GEORGIAN MATHEMATICAL JOURNAL: Vol. 5, No. 3, 1998, 243-250

THE CONTACT PROBLEM FOR AN ELASTIC ORTHOTROPIC PLATE SUPPORTED BY PERIODICALLY LOCATED BARS OF EQUAL RESISTANCE

L. GOGOLAURI

Abstract. The contact problem of the plane theory of elasticity is studied for an elastic orthotropic half-plane supported by periodically located (infinitely many) stringers of equal resistance. Using the methods of the theory of a complex variable, the problem is reduced to the Keldysh–Sedov type problem for a circle. The solution of the problem is constructed.

Let an elastic orthotropic plate occupying a lower half-plane of a complex plane $z = x + iy$ be supported by periodically located elastic absolutely flexible (infinitely many) bars of equal resistance. Longitudinal forces p and q are applied to the bar ends. The bars are to be free assumed from other external loads. The problem consists in finding the cross-sectional areas $S(x)$ of the bars and the contact the tangential stresses $\tau_{xy}(x,0)$ provided that the longitudinal stresses $\sigma_x^{(0)}(x)$ in the bars are constant and equal to a.

Similar problems for isotropic elastic domains have been investigated in [1–4]. In the case of an anisotropic half-plane this problem has been studied by the author in [5]. Periodic problems dealing with springers of constant rigidity can be found in [6–7].

Without restriction of generality, the length of the stringer bases is assumed to be equal to unity. Denote the distance between the stringers by $2l$. The stringers are located symmetrically with respect to the ordinate axis. In such a case the stringers will be located as follows: $[(2k+1)l + k; (2k+1)l + k + 1]$, $k = 0, \pm 1, \pm 2, \dots$.

1991 *Mathematics Subject Classification.* 73C02, 30E25.
Key words and phrases. Orthotropic plate, periodically located bars of equal resistance, conformal mapping, Keldysh–Sedov type problem for a circle.

From the equilibrium condition of stringer elements on the reinforced sections we obtain

$$S(x)\sigma_x^{(0)}(x) - h\int_{(2k+1)l+k}^{x} \tau_{xy}(s)ds - q = 0, \quad x \in L_k, \tag{1}$$

where h is the bar thickness and L_k denotes a segment $[(2k+1)l+k;(2k+1)l+k+1]$.

Taking into account the fact that the bars are absolutely flexible and their resistance under bending is a negligibly small value, we may assume that $\sigma_y = \sigma_y^{(0)} = 0$ for $-\infty < x < \infty$. As far as the stringers are located periodically, we may consider the problem on a half-strip $-\infty < y < 0$, $0 < x < 2l+1$.

On the boundary we have the following conditions:

$$S(x)\sigma_x^{(0)}(x) - h\int_{l}^{x} \tau_{xy}(s)ds - q = 0, \ \ x \in (l;l+1); \tag{2}$$

$$\begin{aligned} &\sigma_y(x) = 0, \quad x \in (0;2l+1),\\ &\tau_{xy} = 0, \qquad x \in (0;l) \cup (l+1;2l+1),\\ &\sigma_x^{(0)}(x) = a, \quad x \in (l;l+1); \end{aligned} \tag{3}$$

$$\begin{aligned} &\tau_{xy}(0;y) - \tau_{xy}(2l+1;y) = \sigma_y(0,y) - \sigma_y(2l+1;y) = \sigma_x(0;y) - \\ &-\sigma_x(2l+1;y) = u(0;y) - u(2l+1;y) = v(0;y) - v(2l+1;y) = 0. \end{aligned} \tag{4}$$

According to Hooke's law, we have respectively for the bar and for the plate:

$$\frac{du_0(x)}{dx} = \frac{\sigma_x^{(0)}(x)}{E_0}; \quad \frac{du(x,0)}{dx} = \frac{\sigma_x(x,0)}{E_1},$$

where E_0 is the modulus of elasticity of the bar; $a_{11} = 1/E_1$ is the elastic constant of the plate.

