GEORGIAN MATHEMATICAL JOURNAL: Vol. 5, No. 4, 1998, 301-314

COMMUTATIVITY FOR A CERTAIN CLASS OF RINGS

H. A. S. ABUJABAL AND M. A. KHAN

Abstract. We first establish the commutativity for the semiprime ring satisfying $[x^n, y]x^r = \pm y^s[x, y^m]y^t$ for all x, y in R, where m, n, r, s and t are fixed non-negative integers, and further, we investigate the commutativity of rings with unity under some additional hypotheses. Moreover, it is also shown that the above result is true for s-unital rings. Also, we provide some counterexamples which show that the hypotheses of our theorems are not altogether superfluous. The results of this paper generalize some of the well-known commutativity theorems for rings which are right s-unital.

1. Introduction

Let R be an associative ring with $N(R)$, $Z(R)$, $C(R)$, $N'(R)$, and R^+ denoting the set of nilpotent elements, the center, the commutator ideal, the set of all zero divisors, and the additive group of R, respectively. For any x, y in R, $[x, y] = xy - yx$. By $GF(q)$ we mean the Galois field (finite field) with q elements, and by $(GF(q))_2$ the ring of all 2×2 matrices over $GF(q)$. We set

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

in $(GF(p))_2$ for prime p.

There are several results dealing with the conditions under which R is commutative. Generally, such conditions are imposed either on the ring itself or on its commutators. Very recently, Abujabal and Perić [1] remarked that if a ring R satisfies either $[x^n, y]y^t = \pm y^s[x, y^m]$ or $[x^n, y]y^t = \pm[x, y^m]y^s$ for all x, y in R, where $m > 1$, $n \geq 1$, and R has the property $Q(m)$ (see below) for $n > 1$, then R is commutative. Also, under different and appropriate constraints, the commutativity of R has been studied for other values of m, n, s and t (see [1]).

1991 *Mathematics Subject Classification.* 16U80.

Key words and phrases. Semiprime rings, heart of a ring, rings with unity, torsion-free rings, commutators, s-unital rings.

1072-947X/98/0700-0301$15.00/0

The objective of this paper is to generalize the above-mentioned commutativity results to a certain class of rings satisfying the property

$$[x^n, y]x^r = \pm y^s[x, y^m]y^t \quad \text{for all} \quad x, y \in R \tag{$*$}$$

for any fixed non-negative integers m, n, r, s, and t (see [2], [3], and [4]).

In Section 2, we shall prove the commutativity of semiprime rings satisfying $(*)$ and, in Section 3, study the commutativity of rings with unity satisfying $(*)$. However, in Section 4, we extend these results to the wider class of rings that are called right s-unital.

2. Commutativity Theorem for Semiprime Rings

Theorem 1. *Suppose that $n > 0, m, r, s,$ and t are fixed non-negative integers such that $(n, r, s, m, t) \neq (1, 0, 0, 1, 0)$. Let R be a semiprime ring satisfying $(*)$. Then R is commutative.*

Proof. Let R be a semiprime ring satisfying the polynomial identity

$$h(x, y) = [x^n, y]x^r \mp y^s[x, y^m]y^t = 0. \tag{1}$$

Then R is isomorphic to a subdirect sum of prime rings R_i, $i \in I$ (the index set), each of which as a homomorphic image of R satisfies the hypothesis placed on R. Thus we can assume that R is a prime ring satisfying (1). By Posner's theorem [5, Sec. 12.6, Theorem 8], the central quotient of R is a central simple algebra over a field. If the ground field is finite, then the center of R is a finite integral domain, and so R is equal to its central quotient and is a matrix ring $M_\alpha(S)$ for some $\alpha \geq 1$ and some finite field S. Further, we prove that $\alpha = 1$.

