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TO THE PROBLEM OF A STRONG DIFFERENTIABILITY OF INTEGRALS ALONG DIFFERENT DIRECTIONS

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Abstract. It is proved that for any given sequence $(\sigma_n, n \in \mathbb{N}) = \Gamma_0 \subset \Gamma$, where Γ is the set of all directions in $\mathbb{R}^2$ (i.e., pairs of orthogonal straight lines) there exists a locally integrable function f on $\mathbb{R}^2$ such that: (1) for almost all directions $\sigma \in \Gamma \backslash \Gamma_0$ the integral $\int f$ is differentiable with respect to the family $B_{2\sigma}$ of open rectangles with sides parallel to the straight lines from σ; (2) for every direction $\sigma_n \in \Gamma_0$ the upper derivative of $\int f$ with respect to $B_{2\sigma_n}$ equals $+\infty$; (3) for every direction $\sigma \in \Gamma$ the upper derivative of $\int |f|$ with respect to $B_{2\sigma}$ equals $+\infty$.

§ 1. Statement of the problem. Formulation of the main result

Let $B(x)$ be a differentation basis at the point $x \in \mathbb{R}^n$ (see [1]). The family $\{B(x) : x \in \mathbb{R}^n\}$ is called a differentiation basis in $\mathbb{R}^n$.

For $f \in L_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ let us denote respectively by $\overline{D}_B(f)(x)$ and $\underline{D}_B(f)(x)$ the upper and the lower derivative of the integral $\int f$ with respect to B at x [1]. When these two derivatives are equal their common value is denoted by $D_B(f)(x)$ and the basis B is said to differentiate $\int f$ if the relation $D_B(f)(x) = f(x)$ holds almost everywhere.

Let B_2 denote the differentiation basis in $\mathbb{R}^n$ consisting of all n-dimensional open intervals, and $B_2(x)$ be the family of sets from B_2 containing x.

Let σ be the union of n mutually orthogonal straight lines in $\mathbb{R}^n$ $(n \geq 2)$ which intersect at the origin. The set of such unions will be denoted by $\Gamma(\mathbb{R}^n)$. Elements of this set will be called directions. Note that $\Gamma(\mathbb{R}^2)$ corresponds in the one-to-one manner to the interval $[0, \frac{\pi}{2})$ (see [2]).

For a fixed direction σ we denote by $B_{2\sigma}$ the differentiation basis in $\Gamma(\mathbb{R}^n)$ which is formed by all n-dimensional open rectangles with the sides parallel

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to the straight lines from σ. If $B_{2\sigma}$ differentiates $\int f$ at x, then the integral $\int f$ is said to be strongly differentiable with respect to σ at x.

The following problem was proposed by Zygmund (see [1], Ch. IV): Given a function $f \in L(\mathbb{R}^2)$, is it possible to choose a direction σ such that $\int f$ would be strongly differentiable with respect to σ?

Let $W(\mathbb{R}^n)$ $(n \geq 2)$ denote a class of locally integrable functions on $\mathbb{R}^n$ whose strong upper derivatives $\overline{D}_{B_{2\sigma}}(f)(x)$ are equal to $+\infty$ almost everywhere along each fixed direction σ. When solving Zygmund's problem, Marstrand [3] showed that the class $W(\mathbb{R}^2)$ is not empty, and thus his answer to the above stated problem was negative. A stronger result was obtained by López Melero [4] and Stokolos [5].

In connection with Zygmund's problem we had the following question [2]: Given a pair of directions σ_1 and σ_2 differing from each other, does there exist an integrable function f such that the integral $\int f$ is strongly differentiable a.e. with respect to σ_1 and strongly differentiable with respect to σ_2 on the null set only? Theorems 1 and 2 from [2] give a positive answer to this question.

It is known ([1], Ch. III) that if $\int |f|$ is strongly differentiable almost everywhere, then the same holds for $\int f$. Papoulis [6] showed that the converse proposition does not hold in general. Namely, there exists an integrable function f on $\mathbb{R}^2$ such that the integral $\int f$ is strongly differentiable almost everywhere, while $\int |f|$ is strongly differentiable on the null set only. Zerekidze [7] has obtained a stronger result from which it follows that for every function f from $W(\mathbb{R}^n)$ there exists a measurable function g such that $|f| = |g|$ and $\int g$ is strongly differentiable almost everywhere along all directions. In other words, changing the sign of the function on some set, we can improve the differentiation properties of the integral in all directions.

There arises a question whether the following alternative holds: Given function f from $W(\mathbb{R}^2)$, can the differentiation properties of the integral $\int f$ after changing the sign of the function be improved in all directions or they do not improve in none of them?

The following theorem gives a negative answer to this question and strengthens the results of Papoulis [6] and Marstrand [3].

Theorem. *Let the sequence of directions $(\sigma_n)_{n=1}^{\infty}$ be given. There exists a locally integrable function f on $\mathbb{R}^2$ such that:*

(1) *for almost all directions σ $(\sigma \neq \sigma_n,\ n \in \mathbb{N})$,*

$$D_{B_{2\sigma}}(f)(x) = f(x) \quad a.e.;$$

(2) *for every direction σ_n $(n \in \mathbb{N})$,*

$$\overline{D}_{B_{2\sigma_n}}(f)(x) = +\infty \quad a.e.;$$

(3) *for every direction* σ,

$$\overline{D}_{B_{2\sigma}}(|f|)(x) = +\infty \quad a.e.$$

Remark. If the sequence $(\sigma_n)_{n=1}^{\infty}$ consists of a finite number of directions, then in item (1) instead of "for almost all directions" it should be written "for all directions".

Corollary. *There is a function* $f \in L_{loc}(\mathbb{R}^2)$ *such that:*

(a) *the integral* $\int |f|$ *is strongly differentiable a.e. in none of the directions;*

(b) *for almost all irrational directions the integral* $\int f$ *is strongly differentiable a.e., while for the rational directions it is strongly differentiable on the null set only.*

§ 2. Auxiliary Assertions. Proof of the Main Result

Before passing to the formulation of auxiliary assertions let us introduce some notation and definitions.

For the set G, $G \subset \mathbb{R}^2$, ∂G is assumed to be the boundary of the set G and $\overline{G}$ its closure. By E we denote the unit square in $\mathbb{R}^2$.

Given a natural number n, let us construct two collections of straight lines: $x = en^{-1}$ and $y = en^{-1}$, $e = 0, 1, \dots, n$, which define the rectangular net E^n in the unit square E and divide it into open square intervals E_k^n, $k = 1, 2, \dots, n^2$, with sides of length n^{-1}.

For the rectifiable curve c denote by $d(c)$ its length.

The set of measurable functions on $\mathbb{R}^n$ taking only the values -1 and 1 will be denoted by $S(\mathbb{R}^n)$.

For the measurable set G, $G \subset \mathbb{R}^2$, the number λ, $0 < \lambda < 1$, and the direction σ, denote by $H^{\sigma}(\chi_G, \lambda)$ the union of all those open rectangles R from $B_{2\sigma}$ for which

$$|R|^{-1} \int_R \chi_G(y)dy \geq \lambda,$$

where χ_G is the characteristic function of the set G. If, moreover, σ is a standard direction, then the set $H^{\sigma}(\chi_G, \lambda)$ will be denoted by $H(\chi_G, \lambda)$.

Furthermore, for the interval $I = (0, e_1) \times (0, e_2)$ and the numbers λ and c $(0 < \lambda < 1,\ 1 < c < \infty)$ we define the interval $Q(I, \lambda, c)$ as follows:

$$Q(I, \lambda, c) = \left[-c\lambda^{-1}e_1, (1 + c\lambda^{-1})e_1 \right] \times \left[-e_2, 2e_2 \right].$$

Let σ be a fixed direction and let $f \in L_{loc}(\mathbb{R}^n)$. In the present work we consider the following maximal Hardy–Littlewood functions:

$$M_{B_{2\sigma}} f(x) = \sup_{R \in B_{2\sigma}(x)} |R|^{-1} \int_R |f(y)| dy,$$

$$M^*_{B_{2\sigma}} f(x) = \sup_{R \in B_{2\sigma}(x)} |R|^{-1} \Big| \int_R f(y) dy \Big|.$$

The validity of the following two assertions can be easily verified.