The conditions of full contact between the elastic bar and the plate

$$\frac{du_0(x)}{dx} = \frac{du(x,0)}{dx}, \quad \tau_{xy}^{(0)}(x) = \tau_{xy}(x)$$

result in the equality $\sigma_x^{(0)}(x) = \frac{E_0}{E_1}\,\sigma_x(x,0)$. Now the boundary conditions (2) and (3) can be written as

$$\begin{aligned} &\frac{E_0}{E_1}\sigma_x(x,0) = a, \quad x \in (l;l+1),\\ &\sigma_y = 0, \quad x \in (0;2l+1), \quad \tau_{xy} = 0, \quad x \in (0;l) \cup (l+1;2l+1),\\ &aS(x) - h\int_{l}^{x} \tau_{xy}(s)ds = q, \quad x \in (l;l+1). \end{aligned} \tag{5}$$

As is known, the stress components are calculated by the formulas [8]

$$\begin{aligned}\sigma_x &= 2\,\mathrm{Re}\left[\mu_1^2\Phi_1(z_1) + \mu_2^2\Phi_2(z_2)\right],\\ \sigma_y &= 2\,\mathrm{Re}\left[\Phi_1(z_1) + \Phi_2(z_2)\right],\\ \tau_{xy} &= -2\,\mathrm{Re}\left[\mu_1\Phi_1(z_1) + \mu_2\Phi_2(z_2)\right],\end{aligned} \tag{6}$$

where $z_k = x_k + \mu_k y$, $k = 1, 2$, and μ_k are the roots of the characteristic equation corresponding to the generalized biharmonic equation.

Due to the periodicity of the boundary conditions, the functions $\Phi_1(z)$ and $\Phi_2(z)$ are also periodic in the half-plane $y < 0$ with period $2l + 1$, that is,

$$\Phi_1(i\beta_1 y) = \Phi_1(2l + 1 + i\beta_1 y), \quad \Phi_2(i\beta_2 y) = \Phi_2(2l + 1 + i\beta_2 y). \tag{7}$$

Since the body is orthotropic and the axes of elastic symmetry are parallel to the coordinate axes, $\mu_1 = i\beta_1$, $\mu_2 = i\beta_2$ (we assume $\beta_1 > \beta_2 > 0$), using formulas (6) the boundary conditions take the form

$$\mathrm{Re}\left[\beta_1^2\Phi_1(x) + \beta_2^2\Phi_2(x)\right] = -\frac{E_1 a}{2E_0}, \quad x \in (l; l+1), \tag{8}$$

$$\mathrm{Re}\left[\Phi_1(x) + \Phi_2(x)\right] = 0, \quad x \in (0; 2l+1), \tag{9}$$

$$\mathrm{Im}\left[\beta_1\Phi_1(x) + \beta_2\Phi_2(x)\right] = 0, \quad x \in (0; l) \cup (l+1; 2l+1), \tag{10}$$

$$aS(x) - h\int_l^x \tau_{xy}(s)ds = q, \quad x \in (l; l+1).$$

Let us prove the validity of the following proposition.

Theorem. *If the boundary conditions* (8), (9), *and* (10) *are fulfilled, then the stress components are expressed in terms of one analytic function.*

Proof. The function $\Phi_1(x) + \Phi_2(x)$ is a boundary value of the function $\mathrm{Im}\, z < 0$ which is holomorphic in the half-plane $\Phi_1(z) + \Phi_2(z)$ and periodic with period $2l + 1$, that is, $\Phi_1(x + 2l + 1) + \Phi_2(x + 2l + 1) = \Phi_1(x) + \Phi_2(x)$, bounded in the half-strip $0 \le x \le 2l + 1$, $y < 0$, continuously extendible to the boundary $0 \le x \le 2l + 1$, with the exclusion maybe of the points $x = l$, $x = l + 1$. In the vicinity of these points the function under consideration satisfies the condition

$$|\Phi_1(z) + \Phi_2(z)| < \frac{c}{|z - (l+k)|^\delta}, \quad k = 0; 1, \quad 0 \le \delta < 1. \tag{11}$$

Since the function $\Phi_1(z) + \Phi_2(z)$ takes imaginary values on the real axis, on the basis of the Riemann-Schwarz symmetry principle it is analytically extendible on the whole strip $0 < x < 2l + 1$, $-\infty < y < \infty$, with the exclusion maybe of the above-mentioned points in whose vicinity the estimate (11) holds.

