\tag{7}$$

hold, where $\mathcal{P}_{ij}:]a,b[\to R_{+}^{n\times n}$ $(i=1,\dots,m;\ j=1,\dots,k)$ *are measurable matrix functions satisfying*

$$\int\limits_a^b\big(\tau_{0ij}(t)-a\big)^{m-i}\|\mathcal{P}_{ij}(t)\|dt<+\infty\quad(i=1,\dots,m;\ j=1,\dots,k)\tag{8}$$

and $q :]a,b[\to R^n_+$ *is a summable vector function. Let, moreover, the problem*

$$|x'(t)| \le \sum_{i=1}^{m-1}\sum_{l=1}^{k}\frac{1}{(m-1-i)!}\mathcal{P}_{ij}(t)\int\limits_a^{\tau_{0ij}(t)}(\tau_{0ij}(t)-s)^{m-1-i}|x(s)|ds + \\ + \sum_{j=1}^{k}\mathcal{P}_{mj}(t)|x(\tau_{0mj}(t))|, \tag{9}$$

$$x(b) = 0 \tag{10}$$

have only the trivial solution. Then problem (1), (2), (3) *is solvable.*

Proof. For any continuous vector function $x : [a,b] \to R^n$, assume

$$\sigma_{ij}(x)(t) = \frac{1}{(m-1-i)!}\int\limits_a^{\tau_{0ij}(t)}(\tau_{0ij}(t)-s)^{m-1-i}x(s)ds \\ (i=1,\dots,m-1;\, j=1,\dots,k),$$

$$\sigma_{mj}(x)(t) = x(\tau_{0mj}(t)) \quad (j=1,\dots,k),$$

$$g(x)(t) = f_0\big(t,\sigma_{11}(x)(t),\dots,\sigma_{1k}(x)(t),\dots,\sigma_{m1}(x)(t),\dots,\sigma_{mk}(x)(t)\big)$$

and consider the functional differential equation

$$x'(t) = g(x)(t). \tag{11}$$

It is easily seen that if u is a solution of problem (1), (2), (3), then the vector function $x(t) = u^{(m-1)}(t)$ is a solution of problem (11), (10), and vice versa if x is a solution of problem (11), (10) and

$$u(t) = \frac{1}{(m-2)!}\int\limits_a^t (t-s)^{m-2}x(s)\,ds,$$

then u is a solution of problem (1), (2), (3). Therefore, to prove the theorem, it suffices to state that problem (11), (10) is solvable.

Owing to (7), for any continuous vector function $x : I \to R^n$ the inequality

$$|g(x)(t)| \le \sum_{i=1}^{m-1}\sum_{l=1}^{k}\frac{1}{(m-1-i)!}\mathcal{P}_{ij}(t)\int\limits_a^{\tau_{0ij}(t)}(\tau_{0ij}(t)-s)^{m-1-i}|x(s)|ds + \\ + \sum_{j=1}^{k}\mathcal{P}_{mj}(t)|x(\tau_{0mj}(t))| + q(t) \tag{12}$$

is fulfilled almost everywhere on I. If along with the above inequality we take into account (8) and the fact that $f_0(t,\cdot,\dots,\cdot) : R^{kmn} \to R^n$ is continuous for almost all $t \in]a,b[$, then it becomes clear that $g : C(I;R^n) \to L(I;R^n)$ is a continuous operator, where $C(I;R^n)$ and $L(I;R^n)$ are, respectively, the spaces of continuous and summable on I n-dimensional real vector functions.

By Theorem 1.1 of [6], the continuity of the operator $g : C(I;R^n) \to L(I;R^n)$, conditions (8), (12), and the absence of a nontrivial solution of problem (9), (10) ensure the solvability of problem (11), (10). □