Lemma 1. *Let $0 < \varepsilon < 1$ and $n_\varepsilon = 9\varepsilon^{-1}$. Let, moreover, c be a continuous rectifiable curve in the unit square E and let $d(c) < 1$. Then for every natural number n, $n \geq n_\varepsilon$, the following relation holds:*

$$\Big| \bigcup_{k \in \tau_n} E_k^n \Big| \leq \varepsilon,$$

where τ_n is a collection of those natural numbers for which the square E_k^n from the rectangular net E^n intersects with the curve c.

Lemma 2. *Let σ_1 be a fixed direction from $\Gamma(\mathbb{R}^2)$. Let I^{σ_1} be a rectangle from $B_{2\sigma_1}$ and B be a circle (on the plane). Then:*
(1) $\big|H^{\sigma_1}(\chi_{I^{\sigma_1}}, \lambda)\big| > \lambda^{-1} \ln(\lambda^{-1}) |I^{\sigma_1}|, \quad 0 < \lambda < 1;$
(2) *for every direction σ,*

$$\big|H^{\sigma}(\chi_B, \lambda)\big| > \lambda^{-1} \ln(\lambda^{-1}) |B|, \quad 0 < \lambda < 2^{-1}.$$

Lemma 3 (Zerekidze [7]). *Let $\varepsilon > 0$. There exists a function $s \in S(\mathbb{R}^n)$ such that*

$$\Big| \int_\gamma s(x) d\eta \Big| < \varepsilon,$$

where γ is an arbitrary interval in $\mathbb{R}^n$ and $d\eta$ is the Lebesgue linear measure on γ.

Lemma 4 ([2]). *Let $I = (0, e_1) \times (0, e_2)$ and let σ be an arbitrary non-standard direction from $\Gamma(\mathbb{R}^2)$. There exists a number $c(\sigma)$, $1 < c(\sigma) < \infty$, such that for every λ, $0 < \lambda < 1$,*

$$H^\sigma(\chi_I, \lambda) \subset Q(I, \lambda, c(\sigma)).$$

If, moreover, $c(\sigma)\lambda^{-1} e_1 \leq e_2$, then

$$\big|H^\sigma(\chi_I, \lambda)\big| \leq 9c(\sigma)\lambda^{-1}|I|.$$

Proof of the theorem. The proof of the theorem is divided into several parts.

1°. We define some auxiliary sets.

For any natural $n \geq 2$ denote

$$\Gamma^n = \{\sigma \in \Gamma(\mathbb{R}^2) : 0 \leq \alpha(\sigma) < 2^{-(n+1)} \cdot n^{-1}\} \cup$$
$$\cup \{\sigma \in \Gamma(\mathbb{R}^2) : \pi 2^{-1} - 2^{-(n+1)} \cdot n^{-1} < \alpha(\sigma) < \pi 2^{-1}\},[1]$$
$$c_n = \sup\{c(\sigma) : \sigma \in \Gamma(\mathbb{R}^2) \backslash \Gamma^n\},$$
$$\beta_n = \max\left\{\exp(c_n n^2 2^{2n}); 2\Big(\sum_{n_1=2}^{n-1} n_1 \beta_{n_1} + \sum_{n_1=2}^{n-1} \lambda_{n1}\Big)\right\}, \tag{1}$$
$$\lambda_n = 2\Big(n\beta_n + \sum_{n_1=2}^{n-1} (n_1 \beta_{n_1} + \lambda_{n_1})\Big). \tag{2}$$

Let $I^n = (0, e_1^n) \times (0, e_2^n)$ be the interval for which

$$e_2^n = c_n n 2^n \beta_n e_1^n.$$

Assume

$$Q^n = Q(I^n, (n 2^n \beta_n)^{-1}, c_n).$$

Denote by Q^{*n} the interval with the same center of symmetry as Q^n but with edges four times larger. Next, for $e = 1, 2, \dots, n$ denote by I_e^n, Q_e^n, and Q_e^{*n} those rectangles from $B_{2\sigma_e}$ which are obtained from the intervals I^n, Q^n, and Q^{*n} by rotation with respect to the center of symmetry (the centers of symmetry of the intervals I^n, Q^n, and Q^{*n} coincide).

Assume

$$H_e^n = H^{\sigma_e}(\chi_{I_e^n}, \beta_n^{-1}) \cup Q_e^{*n}, \quad e = 1, 2, \dots, n,$$
$$a_2 = 1/2, \quad a_n = r_{n-1}(M_{m_{n-1}}^{n-1})^{-1}, \quad n > 2.$$

The sets H_e^n $(e = 1, 2, \dots, n)$ are compact. We use Lemma 1.3 from [1] and cover almost the whole unit square E by the sequence of non-intersecting sets H_{1j}^n, $j = 1, 2, \dots$, homothetic to H_1^n such that all sets H_{1j}^n are contained in the unit square and have a diameter less than a_n. By applying a similar treatment to the sets H_e^n, $e = 2, \dots, n$, we obtain

$$\Big|E \backslash \bigcup_{j=1}^{\infty} H_{ej}^n\Big| = 0, \ \ \operatorname{diam}(H_{ej}^n) \leq a_n, \ \ e = 1, 2, \dots, n, \ \ j = 1, 2, \dots. \tag{3}$$

Let P_{ej}^n $(e = 1, 2, \dots, n, j = 1, 2, \dots)$ denote the homothety transforming the set H_e^n to H_{ej}^n. Assume

$$I_{ej}^n = P_{ej}^n(I_e^n), \quad Q_{ej}^n = P_{ej}^n(Q_e^n) \quad Q_{ej}^{*n} = P_{ej}^n(Q_e^{*n}).$$

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[1]For the direction σ, the number $0 \leq \alpha < \frac{\pi}{2}$ is defined as the angle between the positive direction of the axis ox and the straight line from σ lying in the first quadrant of the plane.

Denote by b_n the circle with center at the point $(2^{-1},2^{-1})$ and of radius r_n, $0<r_n<1$. Choose a number r_n so small that the conditions $r_n<r_{n-1}$ and $H(\chi_{b_n},\lambda_n^{-1})\subset E$ are fulfilled.

For the direction σ denote the set $H^\sigma(\chi_{b_n},\lambda_n^{-1})$ by $h^n(\sigma)$ and for the standard direction use the notation $h^n=H(\chi_{b_n},\lambda_n^{-1})$. Let m_n be a fixed natural number satisfying the condition

$$1\le m_n|h^n|\le 2. \tag{4}$$

Let $M_1^n,M_2^n,\dots,M_{m_n}^n$ be a collection of natural numbers such that $M_1^n<M_2^n<\dots<M_{m_n}^n$. Let us consider the rectangular nets $E^{M_1^n}$, $E^{M_2^n},\dots,E^{M_{m_n}^n}$. Denote by q_{ki}^n $(k=1,2,\dots,m_n,\ i=1,2,\dots,(M_k^n)^2)$ the homothety transforming the unit square E to the square $E_i^{M_k^n}$ from the rectangular net $E^{M_k^n}$. Assume $(\sigma\in\Gamma(\mathbb{R}^2))$,

$$\begin{aligned}
B_{ki}^n&=q_{ki}^n(b_n),\quad h_{ki}^n(\sigma)=q_{ki}^n(h^n(\sigma)),\\
G_k^n(\sigma)&=\bigcup_{i=1}^{(M_k^n)^2}h_{ki}^n(\sigma),\\
\Omega_k^2(\sigma)&=\bigcup_{k_1=1}^{k-1}G_{k_1}^2(\sigma),\quad k>1,\\
\Omega_k^n(\sigma)&=\bigcup_{n_1=2}^{n-1}\bigcup_{k_1=1}^{m_{n_1}}G_{k_1}^{n_1}(\sigma)\cup\bigcup_{k_2=1}^{k-1}G_{k_2}^n(\sigma),\quad n>2,\\
\theta_k^2&=\sum_{k_1=1}^{k-1}(M_{k_1}^2)^2,\quad k>1,\\
\theta_k^n&=\sum_{n_1=2}^{n-1}\sum_{k_1=1}^{m_{n_1}}(M_{k_1}^{n_1})^2+\sum_{k_2=1}^{k-1}(M_{k_2}^n)^2,\quad n>2,\\
w_k^n&=2r_n^{-1}M_{k-1}^n,\quad k>1,\\
\omega_k^n&=2\theta_k^n\sup_{\sigma\in\Gamma(\mathbb{R}^2)}\big\{|E\backslash\Omega_k^n(\sigma)|^{-1}\big\}.
\end{aligned}$$

Choose numbers M_k^n, $k=1,2,\dots,m_n$, increasing so rapidly that the relation

$$M_k^n\ge\max\big\{2\eta_n^{-1};d(\partial h^n);9\omega_k^n;M_{m_{n-1}}^{n-1};k^n\big\} \tag{5}$$

is fulfilled.