It follows from the above that these points are removable. Since the function $\Phi_1(z)+\Phi_2(z)$ is periodic, it is bounded on the whole plane.

According to Liouville's theorem, we can conclude that the function $\Phi_1(z)+\Phi_2(z)$ is constant. If we use the conditions (9) and (10), then we can say that the function $\Phi_1(z)+\Phi_2(z)$ equals zero on the whole plane.

$$\Phi_1(z)=-\Phi_2(z) \quad \text{for} \quad \operatorname{Im} z\le 0. \quad \square \tag{12}$$

Applying the above-obtained equality, the boundary conditions (8) and (10) can be written as

$$\begin{gathered}(\beta_1^2-\beta_2^2)\operatorname{Re}\Phi_1(x)=-\frac{aE_1}{2E_0}, \quad x\in(l;l+1),\\ \operatorname{Im}\Phi_1(x)=0, \quad x\in(0;l)\cup(l+1;2l+1).\end{gathered} \tag{13}$$

Thus the problem under consideration is reduced to the problem of finding an analytic in the half-strip $0<x<2l+1$, $y<0$ function $\Phi(z)$ with the boundary conditions (13).

The function

$$z=(2l+1)\Big(1-\frac{1}{2\pi i}\ln\zeta\Big) \tag{14}$$

maps the half-strip $0<\operatorname{Re} z<2l+1$, $\operatorname{Im} z<0$ onto a circle $|\zeta|<1$ cut along the segment $(0;1)$; besides, the point $x=2l+1$ transfers to the point $\zeta=1$, the segment $(0;2l+1)$ maps onto the circumference $|\zeta|=1$, the half-line $x=0$, $y<0$ transfers to the lower end of the cut, and the half-line $x=2l+1$, $y<0$ to the upper end of the cut.

We introduce the notation

$$\Psi(\zeta)=\Phi_1\Big[(2l+1)\Big(1-\frac{1}{2\pi i}\ln\zeta\Big)\Big]. \tag{15}$$

The function $\Psi(\zeta)$ is holomorphic in the circle $|\zeta|<1$ cut along the segment $0<\zeta<1$. From the periodicity of the function $\Phi_1(z)$ we find the equality $\Psi^+(\zeta)=\Psi^-(\zeta)$, $0<\zeta<1$, where $\Psi^+(\zeta)$ and $\Psi^-(\zeta)$ denote the boundary values of the function $\Psi(\zeta)$ on the upper and lower ends, respectively.

From the above we conclude that the function $\Psi(\zeta)$ is holomorphic in the circle $|\zeta|<1$. In this case the boundary conditions (13) take the form

$$\operatorname{Re}[\Psi(\zeta)]=-\frac{aE_1}{2E_0(\beta_1^2-\beta_2^2)}, \quad \zeta\in\gamma_1, \tag{16}$$

$$\operatorname{Im}[\Psi(\zeta)]=0, \quad \zeta\in\gamma_2, \tag{17}$$

where γ_1 denotes an arc of the circumference of unit radius which is the mapping of the segment $(l;l+1)$, and γ_2 denotes the remaining part of the circumference onto which the segments $(0;l)\cup(l+1;2l+1)$ are mapped.

Moreover, to the points $x = 0$ and $x = 2l + 1$ there corresponds the point $\zeta = 1$, i.e., γ_2 is a continuous arc.