Theorem 2. *Let on the set* $]a,b[\times R^{kmn}$ *the inequality*

$$\big|f_0(t,x_{11},\dots,x_{1k},\dots,x_{m1},\dots,x_{mk})-f_0(t,y_{11},\dots,y_{1k},\dots,y_{m1},\dots,y_{mk})\big| \le$$
$$\le \sum_{i=1}^{m}\sum_{l=1}^{k}\mathcal{P}_{ij}(t)|x_{ij}-y_{ij}| \tag{13}$$

hold, where $\mathcal{P}_{ij} :]a,b[\to R_+^{n\times n}$ $(i=1,\dots,m;\ j=1,\dots,k)$ *are measurable matrix functions satisfying conditions* (8). *Let, moreover,*

$$\int_a^b \big\|f(t,0,\dots,0)\big\|dt < +\infty \tag{14}$$

and let problem (9), (10) *have only the trivial solution. Then problem* (1), (2), (3) *has a unique solution.*

Proof. (13) and (14) imply inequality (7), where $q(t) = |f_0(t,0,\dots,0)| = |f(t,0,\dots,0)|$ and the vector function $q :]a,b[\to R_+^n$ is summable. Consequently, all the conditions of Theorem 1 are fulfilled, which ensures the solvability of problem (1), (2), (3).

To complete the proof of the theorem, it remains to show that problem (1), (2), (3) has at most one solution. Let u and v be arbitrary solutions of this problem. Assume

$$x(t) = u^{(m-1)}(t) - v^{(m-1)}(t).$$

Then

$$u^{(i-1)}(t)-v^{(i-1)}(t) = \frac{1}{(m-1-i)!}\int_a^t (t-s)^{m-1-i}x(s)ds \quad (i=1,\dots,m-1),$$

and $x(b) = 0$. Taking into account this fact and condition (13), we can conclude that x is a solution of problem (9), (10). However, according to one of the conditions of the theorem, this problem has only the trivial solution. Hence $x(t) \equiv 0$ and $u(t) \equiv v(t)$. □

To obtain effective conditions for the solvability and unique solvability of problem (1), (2), (3), we will need

Lemma 1. *Let the matrix functions* $\mathcal{P}_{ij} :]a,b[\to R_+^{n\times n}$ $(i=1,\dots,m;$ $j=1,\dots,k)$ *be measurable, satisfy conditions* (8), *and*

$$r\bigg(\sum_{i=1}^{m}\sum_{j=1}^{k}\frac{1}{(m-i)!}\int_a^b(\tau_{0ij}(t)-a)^{m-i}\,\mathcal{P}_{ij}(t)\,dt\bigg)<1. \tag{15}$$

Then problem (9), (10) *has only the trivial solution.*

Proof. Let $x=(x_i)_{i=1}^n$ be a solution of problem (9), (10),

$$\rho_i=\max\big\{|x_i(t)|:t\in I\big\}\quad\text{and}\quad\rho=(\rho_i)_{i=1}^n.$$

Then

$$|x'(t)|\le\bigg[\sum_{i=1}^{m}\sum_{j=1}^{k}\frac{1}{(m-i)!}(\tau_{0ij}(t)-a)^{m-i}\,\mathcal{P}_{ij}(t)\bigg]\rho$$

and

$$|x(t)|\le Q\rho\quad\text{for}\quad t\in I, \tag{16}$$

where

$$Q=\sum_{i=1}^{m}\sum_{j=1}^{k}\frac{1}{(m-i)!}\int_a^b(\tau_{0ij}(t)-a)^{m-i}\,\mathcal{P}_{ij}(t)\,dt.$$

Due to the definition of ρ, we find from (16) that $\rho\le Q\rho$ so that

$$(E-Q)\rho\le 0. \tag{17}$$

The matrix Q is non-negative, and by condition (15) we have $r(Q)<1$. Therefore $(E-Q)^{-1}$ is non-negative. If we multiply both parts of (17) by $(E-Q)^{-1}$, then we obtain $\rho\le 0$, which implies $\rho=0$ and $x(t)\equiv 0$. □

By virtue of the above-proven lemma, from Theorems 1 and 2 we have the following propositions.