2°. We shall construct the function sought for.

Let B_{ki}^{*n} be the circle with the same center as B_{ki}^n and a twofold larger radius. Assume

$$A_1^n=\big\{1,2,\dots,(M_1^n)^2\big\},$$

$$A_k^n = \left\{ i : E_i^{M_k^*} \cap \left(E \setminus \Big(\bigcup_{k_1=1}^{k-1} \bigcup_{i_1=1}^{(M_{k_1}^n)^2} B_{k_1 i_1}^{*n} \Big) \right) \neq \varnothing \right\} \ (k = 2, \dots, m_2).$$

Let $S_n \in S(\mathbb{R}^2)$. The functions ψ_n, g_n, and f_n will be defined for $n = 2, 3, \dots,$ as follows:

$$\psi_n(x) = \beta_n \sum_{e=1}^{n} \sum_{j=1}^{N_n} \chi_{I_{ej}^n}(x), \quad g_n(x) = \lambda_n S_n(x) \sum_{k=1}^{m_n} \sum_{i \in A_k^n} \chi_{B_{ki}^n}(x),$$
$$f_n(x) = g_n(x) + \psi_n(x).$$

The index function is defined as the series $f(x) = \sum_{n=2}^{\infty} f_n(x)$.

Let us prove that $f \in L(\mathbb{R}^2)$. We have $\|f\|_1 \le \sum_{n=2}^{\infty} \big(\|\psi_n\|_1 + \|g_n\|_1\big)$.

First we estimate $\|\psi_n\|_1$. Using Lemma 2(a) and formula (1), we obtain

$$\|\psi_n\|_1 \le \ln^{-1}(\beta_n) \sum_{e=1}^{n} \sum_{j=1}^{\infty} \beta_n \ln(\beta_n) |I_{ej}^n| \le$$
$$\le \ln^{-1}(\beta_n) \sum_{e=1}^{n} \Big| \bigcup_{j=1}^{\infty} H_{ej}^n \Big| = n \ln^{-1}(\beta_n) < 2^{-n}.$$

Similarly, applying Lemma 2(b) and relations (1), (2), (4) we have

$$\|g_n\|_1 \le \ln^{-1}(\lambda_n) \Big(\sum_{k=1}^{m_n} \sum_{i=1}^{(M_k^n)^2} \lambda_n \ln(\lambda_n) |B_{k_i}^n| \Big) \le$$
$$\le \ln^{-1}(\lambda_n) m_n |h_n| < 2^{-n}.$$

The two latter relations yield the desired inclusion.

3°. Here we shall prove that for almost all directions σ ($\sigma \neq \sigma_n$, $n \in \mathbb{N}$) the integral $\int f$ is strongly differentiable a.e.

(a) Let us first estimate the maximal function $M_{B_{2\sigma}}^* g_n$. Introduce the sets

$$B^n = \operatorname{supp}(g_n) = \bigcup_{k=1}^{m_n} \bigcup_{i \in A_k^n} B_{ki}^n, \quad B^{*n} = \bigcup_{k=1}^{m_n} \bigcup_{i \in A_k^n} B_{ki}^{*n}$$

and let $\rho_n = \sum_{k=1}^{m_n} (M_k^n)^2$.

By Lemma 3 we can assume that

$$\sup_{\gamma} \Big| \int_{\gamma} s_n(x) d\eta \Big| \le r_n (2^{n+1} \lambda_n \rho_n M_{m_n}^n)^{-1}, \tag{6}$$

where γ is an arbitrary interval of an arbitrary straight line in $\mathbb{R}^2$ and $d\eta$ is the Lebesgue measure on γ.

Let us show that for every direction σ and for all x from $\mathbb{R}^2 \backslash B^{*n}$ the inequality

$$M^*_{B_{2\sigma}} g_n(x) \le 2^{-n}, \quad n = 2, 3, \dots . \tag{7}$$

is fulfilled. This inequality will be proved only for the case where σ is a standard direction, since the general case has a similar proved.

Let us fix a natural number n $(n \ge 2)$, a point x from $\mathbb{R}^2 \backslash B^{*n}$, and a interval R from $B_2(x)$. We assume that $R \cap B^n \ne \varnothing$ and (k_1, i) $(1 \le k_1 \le m_n$, $1 \le i \le (M^n_{k_1})^2)$ is a pair of natural numbers for which

$$R \cap B^n_{k_1 i_1} \ne \varnothing. \tag{8}$$

From the inclusion $x \in \mathbb{R}^2 \backslash B^{*n}$ we have

$$\operatorname{dist}(x, B^n_{k_1 i_1}) \ge \operatorname{dist}(\partial B^{*n}_{k_1 i_1}, B^n_{k_1 i_1}) \ge r_n (M^n_{k_1})^{-1} \ge r_n (M^n_{m_n})^{-1}.$$

Taking also the inclusion $x \in R$ and (8) into consideration, we get

$$\operatorname{diam}(R) \ge r_n (M^n_{m_n})^{-1}.$$

Let $R = R_1 \times R_2$. It follows from the last relation that at least for one p $(p = 1, 2)$ the length of the interval R_p is underestimated as follows:

$$|R_p|_1 \ge 2^{-1} r_n (M^n_{m_n})^{-1}. \tag{9}$$

Without loss of generality we assume that $p = 1$. We have (see (9), (6))

$$|R|^{-1} \Big| \int\limits_R g_n(y)dy \Big| \le \lambda_n |R|^{-1} \sum_{k=1}^{m_n} \sum_{i \in A^n_k} \Big| \int\limits_R s_n(y) \chi_{B^n_{ki}}(y) dy \Big| \le$$

$$\le \lambda_n |R|^{-1} \sum_{k=1}^{m_n} \sum_{i \in A^n_k} \int\limits_{R_2} \Big| \int\limits_{R_1} s_n(y_1, y_2) \chi_{R_1}(y_1, y_2) \chi_{B^n_{ki}}(y_1, y_2) dy_1 \Big| dy_2 \le$$

$$\le \lambda_n |R_1|^{-1} \rho_n \sup_\gamma \Big| \int\limits_\gamma s_n(y) d\eta \Big| \le 2^{-n}.$$

To complete the proof of relation (7) it remains to note that $R \in B_2(x)$ and is arbitrary.

Let us now show that $|B^*| = 0$, where $B^* = \lim_{n \to \infty} \sup B^{*n}$.

