GEORGIAN MATHEMATICAL JOURNAL: Vol. 5, No. 3, 1998, 233-241

SPECTRAL AND BOUNDEDNESS RADII IN LOCALLY CONVEX ALGEBRAS

A. EL KINANI, L. OUBBI, AND M. OUDADESS

ABSTRACT. Connections between the spectral radius and the radius of boundedness are studied. Different characterizations of algebras (Q-property, strong saquentiality) are given in terms of these radii. Examples and applications are also provided.

INTRODUCTION

In [1], Allan develops a spectral theory for locally convex algebras (l.c.a.), using a spectrum (of an element) which is a subset of the extended complex plane and considers the corresponding spectral radius r. He studies, in particular, relations between r and the radius of boundedness β. The latter plays an important role in studying sequentiality in l.c.a. In this paper, we consider the classical notion of a spectrum and accordingly the classical spectral radius ρ. We establish connections between ρ and β. Classes of strongly sequential algebras are determined. Two simple conditions concerning the convergence of the series $\sum x^n$ are shown to be essential in this context. Finally, a class of l.c.a.'s not necessarity m-convex, which contains in particular Q-Fréchet locally convex algebras, on which the entire function operates, is found.

I. PRELIMINARIES

A locally convex algebra (l.c.a.) (E,τ) is an algebra over the complex field endowed with a Hausdorff locally convex topology for which the product is separately continuous. The classical spectral radius of an element x will be denoted by $\rho(x)$; it is $\rho(x) := \sup\{|\lambda| : \lambda \in \operatorname{Sp} x\}$, where $\operatorname{Sp} x = \{\lambda \in C : x = \lambda e$ is not invertible in $E\}$. An element x of E is said to be bounded if for some nonzero complex number λ, the set $\{(\lambda x)^n : n = 1, 2, \dots\}$ is a bounded subset of (E,τ). The set of all bounded elements of (E,τ) is

1991 *Mathematics Subject Classification.* 46H99, 46H05.
Key words and phrases. Locally convex algebra, spectral radius, boundedness radius, sequentiality.

1072-947X/98/0500-0233$15.00/0

denoted by E_0. The radius of boundedness β of an element x is defined by $\beta(x) := \inf\{a > 0 : (a^{-1}x)^n,\ n = 1, 2, \dots$ is bounded$\}$, with $\varnothing = +\infty$. A l.c.a. is said to be m-convex (m.c.a.) if its topology can be given by a family of submultiplicative seminorms [2]. A l.c.a. (E, τ) is said to be:

i) strongly sequential if there is a zero-neighborhood U such that $(x^k)_k$ converges to zero for all $x \in U$;

ii) sequential if for each sequence $(x_n)_n$ tending to zero, there exists an element x such that $(x_m^k)_k$ converges to zero;

iii) infra-sequential if for each bounded set B of (E, τ), there is $a > 0$, such that $((ax)^k)_k$ converges to zero for all $x \in B$.

It is clear that i) $\Rightarrow$ ii) $\Rightarrow$ iii) and that ii) $\Rightarrow$ i) in metrizable l.c.a. (cf. [3]).

Recall that a l.c.a. (E, τ) is said to be a Q-algebra if the group $G(E)$ of its invertible elements is open.

II. Comparison of ρ and β

First observe that ρ and β are not always comparable. Let E be the field $\mathbb{C}(X)$ of rational fractions of the ring of complex polynomials endowed with its strongest locally convex topology. One has $\rho(X) = 0$ and $\beta(X) = +\infty$, hence ρ and β are different. Now let F be a normed non Q-algebra. In F, we have $\beta \leq \rho$. In the product algebra $E \times F$, ρ and β are not comparable. Note that E yields an example of a complete commutative l.c.a. with unity in which neither ρ nor β are submultiplicative seminorms. We begin with a lemma which will be useful in the sequel. Denote by $D(E)$ and $S(E)$ the set of all elements $x \in E$ such that $\beta(x) < 1$ and $\rho(x) \leq 1$ respectively.