If we introduce the notation

$$\psi(\zeta) = \Psi(\zeta) + \frac{E_1 a}{2E_0(\beta_1^2 - \beta_2^2)}, \tag{18}$$

then we obtain

$$\operatorname{Re}\psi(\zeta) = 0 \quad \text{for} \quad \zeta \in \gamma_1, \quad \operatorname{Im}\psi(\zeta) = 0 \quad \text{for} \quad \zeta \in \gamma_2, \tag{19}$$

or

$$\psi(\zeta) + \overline{\psi(\zeta)} = 0 \quad \text{for} \quad \zeta \in \gamma_1, \quad \psi(\zeta) - \overline{\psi(\zeta)} = 0 \quad \text{for} \quad \zeta \in \gamma_2. \tag{20}$$

Introducing a piecewise holomorphic function

$$W(\zeta) = \begin{cases} \psi(\zeta) & \text{for} \quad |\zeta| < 1, \\ \overline{\psi(\overline{\zeta})} & \text{for} \quad |\zeta| > 1, \end{cases} \tag{21}$$

we obtain the problem

$$\begin{cases} W^+(\sigma) + W^-(\sigma) = 0 & \text{for} \quad \sigma \in \gamma_1, \\ W^-(\sigma) - W^-(\sigma) = 0 & \text{for} \quad \sigma \in \gamma_2. \end{cases} \tag{22}$$

A general solution of problem (22) belonging to the class h_0 and bounded at infinity is given by the formula [9]

$$W(\zeta) = \frac{c_0\zeta + \overline{c}_0}{\sqrt{(\zeta - \sigma_1)(\zeta - \sigma_2)}}, \tag{23}$$

where σ_1 and σ_2 are the ends of the arc γ_1, $\sigma_1 = e^{\frac{2\pi l i}{2l+1}}$, $\sigma_2 = e^{\frac{2\pi(l+1)i}{2l+1}}$.

By $\sqrt{(\zeta - \sigma_1)(\zeta - \sigma_2)}$ we mean a function which is holomorphic on the plane cut along γ_1 and satisfies the condition

$$\frac{\zeta}{\sqrt{(\zeta - \sigma_1)(\zeta - \sigma_2)}} \to 1 \quad \text{as} \quad \zeta \to \infty.$$

Taking into account the equalities (21) and (23), from the equality (18) we obtain

$$\Psi(\zeta) = \frac{c_0\zeta + \overline{c}_0}{\sqrt{(\zeta - \sigma_1)(\zeta - \sigma_2)}} - M, \quad M = \frac{E_1 a}{2E_0(\beta_1^2 - \beta_2^2)}. \tag{24}$$

Getting back to the variable z which is connected with the variable ζ by the relation (14), i.e., $\zeta = \exp 2\pi i\big(1 - \frac{z}{2l+1}\big)$, and introducing the variables

$$\rho = e^{\frac{2\pi y}{2l+1}}, \quad \theta = 2\pi\Big(1 - \frac{x}{2l+1}\Big), \tag{25}$$

from formula (24) we find that

$$\Psi[\rho e^{i\theta}] = \Phi_1(z) = \frac{c_0\rho e^{i\theta} + \overline{c_0}}{\sqrt{(\rho e^{i\theta} - e^{i\theta_1})(\rho e^{i\theta} - e^{i\theta_2})}} - M, \qquad (26)$$

where $\theta_1 = \frac{2\pi l}{2l+1}$, $\theta_2 = \frac{2\pi(l+1)}{2l+1}$.

If now instead of y we substitute in the formula (26) the values $\beta_1 y$ and $\beta_2 y$, then applying equalities (25) we get

$$\Phi_1(z) = \frac{c_0\rho_1 e^{i\theta} + \overline{c_0}}{\sqrt{(\rho_1 e^{i\theta} - e^{i\theta_1})(\rho_1 e^{i\theta} - e^{i\theta_2})}} - M, \quad \theta_1 < \theta < \theta_2, \qquad (27)$$

where $\rho_1 = e^{\frac{2\pi\beta_1 y}{2l+1}}$.