Indeed, using relations (4) and Lemma 2 (b), we obtain

$$\sum_{n=2}^{\infty} |B^{*n}| \le \sum_{n=2}^{\infty} \beta_n^{-1} \ln^{-1}(\beta_n) \sum_{k=1}^{m_n} \sum_{i=1}^{(M^n_k)^2} \lambda_n \ln(\lambda_n) |B^{*n}_{ki}| \le$$

$$\le 4 \sum_{n=2}^{\infty} \beta_n^{-1} \ln^{-1}(\beta_n) \sum_{k=1}^{m_n} |h^n| \le 8 \sum_{n=2}^{\infty} \beta_n^{-1} \ln^{-1}(\beta_n) < \infty.$$

Thus $|B^*| = 0$, and hence for every $x \in \mathbb{R}^2 \backslash B^*$ there exists a number $p_1(x)$ such that

$$x \in \mathbb{R}^2 \backslash B^{*n} \quad \text{for} \quad n \geq p_1(x). \tag{10}$$

This and inequality (7) imply that for every direction σ and for all x from $\mathbb{R}^2 \backslash B^*$ the following relation is fulfilled:

$$M^*_{B_{2\sigma}} g_n(x) \leq 2^{-n} \quad \text{for} \quad n \geq p_1(x). \tag{11}$$

(b) We will now proceed to the estimation of $M_{B_{2\sigma}} \psi_n$. Taking into account Lemma 4, we find that each one of the following inclusions are fulfilled:

$$H^\sigma(\chi_{I^n}, (n2^n\beta_n)^{-1}) \subset Q(I^n, (n2^n\beta_n)^{-1}, c(\sigma)) \subset Q^n$$
$$\text{for} \quad \sigma \in \Gamma(\mathbb{R}^2) \backslash \Gamma^n.$$

Without loss of generality, we assume that every direction σ_n, $n \in \mathbb{N}$, is not standard. Suppose $(k = 1, 2, \dots, n)$,

$$\Gamma^n_k = \big\{\sigma \in \Gamma(\mathbb{R}^2); |\alpha(\sigma) - \alpha(\sigma_k)| < 2^{-(n+1)} n^{-1}\big\}.$$

Since the rotation is a measure-preserving transformation and the centers of symmetry of the intervals I^n and Q^n coincide, from the previous inclusion it follows (by virtue of the homothety properties) that the following inclusions hold:

$$H^\sigma(\chi_{I^n_{ki}}, (n2^n\beta_n)^{-1}) \subset Q^n_{ki} \quad \text{for} \quad \sigma \in r(\mathbb{R}^2) \backslash \bigcup_{e=1}^{n} \Gamma^n_e, \tag{12}$$
$$k = 1, 2, \dots, n, \quad j = 1, 2, \dots.$$

The definition of the rectangles Q^n_{ej} and Q^{*n}_{ej} immediately implies that for every direction σ there exists a rectangle $E^n_{ej}(\sigma) \in B_{2\sigma}$ possessing the property

$$Q^n_{ej} \subset E^n_{ej}(\sigma) \subset Q^{*n}_{ej}. \tag{13}$$

We have $|Q^{*n}_{ej}| = 16|Q^n_{ej}| \leq 144 c_n \beta_n 2^n n |I^n_{ej}|$. On the other hand, by Lemma 2 (a) we have

$$\big|H^{\sigma_e}(\chi_{I^n_{ej}}, \beta_n^{-1})\big| \geq \beta_n \ln(\beta_n)|I^n_{ej}| \geq c_n n^2 2^{2n} \beta_n |I^n_{ej}|.$$

The last two relations imply

$$\sum_{n=2}^{\infty} \Big| \bigcup_{e=1}^{n} \bigcup_{j=1}^{\infty} Q^{*n}_{ej} \Big| \leq 144 \sum_{n=2}^{\infty} n^{-1} 2^{-n} \sum_{e=1}^{n} \sum_{j=1}^{\infty} |H^n_{ej}| =$$
$$= 144 \sum_{n=2}^{\infty} n^{-1} 2^{-n} \sum_{e=1}^{n} \Big| \bigcup_{j=1}^{\infty} H^n_{ej} \Big| \leq 144 \sum_{n=2}^{\infty} 2^{-n} < \infty.$$

Hence $|Q^*|=0$, where $Q^*=\lim_{n\to\infty}\sup \bigcup\limits_{e=1}^{n}\bigcup\limits_{j=1}^{\infty} Q_{ej}^{*n}$.

This in turn implies that for all points x from $E\backslash Q^*$ there exists a number $P_2(x)$ such that

$$x\in E\backslash \bigcup_{e=1}^{n}\bigcup_{j=1}^{\infty} Q_{ej}^{*n} \quad \text{for} \quad n\geq P_2(x). \tag{14}$$

Further we have

$$\Big|\bigcup_{e=1}^{n}\Gamma_e^n\Big|\leq \sum_{e=1}^{n}|\Gamma_e^n|=\sum_{e=1}^{n}n^{-1}2^{-n}=2^{-n}.$$

Consequently $|\Gamma|=0$, where $\Gamma=\lim_{n\to\infty}\sup \bigcup\limits_{e=1}^{n}\Gamma_e^n$.

This implies that for every direction σ from $\Gamma(\mathbb{R}^2)\backslash\Gamma$ there exists a number $n(\sigma)$ such that

$$\sigma\in\Gamma(\mathbb{R}^2)\backslash \bigcup_{e=1}^{n}\Gamma_e^n \quad \text{for} \quad n\geq n(\sigma). \tag{15}$$

Now let us show that if $\sigma\in\Gamma(\mathbb{R}^2)\backslash\Gamma$ and $x\in E\backslash Q^*$, then

$$M_{B_{2\sigma}}\psi_n(x)\leq 2^{-n} \quad \text{for} \quad n\geq P_2(x,\sigma), \tag{16}$$

where $P_2(x,\sigma)=\max\big\{P_2(x);n(\sigma)\big\}$.

Indeed, let us fix a direction σ from $\Gamma(\mathbb{R}^2)\backslash\Gamma$, a point x from $E\backslash Q^*$ and a rectangle R^σ from $B_{2\sigma}(x)$. Let n be a fixed natural number and $n\geq P_2(x,\sigma)$. Since $\sigma\in\Gamma(\mathbb{R}^2)\backslash\Gamma$ and $n\geq P_2(x,\sigma)\geq n(\sigma)$, it follows from (12), (13), and (15) that for all e, j ($e=1,2,\dots,n$, $j=1,2,\dots$) the chain of inclusions

$$H^\sigma(\chi_{I_{ej}^n},(n2^n\beta_n)^{-1})\subset Q_{ej}^n\subset E_{ej}^n(\sigma)\subset Q_{ej}^{*n} \tag{17}$$

is fulfilled. Since $x\in E\backslash Q^*$ and $n\geq P_2(x,\sigma)\geq P_2(x)$, from (14) and (13) we have

$$x\in E\backslash \bigcup_{e=1}^{n}\bigcup_{j=1}^{\infty} Q_{ej}^{*n}\subset E\backslash \bigcup_{e=1}^{n}\bigcup_{j=1}^{\infty} E_{ej}^n(\sigma).$$

Let $\{j_1,\dots,j_{s_e}\}$ ($e=1,2,\dots,n$) be a set of natural numbers for which $|R^\sigma\cap I_{ej_i}^n|>0$, $i=1,2,\dots,s_e$. If we observe that the set $R^\sigma\cap E_{ej_i}^n$ ($i=1,2,\dots,s_e$, $e=1,2,\dots,n$) is a rectangle from $B_{2\sigma}$ containing at least one point from $E\backslash H^\sigma(\chi_{I_{ej}^n},(n2^n\beta_n)^{-1})$ (see (17)), then we obtain

$$\begin{aligned}&\big|R^\sigma\cap E_{ej_i}^n(\sigma)\big|^{-1}\int\limits_{R^\sigma\cap E_{ej_i}^n(\sigma)}\chi_{I_{ej_i}^n}(y)dy=\\&=\big|R^\sigma\cap E_{ej_i}^n(\sigma)\big|^{-1}\big|R^\sigma\cap I_{ej_i}^n\big|\leq(n2^n\beta_n)^{-1},\end{aligned}$$

and consequently

$$\left|R^\sigma \cap I^n_{ej_i}\right| \le (n2^n\beta_n)^{-1}\left|R^\sigma \cap E^n_{ej_i}(\sigma)\right|, \quad e=1,2,\dots,n, \;\; i=1,2,\dots,s_e.$$

Next, since the rectangles Q^{*n}_{ej}, $j=1,2,\dots$, do not intersect for every fixed e (and hence the rectangles $E^n_{ej}(\sigma)$, $j=1,2,\dots$), we have

$$\Big|\bigcup_{i=1}^{s_e}\big(R^\sigma \cap E^n_{ej_i}(\sigma)\big)\Big| \le |R^\sigma|, \quad e=1,2,\dots,n.$$

The two last relations yield

$$|R^\sigma|^{-1}\int\limits_{R^\sigma}\psi_n(y)dy = \beta_n|R^\sigma|^{-1}\sum_{e=1}^{n}\sum_{j=1}^{\infty}\left|R^\sigma \cap I^n_{ej_i}\right| \le$$
$$\le \beta_n|R^\sigma|^{-1}\sum_{e=1}^{n}\sum_{i=1}^{s_e}(n2^n\beta_n)^{-1}\left|R^\sigma \cap E^n_{ej_i}(\sigma)\right| =$$
$$= n^{-1}2^{-n}|R^\sigma|^{-1}\sum_{e=1}^{n}\Big|\bigcup_{i=1}^{s_e}\big(R^\sigma \cap E^n_{ej_i}(\sigma)\big)\Big| \le 2^{-n}.$$

To complete the proof of (16), it remains to note that $R^\sigma \in B_{2\sigma}(x)$ and is arbitrary.