Lemma II.1. *Let (E, τ) be a l.c.a. with unity e. If (E, τ) is pseudo-complete, then $e + D(E) \subset G(E)$.*

Proof. We will show that $e - x$ is invertible for every $x \in D(E)$ and that the inverse is the classical series $\sum x^n$. We have $\left(\sum_0^N x^n\right)(e - x) = (e - x)\left(\sum_0^N x^n\right) = e - x^{N+1}$. But x^{N+1} tends to zero for $x \in D(E)$. It remains then only to show that the series converges. Consider a such that $\beta(x) < a < 1$. The closed absolutely convex hull B of $\{(a^{-1}x)^n : n = 1, 2, \dots\}$ is an idempotent Banach disk. In the Banach algebra $(E_B, \|\ \|_B)$, one has $\|x\|_B \leq a < 1$, where E_B is the span of B and $\|\ \|_B$ is the gauge of B. Therefore the series $\sum x^n$ converges in $(E_B, \|\ \|_B)$, and hence in (E, τ) as well. $\square$

We now recall the lemma of Michael ([2], p. 58) with the term "absolutely convex" replaced by "balanced" and the proof remaining the same:

Lemma II.2. *Let (E, τ) be a unitary l.c.a. If U is a balanced subset of E such that $e + U \subset G(E)$, then $U \subset S(E)$.*

Here is the first result concerning the comparison of ρ and β.

Proposition II.3. *Let (E, τ) be a unitary pseudo-complete l.c.a. Then $\rho \leq \beta$.*

Proof. By the previous lemmas, one has $D(E) \subset S(E)$, whence follows the conclusion. □

Remark II.4. The preceding proposition can be seen as a consequence of the result of [1]. But the proof given here is short and much easier, and it does not, in particular, appeal to the notion of resolvent.

With no notion of completeness, we provide a condition under which we still have $\rho \leq \beta$. An element x of a l.c.a. is said to be sequentially topologically invertible (s.t.-invertible in short) if there is a sequence $(x_n)_n$ in E such that $(x_n x)_n$ and $(x x_n)_n$ converge to the unit element.

Proposition II.5. *Let (E, τ) be a unitary l.c.a. If every s.t.-invertible element is invertible, then $\rho \leq \beta$; this is the case, in particular, for Q-algebras.*

Proof. It is sufficient to show that $D(E) \subset S(E)$ and, by Lemma II.2, we just have to show that $e + D(E) \subset G(E)$. Let further x be in $D(E)$. Since the sequence $(x^n)_n$ tends to zero, the element $e - x$ is s.t.-invertible, and hence it is invertible by hypothesis. □

Remark II.6. In the normed case, the condition $\rho \leq \beta$ is equivalent to the fact that E is a Q-algebra. In a general l.c.a., the equivalence does not hold. In fact, we have $\rho \leq \beta$ in any unitary and pseudo-complete l.c.a. (cf. Proposition II.3). However, we obtain the following results:

Lemma II.7. *Let (E, τ) be a unitary l.c.a. Then $\rho \leq \beta$ if and only if there is $a > 0$ such that $e - aD(E) \subset G(E)$.*

Proof. If $\rho \leq \beta$, then $D(E) \subset S'(E)$, where $S'(E) = \{x \in E : \rho(x) < 1\}$. But $e - S'(E) \subset G(E)$ and hence $e - D(E) \subset D(E)$. Conversely, let $a > 0$ be such that $e - aD(E) \subset G(E)$. By Lemma II.2, $aD(E) \subset S(E)$. Whence $a\rho \leq \beta$. Then, taking x^n, $n \geq 1$, and letting n tend to infinity, we get $\rho \leq \beta$. □

Proposition II.8. *Let (E, τ) be a unitary l.c.a. such that β is an algebra semi-norm. Then $\rho \leq \beta$ if and only if $G(E)$ is β-open.*

Using Warner's techniques [4], one can obtain the following results providing situations where Proposition II.8 can be applied.

Lemma II.9. *Let (E, τ) be a unitary and commutative l.c.a. with a continuous product. Then $\beta(x + y) \leq \beta(x) + \beta(y)$ and $\beta(xy) \leq \beta(x)\beta(y)$, for every x and y in E.*

Proof. Let us first observe that if $\beta(x)$ or $\beta(y)$ is infinite, there is nothing to show. Now consider the set $A = \{x \in E : (x^n)_n \text{ is bounded}\}$. It is an idempotent disk. Indeed, let x and y be in A and $0 \leq a \leq 1$. We have to show that $(z^n)_n$ where $z = ax + (1-a)y$ is bounded. For a 0-neighborhood V there is a 0-neighborhood U such that $UU \subset V$. By hypothesis, there is $\lambda > 0$ such that $(x^n)_n \subset \lambda U$ and $(y^n)_n \subset \lambda U$. Using the binomial formula, we find that A is convex. On the other hand, it is easily seen that A is balanced and idempotent. Now the gauge P_A of A is exactly the boundedness radius β, and P_A verifies the required inequalities. □