Corollary 1. *Let on the set* $]a,b[\times R^{kmn}$ *inequality* (7) *be fulfilled, where* $\mathcal{P}_{ij} :]a,b[\to R_+^{n\times n}$ $(i=1,\dots,m;\ j=1,\dots,k)$ *are measurable matrix functions satisfying conditions* (8) *and* (15), *and let* $q :]a,b[\to R_+^n$ *be a summable vector function. Then problem* (1), (2), (3) *is solvable.*

Corollary 2. *Let on the set* $]a,b[\times R^{kmn}$ *condition* (13) *be fulfilled, where* $\mathcal{P}_{ij} :]a,b[\to R_+^{n\times n}$ $(i,j=1,\dots,n)$ *are measurable matrix functions satisfying conditions* (8) *and* (15). *If, moreover, condition* (14) *is fulfilled, then problem* (1), (2), (3) *has a unique solution.*

Remark 1. Example 1 given in [1] shows that the strict inequality (15) in Corollaries 1 and 2 cannot be replaced by a non-strict one.

Example. Let $t_{ij} \in [0,1]$, $\alpha_{ij} \geq 0$, and let $f_{ij} : R^n \to R^n$ $(i = 1,\dots,m;\ j = 1,\dots,k)$ be vector functions satisfying

$$\big|f_{ij}(x) - f_{ij}(y)\big| \leq A_{ij}|x-y| \quad (i = 1,\dots,m;\ j = 1,\dots,k),$$

where $A_{ij} \in R_+^{n\times n}$ and

$$r\bigg(\sum_{i=1}^{m}\sum_{j=1}^{k} \frac{1}{(m-i)!}\, A_{ij}\bigg) < 1.$$

Let, moreover, $q : \,]0,1[\to R^n$ be a summable vector function. Then, by Corollary 2, the problem

$$u^{(m)}(t) = \sum_{i=1}^{m}\sum_{j=1}^{k} |t-t_{ij}|^{(i-m)\alpha_{ij}} f_{ij}\big(u^{(i-1)}(|t-t_{ij}|^{\alpha_{ij}})\big) + q(t),$$
$$u^{(i-1)}(0) = 0 \quad (i = 1,\dots,m-1), \quad u^{(m-1)}(1)$$

has a unique solution.

This example shows that in the conditions of Theorems 1 and 2 as well as of Corollaries 1 and 2, the right-hand side of equation (1) may have with respect to the first argument a non-integrable singularity of any order at any point of the interval $[a,b]$.

Lemma 2. *Let* $n = 1$,

$$\tau_{ij}(t) \geq t \quad \text{for} \quad t \in I \quad (i = 1,\dots,m;\ j = 1,\dots,k), \tag{18}$$

let the functions $\mathcal{P}_{mj} : \,]a,b[\to R_+$ $(j = 1,\dots,k)$ *be summable, the functions* $\mathcal{P}_{ij} : \,]a,b[\to R_+$ $(i = 1,\dots,m-1;\ j = 1,\dots,k)$ *be measurable, and*

$$\sum_{i=1}^{m-1}\sum_{j=1}^{k} \frac{1}{(m-i)!}\int_a^b (\tau_{0ij}(t)-a)^{m-i}\mathcal{P}_{ij}(t)\,dt \leq$$
$$\leq \exp\bigg(-\sum_{j=1}^{k}\int_a^b \mathcal{P}_{mj}(t)\,dt\bigg). \tag{19}$$

Then problem (9), (10) *has only the trivial solution.*

Proof. Suppose on the contrary that problem (9), (10) has a nontrivial solution x. Then there exists a point $t_0 \in [a,b[$ such that

$$|x(t_0)| = \max\big\{|x(t)| : t \in I\big\}, \quad |x(t)| < |x(t_0)| \quad \text{for} \quad t_0 < t \leq b.$$

Assume

$$y(t)=\max\big\{|x(s)|:t\le s\le b\big\}.$$

Then

$$y(t)<y(t_0)\quad\text{for}\quad t_0<t\le b,\quad y(t)=y(t_0)\quad\text{for}\quad a\le t\le t_0. \tag{20}$$

On the other hand, taking into account (18), we find from (9) and (10) that

$$y(t)\le c_0+\sum_{j=1}^{k}\int_t^b \mathcal{P}_{mj}(s)y(s)\,ds,\quad\text{for}\quad t_0\le t\le b, \tag{21}$$

where

$$c_0=\sum_{i=1}^{m-1}\sum_{j=1}^{k}\frac{1}{(m-1-i)!}\int_{t_0}^{b}\mathcal{P}_{ij}(s)\int_a^{\tau_{0ij}(s)}\big(\tau_{0ij}(s)-\xi\big)^{m-1-i}y(\xi)d\xi ds. \tag{22}$$