(c) Let us show that for almost all directions σ the maximal function $M^*_{B_{2\sigma}}f$ is finite a.e. on $\mathbb{R}^2$. Suppose

$$P(x,\sigma) = \max\{P_1(x); P_2(x,\sigma)\}.$$

Fix a direction σ from $\Gamma(\mathbb{R}^2)\backslash\Gamma$, a point x from $E\backslash(Q^*\cup B^*)$, and a rectangle R^σ from $B_{2\sigma}(x)$.

We have

$$|R^\sigma|^{-1}\Big|\int\limits_{R^\sigma} f(y)dy\Big| \le \sum_{n=2}^{P(x,\sigma)}|R^\sigma|^{-1}\int\limits_{R^\sigma}|f_n(y)|dy +$$
$$+|R^\sigma|^{-1}\Big|\int\limits_{R^\sigma}\sum_{n=p(x;\sigma)+1} f_n(y)dy\Big| = a_1(x,R^\sigma) + a_2(x,R^\sigma)$$

and

$$a_1(x,R^\sigma) \le \sum_{n=2}^{P(x,\sigma)} M_{B_{2\sigma}}\psi_n(x) + \sum_{n=2}^{P(x,\sigma)} M_{B_{2\sigma}}g_n(x) \le$$
$$\le \sum_{n=2}^{P(x,\sigma)}\|\psi_n\|_{L^\infty} + \sum_{n=2}^{P(x,\sigma)}\|g_n\|_{L^\infty} \le \sum_{n=2}^{P(x,\sigma)}(n\beta_n + m_n\lambda_n).$$

Estimate now $a_2(x, R^\sigma)$. Using the theorem on the passage to the limit under the integral sign as well as relations (7) and (16), we obtain

$$a_2(x,R^\sigma) \le \sum_{n=p(x,\sigma)+1} |R^\sigma|^{-1} \Big| \int\limits_{R^\sigma} f_n(y)dy \Big| \le$$
$$\le \sum_{n=p(x,\sigma)+1}^{\infty} (M_{B_{2\sigma}} \psi_n(x) + M^*_{B_{2\sigma}} g_n(x)) \le 2.$$

Hence

$$|R^\sigma|^{-1} \Big| \int\limits_{R^\sigma} f(y)dy \Big| \le \sum_{n=2}^{p(x,\sigma)} (n\beta_n + m_n \lambda_n) + 2 < \infty.$$

Since the right-hand side of this inequality does not depend on a choice of rectangles from $B_{2\sigma}(x)$, we get

$$M^*_{B_{2\sigma}} f(x) < \infty, \quad \sigma \in \Gamma(\mathbb{R}^2)\backslash\Gamma, \quad x \in E\backslash(Q^* \cup B^*).$$

Consequently for $\sigma \in \Gamma(\mathbb{R}^2)\backslash\Gamma$ and for $x \in E\backslash(Q^* \cup B^*)$ we have

$$-\infty < \underline{D}_{B_{2\sigma}}(f)(x) \le \overline{D}_{B_{2\sigma}}(f)(x) < +\infty.$$

Using now the Besicovitch theorem on possible values of upper and lower derivatives (see [1], Ch. V), we obtain ($|\Gamma| = |Q^* \cup B^*| = 0$) and for almost every direction σ ($\sigma \ne \sigma_n$, $n \in \mathbb{N}$) the relation $D_{B_{2\sigma}}(f)(x) = f(x)$ is fulfilled a.e.

4°. It will now be shown that for every direction σ_s ($s \in \mathbb{N}$) the strong upper derivative of the integral $\int f$ is equal to $+\infty$ a.e. on E.

To this end we fix a natural number s and notice that $|J_s| = 1$, where $J_s = \lim_{n\to\infty} \sup \bigcup\limits_{j=1}^{N_n} H^{\sigma_s}(\chi_{I^n_{sj}}, \beta_n^{-1})$. Indeed,

$$1 = \Big| \lim_{n\to\infty} \sup \bigcup_{j=1}^{N_n} H^n_{sj} \Big| \le \Big| \lim_{n\to\infty} \sup \bigcup_{j=1}^{N_n} H^{\sigma_s}(\chi_{I^n_{sj}}, \beta_n^{-1} \Big| +$$
$$+ \Big| \lim_{n\to\infty} \sup \bigcup_{j=1}^{N_n} Q^{*n}_{sj} \Big| = |J_s| + |Q^*| = |J_s|.$$

Let $D_s = \bigcap\limits_{n=2}^{\infty} D^n_s$, where $D^n_s = \big\{ y \in E : D_{B_2,\sigma_s}(f_n)(y) = f_n(y) \big\}$.

Since the basis B_2 differentiates the integrals of the bounded functions (see [1], Ch. III), it is evident that $|D^n_s| = 1$ for every $n = 2, 3, \dots$.

Let us fix a point x from $J_s \cap D_s \backslash (Q^* \cup B^*)$ and prove that

$$\overline{D}_{B_{2\sigma_s}}(f)(x) = +\infty. \tag{18}$$

Since $x \in J_s$, it is clear that there exists a sequence of pairs of natural numbers $(n_q, i_q)_{q=1}^{\infty}$ such that

$$x \in H^{\sigma_s}(\chi_{I^{n_q}_{si_q}}, \beta^{-1}_{n_q}), \quad q = 1, 2, \dots$$

which by the construction of the sets $H^{\sigma_s}(\chi_{I^{n_q}_{si_q}}, \beta^{-1}_{n_q})$ implies that there exists a rectangle $R^{\sigma_s}_q$ from $B_{2\sigma_s}(x)$ such that $R^{\sigma_s}_q \subset H^{n_q}_{si_q}$, $R^{\sigma_s}_q \supset I^{n_q}_{si_q}$ and

$$|R^{\sigma_s}_q|^{-1} \int\limits_{R^{\sigma_s}_q} \chi_{I^{n_q}_{si_q}}(y)dy \geq \beta^{-1}_{n_q}, \quad q = 1, 2, \dots. \tag{19}$$

Without loss of generality we may assume that $n_q \geq p(x, s)$, $q = 1, 2, \dots$, where $p(x, s) = \max\{p_1(x); p_2(x); s\}$. We have

$$|R^{\sigma_s}_q|^{-1} \int\limits_{R^{\sigma_s}_q} f_n(y)dy = |R^{\sigma_s}_q|^{-1} \int\limits_{R^{\sigma_s}_q} \Big(\sum_{n=2}^{p(x,s)} f_n(y)\Big)dy +$$

$$+|R^{\sigma_s}_q|^{-1} \int\limits_{R^{\sigma_s}_q} \Big(\sum_{p(x,s)<n<n_q} f_n(y)\Big)dy + |R^{\sigma_s}_q|^{-1} \int\limits_{R^{\sigma_s}_q} \Big(\sum_{n=n_q}^{\infty} f_n(y)\Big)dy =$$

$$= a_1(x_1, R^{\sigma_s}_q) + a_2(x_1, R^{\sigma_s}_q) + a_3(x_1, R^{\sigma_s}_q). \tag{20}$$

Consider the limits $\lim_{q\to\infty} a_i(x, R^{\sigma_s}_a)$, $\quad i = 1, 2, 3$,

(a) First let us show that the limit of $a_1(x, R^{\sigma_s}_q)$ for $q \to \infty$ is equal to $f(x)$. Indeed, since $x \in E\backslash D^n_s$, $\operatorname{diam}(R^{\sigma_s}_q) < a_{n_q} \searrow 0, q \to \infty$, we have

$$\lim_{q\to\infty} a_1(x, R^{\sigma_s}_q) = \sum_{n=2}^{p(x,s)} \lim_{q\to\infty} |R^{\sigma_s}_q|^{-1} \int\limits_{R^{\sigma_s}_q} f_n(y)dy =$$

$$= \sum_{n=2}^{p(x,s)} D_{B_{2\sigma_s}}(f_n)(x) = \sum_{n=2}^{p(x,s)} f_n(x).$$