For every bounded element we have

Proposition II.10. *Let (E, τ) be a unitary commutative l.c.a. with continuous product, and in which every element is bounded. Then $\rho \leq \beta$ if and only if $G(E)$ is β-open.*

III. Strong Sequentiality

As far as we know, S. Warner [4] was the first to introduce this notion in m-convex algebras. He used the term "P-algebras". In connection with Michael's problem, Husain considered this notion in general topological algebras [3]. In [5], Oudadess gave several classes of strongly sequential algebras. We develop this idea in a larger setting. Here is a characterization of strongly sequential l.c.a.'s. The proof, being straightforward, is omitted.

Proposition III.1. *Let (E, τ) be a unitary l.c.a. Then (E, τ) is strongly sequential if and only if β is continuous at 0.*

Note that β may be continuous at 0 but not everywhere even in the Banach case ([6], p. 88). In relation to Remark II.6, we have

Proposition III.2. *Let (E, τ) be a unitary strongly sequential l.c.a. Then $\rho \leq \beta$ if and only if (E, τ) is a Q-algebra. In particular, if (E, τ) is pseudo-complete, then it is a Q-algebra.*

Proof. We have seen that $\rho \leq \beta$ is any Q-algebra. Conversely, if $\rho \leq \beta$, then ρ is continuous since this is the case for β by the previous proposition Therefore (E, τ) is a Q-algebra. □

We now consider the relation between a strong sequentiality and the Q-algebra property. None of the implications hold, as the following examples show.

Examples III.3. (1) Any normed algebra is strongly sequential but certainly is not necessarily a Q-algebra.

(2) Let $E = \mathbb{C}(X)$ be the field of rational functions endowed with its finest linear locally convex topology. It is a Q-algebra which is not strongly sequential for its elements are not all bounded.

As we will see, under additional conditions, the equivalence may hold. We first give a general result.

Proposition III.4. *Let (E, τ) be a pseudo-complete l.c.a. whose every element is bounded. Then $\rho \leq \beta$; hence (E, τ) is strongly sequential if and only if it is a Q-algebra.*

Proof. First suppose that E is commutative. Since E is pseudo-complete, it can be written as a directed union of Banach algebras, i.e., $E = \cup_{t \in I} E_i$. For every x in E, we have the following formulas: $\rho(x) = \inf\{\rho_i(x) : i \in I(x)\}$ and $\beta(x) = \inf\{\beta_i(x) : i \in I(x)\}$, where $I(x) = \{i \in I : x \in E_i\}$. But $\rho_i = \beta_i$, in each E_i. Whence follows the result. Now if E is not commutative, consider for every x, a maximal commutative subalgebra $C(x)$ containing x. Then one has $\beta(x) = \beta_C(x) = \rho_C(x) = \rho(x)$. □

Remark III.5. The previous result can be obtained from Theorem 3.12 of [1], but our proof is elementary; it makes no use of the holomorphy of the resolvent.

Proposition III.6. *Let (E, τ) be a Hausdorff l.c.a. If (E, τ) is a Q-algebra whose every element is bounded, then it is strongly sequential.*

Proof. Since (E, τ) is a Q-algebra, one has $\rho \leq \beta$ (cf. Proposition II.5). On the other hand, $\beta \leq \rho$ by Theorem 3.12 on p. 411 of [1], for $E = E_0$. So $\rho = \beta$, whence follows the conclusion. □

Corollary III.7. *Let (E, τ) be a pseudo-complete l.c. Q-algebra with the continuous inverse. Then it is strongly sequential.*

Proof. The spectrum of every element of E is compact and hence bounded. Therefore every element is bounded by ([1], 4.2, p. 414). □

In any normed algebra we have the property

$$\rho(x) < 1 \Rightarrow (e - x)^{-1} = \sum_{n=0}^{+\infty} x^n. \tag{P1}$$

This property is not always fulfilled in a general l.c.a. (e.g., $\mathbb{C}(X)$ of Example III.3.2). Actually, we have the following characterization:

Proposition III.8. *Let (E, τ) be a unitary Hausdorff l.c.a. Then (E, τ) verifies* (P1) *if and only if $\beta \leq \rho$.*

Proof. Let x be in E. If $\rho(x) = +\infty$, the inequality obviously holds. If $\rho(x) < +\infty$, then for every $a > \rho(x)$, $(a^{-1}x)^n$ tends to 0. Whence $\beta(x) \leq a$. Conversely if $\rho(x) < 1$, then $\beta(x) < 1$. Hence $(x^n)_n$ tends to 0. The proof is completed by a classical argument. □

It appears that in the presence of (P1), the Q-algebra property implies a strong sequentiality. An interesting consequence of Proposition III.8 is as follows:

Corollary III.9. *In every unitary Hausdorff l.c.a. we always have* $\beta \leq \rho$.

A property analogous to (P1) for β is

$$\beta(x) < 1 \Rightarrow (e-x)^{-1} = \sum_{n=0}^{+\infty} x^n. \tag{P2}$$

Let us note that under condition (P2), once $e - x$ is invertible, its inverse is exactly the series. So what is assumed in (P2) is actually the existence of $(e-x)^{-1}$.

An analogue of Proposition III. 7 is

Proposition III.10. *Let* (E, τ) *be a unitary Hausdorff l.c.a. Then it verifies* (P2) *if and only if* $\rho \leq \beta$.

Here it appears that in the presence of (P2), a strong sequentiality implies the Q-algebra property. Note that any pseudo-complete l.c.a. and any Q-algebra verify (P2).

Corollary III.11. *In a pseudo-complete Hausdorff unitary l.m.c.a., we have* $\rho = \beta$.

Hence strong sequentiality and the Q-algebra property are the same.

We now give the third property involving both ρ and β:

$$\min(\rho(x), \beta(x)) < 1 \Rightarrow (e-x)^{-1} = \sum_{n=0}^{+\infty} x^n. \tag{P3}$$

Remark III.12. (P3) is equivalent both to (P1) and (P2) together. So in any Hausdorff l.c.a., the property (P3) is equivalent to $\rho = \beta$. Hence in this context, the Q-algebra property and strong sequentiality are the same. As a direct consequence of this, every element is bounded in any Q-l.m.c.a.

IV. Classes of Strong Sequential Algebras

The following observation is given in [5]: A.l.c.a. (E,τ) is infra-sequential if and only if β is bounded. One may define the notion of weak infra-sequentiality, but according to the previous observation, it is the same as infra-sequentiality, since the bounded sets are the same for all topologies compatible with a given duality. In the sequel, if (E,τ) is a locally convex space, denote by τ_i the finest linear locally convex topology on E having the same bounded sets as τ (cf. [6]). If (E,τ) is a l.c.a. then so is (E,τ) [4].

Proposition IV.1. *If (E,τ) is a commutative Hausdorff infra-sequential l.c.a. with a continuous product, then (E,τ_i) is strongly sequential.*

Proof. Since (E,τ) is infra-sequential, β is finite. By Lemma II.9, β is an algebra semi-norm. It is bounded by the hypothesis and therefore τ_i is continuous. □

As a consequence, we find that any bornological infra-sequential commutative l.c.a. with continuous product is strongly sequential.

With a similar proof as for Proposition IV.1, we get the following result:

Proposition IV.2. *Let (E,τ) be a commutative unitary l.c.a. If (E,τ) is either pseudo-normed (i.e., every bounded set is regular) or pseudo-complete and infra-sequential, then (E,τ_i) is strongly sequential.*

Corollary IV.3. *Let (E,τ) be a bornological unitary commutative and pseudo-complete l.c.a. Then (E,τ) is infra-sequential if and only if it is strongly sequential.*

The following facts are worth stating:

Proposition IV.4. *Let (E,τ) be a commutative unitary l.c.a. whose every element is bounded such that β is an algebra semi-norm. Then:*

(i) *E can be endowed with a topology τ' finer than τ such that (E,τ') is strongly sequential,*

(ii) *τ and τ' have always the same regular bounded sets (i-bounded sets of [4]),*

(iii) *τ and τ' have the same bounded sets if and only if (E,τ) is infra-sequential,*