With regard for the condition $\Phi_2(z) = -\Phi_1(z)$ for the function $\Phi_2(z)$ we obtain the formula

$$\Phi_2(z) = -\frac{c_0\rho_2 e^{i\theta} + \overline{c_0}}{\sqrt{(\rho_2 e^{i\theta} - e^{i\theta_1})(\rho_2 e^{i\theta} - e^{i\theta_2})}} + M, \quad \theta_1 < \theta < \theta_2, \qquad (28)$$

where $\rho_2 = e^{\frac{2\pi\beta_2 y}{2l+1}}$.

To find a complex constant c_0, we take advantage of the conditions $\sigma_x(x-i\infty) = \sigma_y(x-i\infty) = 0$.

The external forces acting on the stringer are, in general, not in equilibrium ($p - q \neq 0$), that is, the principal vector of tangential stresses does not equal to zero. Therefore, the tangential stresses do not tend to zero as $y \to -\infty$, while σ_y and σ_x vanish as $y \to -\infty$.

Passing to the limit in the equalities (27) and (28), we obtain the following relations:

$$\Phi_1(x - i\infty) = -\Phi_2(x - i\infty) = \overline{c}_0 - M. \qquad (29)$$

Using now formulas (6), we arrive at

$$\operatorname{Re}\left[\beta_1^2\Phi_1(x - i\infty) + \beta_2^2\Phi_2(x - i\infty)\right] = 0,$$
$$\operatorname{Re}\left[\Phi_1(x - i\infty) + \Phi_2(x - i\infty)\right] = 0.$$

The second condition, due to the equalities (29), is satisfied for any $\overline{c}_0$, and from the first condition it follows that

$$\operatorname{Re} c_0 = M. \qquad (30)$$

We use the following equilibrium condition of the stringer:

$$\int_l^{l+1} \tau_{xy}(s)ds = \frac{p-q}{h}. \qquad (31)$$

Applying the third equality from formulas (6), we obtain the following expression for contact tangential stresses:

$$\tau_{xy}(x) = -\frac{2(\beta_1-\beta_2)[\operatorname{Re} c_0 \cos\frac{\theta}{2} - \operatorname{Im} c_0 \sin\frac{\theta}{2}]}{\sqrt{\sin\frac{\theta-\theta_1}{2}\sin\frac{\theta_2-\theta_1}{2}}}. \tag{32}$$

Since $\theta_2-\theta_1=\frac{2\pi}{2l+1}<2\pi$, the value under the radical sign is positive.

In order to determine a form of the stringer, we have to calculate the integral $\int_l^x \tau_{xy}(s)ds$.

With the use of the condition $\theta_1+\theta_2=2\pi$, after elementary transformations we find from the equality (32) that

$$\tau_{xy}(x) = \frac{2(\beta_1-\beta_2)\operatorname{Re} c_0\cos\frac{\theta}{2}}{\sqrt{\sin^2\frac{\theta}{2}-\sin^2\frac{\theta_1}{2}}} + \frac{2(\beta_1-\beta_2)\operatorname{Im} c_0\cos\frac{\theta}{2}}{\sqrt{\cos^2\frac{\theta_1}{2}-\cos^2\frac{\theta}{2}}}.$$