If the ground field is infinite and $h(x, y) = 0$ is the polynomial identity for R, we write $h = h_0 + h_1 + h_2 + h_3 + \cdots + h_{m-1} + h_m$ where h_j, $j = 0, 1, 2, \ldots, m$, is a homogeneous polynomial in x, y. Then $g_0(x, y) = g_1(x, y) = \cdots = g_m(x, y) = 0$ for every x, y in R, since the center of R is infinite. Hence $h_0 = h_1 = h_2 = \cdots = h_m = 0$ is also valid in the central quotient of R. Thus $h = h_0 + h_1 + h_2 + \cdots + h_m = 0$ is satisfied by elements in the central quotient of R. Moreover, $h = 0$ is satisfied by elements in $A \otimes_S B$, where A is the central quotient of R, S the center of A, and B any field extension of S [5, Sec. 12.5, Proposition 3]. As a special case, choosing B to be a splitting field of A, we have $A \otimes_S B \simeq M_\alpha(S)$. Now $f = 0$ is satisfied by the elements in $M_\alpha(S)$. So it is enough to prove $\alpha = 1$. Let e_{ij}, $1 \leq i, j \leq \alpha$, be the matrix units in the ring of $\alpha \times \alpha$ matrices. Suppose that $\alpha \geq 2$. Then (1) can be rewritten as $h(x, y) = [x^n, y]x^r \mp y^s[x, y^m]y^t = 0$, which does not hold in $M_\alpha(S)$ because $h(e_{11} + e_{21}, e_{12}) \neq 0$. Hence we get a contradiction, i.e., $\alpha = 1$. So the central quotient of R is contained in the respective ground field. Hence this proves that R itself is commutative. $\square$

Remark 1. In 1989 W. Streb [6] gave some classification of minimal commutative factors of non-commutative rings. This classification is very useful in proving commutativity theorems for rings satisfying conditions that are not necessarily identities (see details in [7]).

If we take $(n, r, s, m, t) = (1, 0, 0, 1, 0)$ in Theorem 1, then $(*)$ becomes an identity.

The following example demonstrates that we cannot extend the above theorem to arbitrary rings.

Example 1. Let $R = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ \beta & \gamma & 0 \end{pmatrix} \middle| \alpha, \beta \text{ and } \gamma \text{ are integers} \right\}$. Then it can be easily verified that R satisfies (1). However, R is not commutative.

One might ask a natural question: "What additional conditions are needed to force the commutativity for arbitrary rings which satisfies $(*)$?". To investigate the commutativity of the ring R with the property $(*)$, we need some extra conditions on R such as the property:

$Q(m)$: for all x, y in $R, m[x, y] = 0$ implies $[x, y] = 0$, where m is some positive integer.

The property $Q(m)$ is an H-property in the sense of [8]. It is easy to check that every m-torsion free ring R has the property $Q(m)$, and every ring has the property $Q(1)$. Further, it is clear that if the ring R has the property $Q(m)$, then R has the property $Q(n)$ for every divisor n of m. In this direction we prove the following theorems.

3. Some Commutativity Theorems for Rings with Unity

The following theorem is a generalization of H. E. Bell and A. Yaqub obtained in the 80s.

Theorem 2. *Suppose that $n > 1, m, r, s,$ and t are fixed non-negative integers, and let R be a ring with unity 1 satisfying $(*)$. Further, if R has property $Q(n)$, then R is commutative.*

We begin with the following well-known result [9, p. 221].

Lemma 1. *Let x, y be elements in a ring R such that $[x, [x, y]] = 0$. Then for any positive integer h, $[x^h, y] = hx^{h-1}[x, y]$.*

The next four lemmas are essentially proved in [3], [10], [11], and [12], respectively.

Lemma 2. *Let R be a ring with unity 1 and x, y in R. If $k[x, y]x^m = 0$ and $k[x, y](x + 1)^m = 0$ for some integers $m \geq 1$ and $k \geq 1$, then necessarily $k[x, y] = 0$.*

Lemma 3. *Let R be a ring with unity 1 and let there exist relatively prime positive integers m and n such that $m[x,y]=0$ and $n[x,y]=0$. Then $[x,y]=0$ for all x,y in R.*

Lemma 4. *Let R be a ring with unity 1. If for each x in R there exists a pair m and n of relative prime positive integers for which $x^m \in Z(R)$ and $x^n \in Z(R)$, then R is commutative.*

Lemma 5. *Let R be a ring with unity 1. If $(1-y^k)=0$, then $(1-y^{km})x=0$ for any positive integers m and k.*

Further, the following results play a key role in proving the main results of this paper. The first and the second are due to Herstein [13] and [14], and the third is due to Kezlan [15].

Theorem A. *Suppose that R is a ring and $n>1$ is an integer. If x^n-x in $Z(R)$ for all x in R, then R is commutative.*

Theorem B. *If for every x and y in a ring R we can find a polynomial $p_{x,y}(z)$ with integral coefficients which depends on x and y such that $[x^2p_{x,y}(x)-x,y]=0$, then R is commutative.*

Theorem C. *Let f be a polynomial in non-commuting indeterminates $x_1,x_2,\ldots,x_n$ with integer coefficients. Then the following statements are equivalent:*
(i) *For any ring R satisfying $f=0$, $C(R)$ is a nil ideal.*
(ii) *For every prime p, $(GF(p))_2$ fails to satisfy $f=0$.*

Now we shall prove the following lemmas:

Lemma 6. *Let $n>1$, m,r,s, and t be fixed non-negative integers, and let R be a ring with unity 1 which satisfies the property $(*)$. Further, if R has $Q(n)$, then $N(R)\subseteq Z(R)$.*

Proof. Let $a\in N(R)$. Then there exists a positive integer p such that

$$a^h \in Z(R) \tag{2}$$

for all $h\geq p$, where p is minimal.