(iv) *(E,τ) has the continuous inverse if and only if it is so for (E,τ').*

Corollary IV.5. *Any commutative unitary pseudo-complete infra-sequential and m-barrelled l.c.a. is strongly sequential.*

Corollary IV.6. *Let (E,τ) be a commutative unitary pseudo-complete l.c.a. whose every element is bounded. If (E,τ) has the continuous inverse, then it can be endowed with an m-convex topology finer than τ and for which E is strongly sequential.*

V. Entire Functions

Blali asserts that entire functions operate in sequentially complete l.c. Q-algebras ([7], Proposition II.6). This is not correct as the following example shows. Let $E = \mathbb{C}(X)$ be the field of rational fractions of the ring of polynomials and E^* its algebraic dual. Consider the Mackey topology $\tau = \tau(E, E^*)$. Then (E,τ) is a complete l.c. Q-algebra. If $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ is an entire function, $f(X)$ makes no sense. Indeed, the sequence $\left(\sum_{n=0}^{p} a_n X^n\right)_p$ should be contained in a finite-dimensional subspace of E and this is not true. Blali's proof relies on the following assertion of [3]: A l.c.a. is a Q-algebra if and only if it is strongly sequential. But this is not correct either, because of the example $\mathbb{C}(X)$ above. Husain's result must be split into two assertions. One of them is Proposition III. 3, and the second is Corollary III. 7. The hypothesis making Blali's proof work is the strong sequentiality of the algebra, which is fulfilled under an additional condition of the inverse. We then get:

Proposition V.1. *Let (E,τ) be a sequentially complete l.c. Q-algebra with the continuous inverse. Then entire functions operate on (E,τ).*

Proof. By Corollary III.7, the algebra is strongly sequential. Then there is a 0-neighbouhood U such that $h^n \to 0$ for every h in U. Let x be an element of E and $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ an entire function. There is h in U and $t > 0$ such that $x = th$. For an absolutely convex 0-neightborhood W, there is n_0 such that $h^n \in W$ and $\sum_{k=p}^{q} |a_k| t^k \leq 1$ for $n, p, q \geq n_0$. Therefore $\sum_{k=p}^{q} a_k x^k \in W$, for $p, q \geq n_0$, whence follows the convegence of the series. □

Corollary V.2 ([8]). *Entire functions operate in any Frèchet Q-algebra.*

Remark V.3. By the result of P. Turpin [9], a commutative l.c.a. which is a Q-algebra with the continuous inverse is actually m-convex and hence entire functions operate in such a complete algebra. In the noncommutative case, W. Zelazko gives in [10] an example of a complete non-m-convex locally convex Q-algebra with the continuous inverse on which entire functions operate. Proposition V.1. shows that entire functions operate on the whole class of such algebras.

References

1. G. R. Allan, A spectral theory for locally convex algebras. *Proc. London Math. Soc.* **115**(1965), 399–421.

2. E. A. Michael, Locally multiplicatively convex topological algebras. *Memoirs Amer. Math. Soc.* 11, *Providence,* 1952.

3. T. Husain, Infra-sequential topological algebra. *Canadian Math. Bull,* **22**(1979), 413–418.

4. S. Warner, Inductive limits of normed algebras. *Trans. Amer. Math. Soc.* **82**(1956), 90–16.

5. M. Oudadess, Rayoin de régularite dans les algèbres infra-séquentielles. *Canadian J. Math.* **36**(1984), 84–94.

6. P. Perez Carreras and J. Bonet, Barrelled locally convex spaces. *Notas de Mathematica, North Holland,* 1987.

7. A. Blali, Fonctions entières dans les algèbras topologiques ou bornologiques. *Thèse de troisième cycle, E.N.S. Rabat* (*Maroc*).

8. W. Zelazko, Selected topics in topological algebras. *Lecture Notes Ser.* 31, *Math. Inst. Aarhus Univ., Aarhus*, 1971.

9. P. Turpin, Une remarque sur les algèbres à inverse continu. *C.R. Acad. Sci. Paris* **270**(1970), 1686–1689.

10. W. Zelazko, Concerning entire functions in B0-algebras. *Studia Math.* **110**(3)(1994), 283–290.

(Received 12.04.1996; revised 07.11.1996)

Authors' address:
Department of Mathematics
Ecole Normale Superieure
B.P. 5118, Takaddoum
10 000 Rabat
Morocco