Let $n \geq p(x, s)$. Then by (10) and (14) we have

$$x \in E\backslash\big(B^{*n} \cup \bigcup_{e=1}^{n} \bigcup_{j=1}^{\infty} Q^{*n}_{ej}\big) \subset E\backslash \operatorname{supp}(f_n).$$

Hence $\sum_{n=p(x,s)}^{\infty} f_n(x) = 0$ and consequently,

$$\lim_{q\to\infty} a_1(x, R^{\sigma_s}_q) = \sum_{n=2}^{\infty} f_n(x) = f(x). \tag{21}$$

(b) Let us now show that the values $a_2(x, R_q^{\sigma_s})$, $q = 1, 2, \dots$, are nonnegative. We shall assume that $\{n : p(x, s) < n < n_q\} \neq \varnothing$. Using the fact that the functions ψ_n $(n = 2, 3, \dots)$ are nonnegative, we obtain

$$a_2(x, R_q^{\sigma_s}) \geq \sum_{p(x,s)<n<n_q} |R_q^{\sigma_s}|^{-1} \int\limits_{R_q^{\sigma_s}} g_n(y)dy =$$

$$= \sum_{p(x,s)<n<n_q} \sum_{k=1}^{m_n} \sum_{i \in A_k^n} \lambda_n |R_q^{\sigma_s}|^{-1} \int s_n(y) \chi_{B_{ki}^n}(y) \chi_{R_q^{\sigma_s}}(y)dy. \tag{22}$$

It is sufficient to show that for $k = 1, \dots, m_n$, $i = 1, \dots, (M_k^n)^2$ we have

$$B_{ki}^n \cap R_q^{\sigma_s} = \varnothing, \quad p(x, s) < n < n_q. \tag{23}$$

Towards this end we fix the natural numbers n, k, i $(p(x, s) < n < n_q$, $k = 1, 2, \dots, m_n$, $i = 1, 2, \dots, (M_k^n)^2)$. Since $p_1(x) \leq p(x, s) < n < n_q$, it is obvious that $x \in E \backslash B^{*n}$ (see (10)) and

$$\operatorname{dist}(x, B_{ki}^n) \geq \operatorname{dist}(\partial B_{ki}^{*n}, B_{ki}^n) = r_n (M_k^n)^{-1}.$$

If, moreover, we recall that $r_n \geq r_{n_q - 1}$ and $M_k^n < M_{m_{n_q-1}}^{n_q-1}$, then we obtain

$$\operatorname{dist}(x, B_{ki}^n) \geq (M_{m_{n_q-1}}^{n_q-1})^{-1} r_{n_q} = a_{n_q}.$$

On the other hand, $R_q^{\sigma_s} \subset H_{si_q}^{n_q}$ and $\operatorname{diam}(R_q^{\sigma_s}) < a_{n_q}$ (see (3)). Hence $\operatorname{diam}(R_q^{\sigma_s}) < \operatorname{dist}(x, B_{ki}^n)$, and since $x \in R_q^{\sigma_s}$, we have $R_q^{\sigma_s} \cap B_{ki}^n = \varnothing$. Thus (23) is valid and hence (see (22))

$$\lim_{q \to \infty} a_2(x, R_q^{\sigma_s}) \geq 0. \tag{24}$$

(c) Let us show that as $q \to \infty$ the limit of the value $a_3(x, R_q^{\sigma_s})$ is more than unity. Using the theorem on the passage to the limit under the integral sign as well as the fact that the function ψ_n is nonnegative, and relations (19), (11),we have

$$a_3(x, R_q^{\sigma_s}) = |R_q^{\sigma_s}|^{-1} \int\limits_{R_q^{\sigma_s}} f_{n_q}(y)dy + \sum_{n=n_q+1}^{\infty} |R_q^{\sigma_s}|^{-1} \int\limits_{R_q^{\sigma_s}} f_n(y)dy \geq$$

$$\geq |R_q^{\sigma_s}|^{-1} \int\limits_{R_q^{\sigma_s}} f_{n_q}(y)dy - \sum_{n=n_q}^{\infty} |R_q^{\sigma_s}|^{-1} \Big| \int\limits_{R_q^{\sigma_s}} g_n(y)dy \Big| \geq$$

$$\geq 1 - \sum_{n=n_q}^{\infty} M_{B_{2\sigma_s}}^* g_n(x) \geq 1 - \sum_{n=n_q}^{\infty} 2^{-n}, \quad q = 1, 2, \dots.$$

Consequently,

$$\lim_{q\to\infty} a_3(x, R_q^{\sigma_s}) \geq 1. \tag{25}$$

Now we are able to establish (18). Indeed (see (20), (21), (24), (25)),

$$\lim_{q\to\infty} |R_q^{\sigma_s}|^{-1} \int_{R_q^{\sigma_s}} f(y)dy = \lim_{q\to\infty} \big(a_1(x, R_q^{\sigma_s}) + a_2(x, R_q^{\sigma_s}) +$$

$$+ a_3(x, R_q^{\sigma_s})\big) \geq f(x) + 1$$

and hence $\overline{D}_{B_{2\sigma_s}}(f)(x) > f(x)$, $x \in J_s \backslash (Q^* \cup B^* \cup D)$ which by virtue of the above-mentioned Besicovitch theorem implies that ($|J_s \backslash (Q^* \cup B^* \cup D)| = 1$)

$$\overline{D}_{B_{2\sigma_s}}(f)(x) = +\infty \quad \text{a.e. on} \quad E.$$

5°. We shall prove that for every direction σ the strong upper derivative of the integral $\int |f|$ in the direction σ is equal to $+\infty$ a.e. on E.

(a) Let us prove first that

$$\Big| \lim_{n\to\infty} \sup \bigcup_{k=1}^{m_n} G_k^n(\sigma) \Big| = 1. \tag{26}$$

Indeed, since

$$\sum_{n=p}^{\infty} \sum_{k=1}^{m_n} |G_k^n(\sigma)| = \sum_{n=p}^{\infty} m_n |h^n| = \infty,$$

for any $p = 2, 3, \dots$, we have

$$\prod_{n=p}^{\infty} \prod_{k=1}^{m_n} (1 - 2^{-1}|G_k^n(\sigma)|) = 0$$

and hence to prove (26) it is sufficient to show that

$$\Big| E \backslash \Big(\bigcup_{n_1=p}^{n-1} \bigcup_{k_1=1}^{m_{n_1}} G_{k_1}^{n_1}(\sigma) \cup \bigcup_{k_2=1}^{k} G_{k_2}^{n}(\sigma) \Big) \Big| \leq$$

$$\leq \prod_{n_1=p}^{n-1} \prod_{k_1=1}^{m_{n_1}} \big(1 - 2^{-1}|G_{k_1}^{n_1}(\sigma)|\big) \prod_{k_2=1}^{k} \big(1 - 2^{-1}|G_{k_2}^{n}(\sigma)|\big) \tag{27}$$

for every fixed direction σ and numbers n, p, k ($n = 2, 3, \dots, p = 2, \dots, n-1$, $k = 1, \dots, m_n$).