Owing to Gronwall's lemma, (21) implies

$$y(t_0)\le c_0\exp\bigg(\sum_{j=1}^{k}\int_{t_0}^{b}\mathcal{P}_{mj}(s)\,ds\bigg) \tag{23}$$

and hence $c_0>0$. Since c_0 is positive, because of (22) there obviously exists a set of positive measure $I_0\subset[t_0,b]$ such that

$$\sum_{i=1}^{m-1}\sum_{j=1}^{k}\frac{1}{(m-1-i)!}\mathcal{P}_{ij}(t)>0\quad\text{for}\quad t\in I_0.$$

If along with this fact we take into consideration conditions (18) and (20), then from (22) we find

$$c_0<y(t_0)\sum_{i=1}^{m-1}\sum_{j=1}^{k}\frac{1}{(m-i)!}\int_{t_0}^{b}\big(\tau_{0ij}(s)-a\big)^{m-i}\mathcal{P}_{ij}(s)\,ds.$$

Due to this inequality and condition (19), (23) yields

$$\begin{aligned}y(t_0)<y(t_0)\exp\bigg(\sum_{j=1}^{k}\int_{t_0}^{b}\mathcal{P}_{mj}(s)\,ds\bigg)\times\\ \times\sum_{i=1}^{m-1}\sum_{j=1}^{k}\frac{1}{(m-i)!}\int_{t_0}^{b}\big(\tau_{0ij}(s)-a\big)^{m-i}\mathcal{P}_{ij}(s)\,ds\le y(t_0).\end{aligned}$$

The obtained contradiction proves the lemma. □

By virtue of Lemma 2, Theorems 1 and 2 result in the following assertions.

Corollary 3. *Let* $n = 1$, $\tau_{ij}(t) \geq t$ *for* $t \in I$ $(i = 1, \dots, m;\ j = 1, \dots, k)$, *and let on the set* $]a, b[\times R^{km}$ *condition* (7) *be fulfilled, where* $\mathcal{P}_{mj} :]a, b[\to R_+$ $(j = 1, \dots, k)$ *and* $q :]a, b[\to R_+$ *are summable functions, while* $\mathcal{P}_{ij} :]a, b[\to R_+$ $(i = 1, \dots, m-1;\ j = 1, \dots, k)$ *are measurable functions satisfying inequality* (19). *Then problem* (1), (2), (3) *is solvable.*

Corollary 4. *Let* $n = 1$, $\tau_{ij}(t) \geq t$ *for* $t \in I$ $(i = 1, \dots, m;\ j = 1, \dots, k)$ *and let on the set* $]a, b[\times R^{km}$ *condition* (13) *be fulfilled, where* $\mathcal{P}_{mj} :]a, b[\to R_+$ $(j = 1, \dots, k)$ *are summable functions, while* $\mathcal{P}_{ij} :]a, b[\to R_+$ $(i = 1, \dots, m-1;\ j = 1, \dots, k)$ *are measurable functions satisfying inequality* (19). *If, moreover,*

$$\int_a^b \big|f(t, 0, \dots, 0)\big|\, dt < +\infty,$$

then problem (1), (2), (3) *has a unique solution.*

Remark 2. Example 3 given in [1] shows that condition (19) in Corollaries 3 and 4 is optimal because for no $i_0 \in \{1, \dots, m-1\}$ and $\varepsilon \in]0, 1[$ can it be replaced by the condition

$$\sum_{i=1}^{m-1} \sum_{j=1}^{k} \frac{\gamma_i}{(m-i)!} \int_a^b \big(\tau_{0ij}(t) - a\big)^{m-i} \mathcal{P}_{ij}(t)\, dt \leq \exp\bigg(-\sum_{j=1}^{k} \int_a^b \mathcal{P}_{ij}(t)\, dt\bigg),$$

where $\gamma_{i_0} = 1 - \varepsilon$ and γ_i $(i \neq i_0;\ i = 1, \dots, m-1)$ are arbitrarily large positive numbers.

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