Integrating the last equality, we obtain

$$\begin{aligned}\int_l^x \tau_{xy}(s)ds &= 2(\beta_1-\beta_2)\operatorname{Re} c_0\int_l^x \frac{\cos\frac{\pi s}{2l+1}ds}{\sqrt{\sin^2\frac{\pi s}{2l+1}-\sin^2\frac{(l+1)\pi}{2l+1}}}+\\ &+2(\beta_1-\beta_2)\operatorname{Im} c_0\int_l^x \frac{\sin^2\frac{\pi s}{2l+1}ds}{\sqrt{\cos^2\frac{(l+1)\pi}{2l+1}-\cos^2\frac{\pi s}{2l+1}}}=\\ &=2(\beta_1-\beta_2)\frac{2l+1}{\pi}\Big[\operatorname{Re} c_0\ln\frac{\sin\frac{\pi x}{2l+1}+\sqrt{\sin^2\frac{\pi x}{2l+1}-\sin^2\frac{(l+1)\pi}{2l+1}}}{\sin\frac{\pi l}{2l+1}}+\\ &+\operatorname{Im} c_0\Big(\arcsin\frac{\cos\frac{\pi x}{2l+1}}{\cos\frac{\pi l}{2l+1}}-\frac{\pi}{2}\Big)\Big]\end{aligned} \tag{33}$$

whence we have the formula $\int_l^{l+1}\tau_{xy}(s)ds = 2(\beta_2-\beta_1)(2l+1)\operatorname{Im} c_0$.

If the use is made of the equilibrium equation (31), then we get

$$\operatorname{Im} c_0 = \frac{p-q}{2h(\beta_2-\beta_1)(2l+1)}. \tag{34}$$

Substituting the integral value defined by the equality (33) into the last of the boundary conditions (formulas (5)) for an elastic half-plane, we obtain for an unknown profile of the stinger the following relation:

$$\begin{aligned}aS(x) &= q+2h(\beta_1-\beta_2)\frac{2l+1}{\pi}\Big[\operatorname{Re} c_0\ln\frac{\sin\frac{\pi x}{2l+1}+\sqrt{\sin^2\frac{\pi x}{2l+1}-\sin^2\frac{(l+1)\pi}{2l+1}}}{\sin\frac{\pi l}{2l+1}}+\\ &+\operatorname{Im} c_0\Big(\arcsin\frac{\cos\frac{\pi x}{2l+1}}{\cos\frac{\pi l}{2l+1}}-\frac{\pi}{2}\Big)\Big]. \quad \square\end{aligned}$$

References

1. B.M. Nuller, Contact problem for an elastic wedge reinforced by a bar of equal resistance. (Russian) *Dokl. Akad. Nauk SSSR*, No. 3, **225**(1975) 532–534.

2. B.M. Nuller, Optimal choice of rigidity of thin-shelled supporting elements in contact problems of the theory of elasticity. (Russian) *Izv. Vsesojuzn. Nauchn.-Issled. Inst. Gidrotekhniki* **108**(1975), 115–123.

3. A.N. Zlatin and B.M. Nuller, Optimal rigidity of circular plates lying on an elastic halfplane. (Russian) *Izv. Vsesojuzn. Nauchn.-Issled. Inst. Gidrotekhniki* **110**(1976) 91–94.

4. E.L. Nakhmein and B.M. Nuller, Periodical contact problems for an elastic half-plane reinforced by stringers or beams of equal resistance. (Russian) *Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela* 1978, No. 5, 81–88.

5. L.A. Gogolauri, Contact problem for an elastic anisotropic plate reinforced by a stringer of equal resistance. (Russian) *Soobshch. Bull. Acad. Sci. Georgia* **141**(1991), No. 1, 57–60.

6. N.Kh. Arutunyan and S.M. Mkhitaryan, Periodical contact problem for a half-plane with elastic straps. (Russian) *Prikl. Mat. Mekh.* **33**(1969), No. 5, 813–843.

7. G.A. Morar' and G.Ya. Popov, To the periodic contact problem for a half-plane with elastic straps. (Russian) *Prikl. Mat. Mekh.* **35**(1971), No. 1, 172–178.

8. S.G. Lekhnitskii, Anisotropic plates. (Russian) *Gostekhizdat, Moscow–Laningrad*, 1947.

9. N.I. Muskhelishvili, Singular integral equations. (Translation from the Russian) *P. Noordhoff, Groningen*, 1953.

(Received 08.05.1996)

Author's address:
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., Tbilisi 380093
Georgia