The proof of this relation will be carried out by induction. Indeed,

$$\big| E \backslash G_1^p(\sigma) \big| = 1 - |G_1^p(\sigma)| \leq 1 - 2^{-1}|G_1^p(\sigma)|.$$

Without loss of generality, we assume that $n > 2$, $k > 1$ and

$$\Big|E\setminus\big(\bigcup_{n_1=p}^{n-1}\bigcup_{k_1=1}^{m_{n_1}} G_{k_1}^{n_1}(\sigma)\cup\bigcup_{k_2=1}^{k-1} G_{k_2}^{n}(\sigma)\big)\Big| \le$$
$$\le \prod_{n_1=p}^{n-1}\prod_{k_1=1}^{m_{n_1}}\big(1-2^{-1}|G_{k_1}^{n_1}(\sigma)|\big)\prod_{k_2=1}^{k-1}\big(1-2^{-1}|G_{k_2}^{n}(\sigma)|\big). \tag{28}$$

Owing to this relation, we can prove (27). Assume

$$a_1(\sigma)=\Big\{i: E_i^{M_k^n}\cap\Big(\bigcup_{n_1=p}^{n-1}\bigcup_{k_1=1}^{m_{n_1}}\partial G_{k_1}^{n_1}(\sigma)\cup\bigcup_{k_2=1}^{k-1}\partial G_{k_2}^{n}(\sigma)\Big)\neq\varnothing\Big\},$$
$$a_2(\sigma)=\Big\{i: E_i^{M_k^n}\subset\bigcup_{n_1=p}^{n-1}\bigcup_{k_1=1}^{m_{n_1}} G_{k_1}^{n_1}(\sigma)\cup\bigcup_{k_2=1}^{k-1} G_{k_2}^{n}(\sigma)\Big\},$$
$$a_3(\sigma)=\Big\{1,2,\dots,(M_k^n)^2\Big\}\setminus(a_1(\sigma)\cup a_2(\sigma)).$$

For $n_1,\ k_1,\ i_1$ ($n_1=2,3,\dots,\ k_1=1,2,\dots,m_n,\ i_1=1,2,\dots.(M_{k_1}^{n_1})^2$) assume further

$$a_{k_1i_1}^{n_1}(\sigma)=\Big\{i: E_i^{M_k^n}\cap\partial h_{k_1i_1}^{n_1}(\sigma)\neq\varnothing\Big\}.$$

Clearly, for every triple of natural numbers $n_1,\ k_1,\ i_1$ ($n_1=2,3,\dots$, $k_1=1,2,\dots,m_n, i_1=1,2,\dots.(M_{k_1}^{n_1})^2$) it follows from the condition $M_{k_1}^{n_1}\ge d(\partial h^{n_1})$ (see (5)) and from the homothety that

$$d(\partial h_{k_1i_1}^{n_1}(\sigma))\le (M_{k_1}^{n_1})^{-1}d(\partial h^{n_1})\le 1,$$

which by the inequality $M_k^n\ge 9\omega_k^n$ (see (5)) and Lemma 1 implies

$$\Big|\bigcup_{i\in a_{k_1i_1}^{n_1}(\sigma)} E_i^{M_k^n}\Big|\le(\omega_k^n)^{-1}.$$

This yields

$$\Big|\bigcup_{i\in a_1(\sigma)} E_i^{M_k^n}\Big|\le\theta_k^n(\omega_k^n)^{-1}\le\theta_k^n\big(2\theta_k^n|E\setminus\Omega_k^n(\sigma)|^{-1}\big)^{-1}\le$$
$$\le 2^{-1}|E\setminus\Omega_k^n(\sigma)|\le 2^{-1}\Big|E\setminus\Big(\bigcup_{n_1=p}^{n-1}\bigcup_{k_1=1}^{m_{n_1}} G_{k_1}^{n_1}(\sigma)\cup\bigcup_{k_2=1}^{k-1} G_{k_2}^{n}(\sigma)\Big)\Big|.$$

On the other hand, it is easily seen that

$$\Big|\bigcup_{i\in a_2(\sigma)} E_i^{M_k^n}\Big|\le\Big|\bigcup_{n_1=p}^{n-1}\bigcup_{k_1=1}^{m_{n_1}} G_{k_1}^{n_1}(\sigma)\cup\bigcup_{k_2=1}^{k-1} G_{k_2}^{n}(\sigma)\Big|.$$

From the last two relations we obtain

$$\Big|\bigcup_{i\in a_3(\sigma)} E_i^{M_k^n}\Big|\ge 2^{-1}\Big|E\setminus\Big(\bigcup_{n_1=p}^{n-1}\bigcup_{k_1=1}^{m_{n_1}} G_{k_1}^{n_1}(\sigma)\cup\bigcup_{k_2=1}^{k-1} G_{k_2}^{n}(\sigma)\Big)\Big|.$$

Now let us derive (27). The last relation and (28) (by virtue of the homothety property) imply

$$\Big|E\backslash\Big(\bigcup_{n_1=p}^{n-1}\bigcup_{k_1=1}^{m_{n_1}} G_{k_1}^{n_1}(\sigma)\cup\bigcup_{k_2=1}^{k} G_{k_2}^{n}(\sigma)\Big)\Big| \le$$

$$\le \Big|E\backslash\Big(\bigcup_{n_1=p}^{n-1}\bigcup_{k_1=1}^{m_{n_1}} G_{k_1}^{n_1}(\sigma)\cup\bigcup_{k_2=1}^{k-1} G_{k_2}^{n}(\sigma)\Big)\Big| - \Big|\bigcup_{i\in a_3(\sigma)} E_i^{M_k^n}\Big| |G_k^n(\sigma)| \le$$

$$\le \prod_{n_1=p}^{n-1}\prod_{k_1=1}^{m_{n_1}}\big(1-2^{-1}|G_{k_1}^{n_1}(\sigma)|\big)\prod_{k_2=1}^{k}\big(1-2^{-1}|G_{k_1}^{n}(\sigma)|\big).$$

(b) Let us strengthen now relation (26) and show that

$$\Big|\lim_{n\to\infty}\bigcup_{k=1}^{m_n}\bigcup_{i\in A_k^n} h_{ki}^n(\sigma)\Big| = 1. \tag{29}$$

which will immediately follow from (26) if we prove that the equality

$$\bigcup_{k=1}^{p}\bigcup_{i=1}^{(M_k^n)^2} h_{ki}^n(\sigma) = \bigcup_{k=1}^{p}\bigcup_{i\in A_k^n} h_{ki}^n(\sigma) \tag{30}$$

holds for any pairs of natural numbers $n,\ p$ $(n=2,3,\dots,p=1,2,\dots,m_n)$. Checking it inductively, we can see that it is fulfilled for $p=1$.

Assume now that $p>1$ and

$$\bigcup_{k=1}^{p-1}\bigcup_{i=1}^{(M_k^n)^2} h_{ki}^n(\sigma) = \bigcup_{k=1}^{p-1}\bigcup_{i\in A_k^n} h_{ki}^n(\sigma).$$

Owing to this equality we can easily obtain (30). For this it is enough to note that

$$h_{pi}^n \subset \bigcup_{k_1=1}^{p-1}\bigcup_{i_1=1}^{(M_k^n)^2} h_{k_1 i_1}^n(\sigma)$$

for $i\in\{1,2,\dots,(M_p^n)^2\}\backslash A_p^n$, since $h_{pi}^n(\sigma)\subset E_i^{M_p^n}$ and $B_{k_1 i_1}^{*n}(\sigma)\subset h_{k_1 i_1}^n(\sigma)$ $(k_1=1,2,\dots,p-1,\ i_1=1,2,\dots,(M_{k_1}^n)^2)$.

(c) Establish that for every natural number n

$$g_n(x)=\lambda\chi_{B^n}(x),\quad x\in\mathbb{R}^2. \tag{31}$$

It is enough to show that for fixed n the circles B_{ki}^n do not intersect. On the one hand, to this end we can show that the equality

$$B_{ki}^n\cap B_{k_1 i_1}^n=\varnothing \tag{32}$$

is fulfilled for any pairs $k,\ i$ and $k_1,\ i_1$ $(k=2,\dots,m_n,\ i\in A_k^n,\ k_1=1,2,\dots,k-1,\ i_1\in A_{k_1}^n)$.

Indeed, it follows from the inclusion $i\in A_k^n$ that one of the two cases may take place:

(1) $E_i^{M_k^n} \cap B^{*n}_{k_1 i_1} = \varnothing$;

(2) $E_i^{M_k^n} \cap \partial B^{*n}_{k_1 i_1} \neq \varnothing$.

Obviously, relation (32) is fulfilled in Case (1). Let us consider Case (2). The condition $M_k^n \geq w_k^n = 2r_n^{-1} M_{k-1}^n \geq 2r_n^{-1} M_{k_1}^n$ (see (6)) implies

$$\operatorname{diam}(E_i^{M_k^n}) \leq 2(M_k^n)^{-1} \leq r_n (M_{k_1}^n)^{-1}.$$

On the other hand, $\operatorname{dist}(\partial B^{*n}_{k_1 i_1}, B^n_{k_1 i_1}) = r_n (M_{k_1}^n)^{-1}$.

It follows from the last two estimates that in Case (2) we have $E_i^{M_k^n} \cap B^n_{k_1 i_1} = \varnothing$. To complete the proof of (31) note that $B^n_{ki} \subset E_i^{M_k^n}$.

(d) Let us give some remarks which will be used in the sequel.

Remark 1. For every point $x \in \operatorname{supp}(f_n)$, $n = 2, 3, \dots$, the following inequality $|f_n(x)| \geq \beta_n$ is fulfilled.

To prove this one should use (31) and (3).

Remark 2. Let m, m_1 be arbitrary natural numbers and $2 \leq m < m_1$. We define the functions f^{m,m_1} and f^m as follows:

$$f^{m,m_1}(x) = \sum_{n=m}^{m_1} f_n(x), \quad f^m(x) = \sum_{n=m}^{\infty} f_n(x).$$

Then the inequality $\left|f^{m,m+p}(x)\right| \geq \left|f^{m,m+p-1}(x)\right|$ is fulfilled for every natural p.

This statement easily follows from the previous remark and relation (31) and (3).

Remark 3. Let m, m_1 be arbitrary natural numbers and $2 \leq m < m_1$. Let, moreover, $x \in B^{m_1}$ and $x \in E \backslash (B^* \cup Q^*)$. $|f^m(x)| \geq 2^{-1}\lambda_{m_1}$.

Indeed, since $B^{*n} \cup \bigcup\limits_{e=1}^{n} \bigcup\limits_{j=1}^{\infty} Q^{*n}_{ej} \supset \operatorname{supp}(f_n)$, from relations (10) and (14) we can conclude that if $n \geq p(x)$, then

$$x \in E \backslash \Big(B^{*n} \cup \bigcup_{e=1}^{n} \bigcup_{j=1}^{\infty} Q^{*n}_{ej}\Big) \subset E \backslash \operatorname{supp}(f_n),$$

where $p(x) = \max\{p_1(x); p_2(x)\}$. On the other hand, the inclusion $x \in B^{m_1}$ implies that $p(x) \geq m_1$. Hence $f^m(x) = f^{m,p(x)}(x)$ and by Remark 2 we obtain

$$\left|f^m(x)\right| = \left|f^{m,p(x)}(x)\right| \geq \left|f^{m,p(x)-1}(x)\right| \geq \cdots \geq \left|f^{m,m_1}(x)\right|.$$

Using now the inclusion $x \in B^{m_1}$, relations (31) and (1), we have

$$\left|f^m(x)\right| \geq \left|f^{m,m_1}(x)\right| \geq \left|f_{m_1}(x)\right| - \Big|\sum_{n=m}^{m_1-1} f_n(x)\Big| \geq$$

$$\geq |g_{m_1}(x))| - \psi_{m_1}(x) - \sum_{n=2}^{m_1-1} \big(\psi_n(x) + |g_n(x)|\big) \geq$$

$$\geq \lambda_{m_1} - m_1\beta_{m_1} - \sum_{n=2}^{m_1-1} (n\beta_n + \lambda_n) \geq 2^{-1}\lambda_{m_1}.$$

(e) We can now prove that for every direction σ the relation $\overline{D}_{B_{2\sigma}}(|f|)(x) = +\infty$ is fulfilled a.e. on E.

Let us fix the direction σ and the number ε, $0 < \varepsilon < 1$. The natural number m_ε can be defined from the condition $|\operatorname{supp}(f^{m_\varepsilon})| < \varepsilon$. Suppose

$$h(\sigma) = \lim_{n\to\infty} \sup \overset{m_n}{\underset{k=1}{\cup}} \underset{i\in A_k^n}{\cup} h_{ki}^n(\sigma), \quad T = E\backslash(B^* \cup Q^*).$$

Let $x \in h(\sigma)$. Then there exists a sequence (n_p, k_p, i_p) $(k_p = 1, 2, \dots, m_p$, $i_p \in A_{k_p}^{n_p})$, such that $x \in h_{k_p i_p}^{n_p}(\sigma)$, $p = 1, 2, \dots$. From this inclusion and the construction of sets $h_{k_p i_p}^{n_p}(\sigma)$ it follows that there exists a rectangle R_p^σ from $B_{2\sigma}(x)$ such that $R_p^\sigma \supset B_{k_p i_p}^{n_p}$, $R_p^\sigma \subset E_{i_p}^{M_{k_p}^{n_p}}$ and

$$|R_p^\sigma|^{-1}\lambda_{n_p} \int\limits_{R_p^\sigma} \chi_{B_{k_p i_p}^{n_p}}(y)dy \geq 2^{-1}. \tag{33}$$

We have $(|T| = 1)$

$$|R_p^\sigma|^{-1} \int\limits_{R_p^\sigma} |f^{m_\varepsilon}(y)|dy \geq |R_p^\sigma|^{-1} \int |f^{m_\varepsilon}(y)dy|\chi_{B_{k_p i_p}^{n_p}}(y)\chi_T(y)dy.$$

Assume that $n_p > m_\varepsilon$. Then by Remark 3 we obtain

$$|f^{m_\varepsilon}(y)| > 2^{-1}\lambda_{n_p}, \quad y \in B_{k_p i_p}^{n_p} \cap T, \quad p = 1, 2, \dots,$$

which by virtue of (33) yields

$$|R_p^\sigma|^{-1} \int\limits_{R_p^\sigma} |f^{m_\varepsilon}(y)|dy > 2^{-1}|R_p^\sigma|^{-1}\lambda_{n_p} \int\limits_{R_p^\sigma} \chi_{B_{k_p i_p}^{n_p}}(y)dy \geq 2^{-2}.$$

Consequently, $\operatorname{diam}(R_p^\sigma) \leq 2(M_{k_p}^{n_p})^{-1} \searrow 0$.

$$\overline{D}_{B_{2\sigma}}(|f^{m_\varepsilon}|)(x) \geq \lim_{p\to\infty} |R_p^\sigma|^{-1} \int\limits_{R_p^\sigma} |f^{m_\varepsilon}(y)dy \geq 2^{-2}.$$

Suppose $z_\varepsilon = \{x \in E : f^{m_\varepsilon}(x) = 0\}$. From the previous relation we obtain

$$\overline{D}_{B_{2\sigma}}(|f^{m_\varepsilon}|)(x) \geq 2^{-2} > |f^{m_\varepsilon}|)(x)|, \quad x \in h(\sigma) \cap z_\varepsilon.$$

Clearly, $z_\varepsilon \supset E\setminus \operatorname{supp}(f^{m_\varepsilon})$ and hence $|z_\varepsilon| > 1-\varepsilon$. Thus $(|h(\sigma)\cap z_\varepsilon| = |z_\varepsilon|)$,

$$\Big|\big\{x\in E : \overline{D}_{B_{2\sigma}}(|f^{m_\varepsilon}|)(x) > |f^{m_\varepsilon}(x)|\big\}\Big| > 1-\varepsilon.$$

Using once again the Besicovitch theorem, we can conclude that

$$\Big|\big\{x\in E : \overline{D}_{B_{2\sigma}}(|f^{m_\varepsilon}|)(x) = +\infty\big\}\Big| > 1-\varepsilon.$$

Further we have

$$\overline{D}_{B_{2\sigma}}(|f|)(x) \geqq \overline{D}_{B_{2\sigma}}(|f^{m_\varepsilon}|)(x) - \sum_{n=2}^{m_\varepsilon} \|f_n\|_{L^\infty}.$$

Since $\|f_n\|_{L^\infty} < \infty$ $(n=2,3,\dots)$, the last two relations imply

$$\big|\big\{x\in E : \overline{D}_{B_{2\sigma}}(|f|)(x) = +\infty\big\}\big| > 1-\varepsilon.$$

Because of the fact that ε is arbitrary, we get

$$\big|\big\{x\in E : \overline{D}_{B_{2\sigma}}(|f|)(x) = +\infty\big\}\big| = 1.$$

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