If $p=1$, then $a\in Z(R)$. Now we assume that $p>1$ and $b=a^{p-1}$. Replacing x by b in $(*)$ we get $[b^n,y]y^r=\pm y^s[b,y^m]y^t$. Using (2) and the fact that $(p-1)n\geq p$ for $n>1$, we obtain

$$\pm y^s[b,y^m]y^t=0 \quad \text{for all} \quad y\in R. \tag{3}$$

Replacing $1+b$ for x in $(*)$, we get $[(1+b)^n,y](1+b)^r=\pm y^s[b,y^m]y^t$.

Since $(1+b)$ is invertible (3) leads to

$$[(1+b)^n,y]=0 \quad \text{for all} \quad y\in R. \tag{4}$$

Combining (2) with (4) yields $0 = [(1+b)^n, y] = [1+nb, y] = n[b,y]$. Using the property $Q(n)$ gives $[b,y] = 0$. Thus $a^{p-1} \in Z(R)$, which is a contradiction for the minimality of p. Hence $p = 1$ and $a \in Z(R)$, which implies $N(R) \subseteq Z(R)$. $\square$

Lemma 7. *Let $n > 0, m, r, s,$ and t be fixed non-negative integers, and let R be a ring with unity 1 satisfying $(*)$. Then $C(R) \subseteq Z(R)$.*

Proof. Let $x = e_{11} + e_{21}$ and $y = e_{12}$. Then x and y fail to satisfy $(*)$ whenever $n > 0$ except for $r = 0$, $s = 0$, and $m = 1$. For other cases we can also choose $x = e_{12}$ and $y = e_{22}$. Thus by Theorem C, $C(R)$ is nil and hence by Lemma 6 we get $C(R) \subseteq Z(R)$. $\square$

Remark 2. In view of the above lemma it is guaranteed that Lemma 1 holds for each pair of elements x and y in the ring R which satisfies $(*)$.

Proof of Theorem 2. To get $[x^n, y]x^r = 0$ let $m = 0$ in $(*)$. By Lemmas 1 and 7 this becomes $n[x,y]x^{r+n-1} = 0$. By Lemma 2 and the property of $Q(n)$ this yields the commutativity of R. Let $m \geq 1$ and $k = (\lambda^{n+r} - \lambda)$, where λ is a prime. Then by $(*)$ we have

$$k[x^n, y]x^r = (\lambda^{n+r} - \lambda)[x^n, y]x^r, \quad [(\lambda x)^n, y](\lambda x)^r \mp y^s[(\lambda x), y]y^t = 0.$$

Again by Lemmas 7 and 2 one gets $0 = kn[x,y] = kn[x,y]x^{r+n-1}$. Suppose that $h = kn$; this gives $h[x,y] = 0$. So $[x^h, y] = hx^{h-1}[x,y] = 0$, whence

$$x^h \in Z(R) \quad \text{for all} \quad x \in R. \tag{5}$$

Now, we consider two cases:

Case (1). If $m = 1$ in $(*)$ we get $[x^n, y]x^r = \pm y^s[x,y]y^t$ for all x, y in R. By Lemmas 1 and 7

$$n[x,y]x^{r+n-1} = \pm y^{s+t}[x,y]. \tag{6}$$

Replacing x by x^n in (6), we have

$$n[x,y]x^{n(r+n-1)} = \pm y^{s+t}[x^n, y] \quad \text{for all} \quad x, y \in R. \tag{7}$$

Using Lemmas 1 and 7 together with $(*)$ (7) yields

$$n[x^n, y]x^{n(r+n-1)} = \pm nx^{n-1}[x,y]y^{s+t} = n[x^n, y]x^{r+n-1} \quad \text{for all} \quad x, y \in R.$$

This implies that

$$n[x^n, y]x^{r+n-1}(1 - x^{(n-1)(r+n-1)}) = 0 \quad \text{for all} \quad x, y \in R.$$

By Lemma 5 one can write

$$n[x^n, y]x^{r+n-1}(1 - x^{h(n-1)(r+n-1)}) = 0 \quad \text{for all} \quad x, y \in R. \tag{8}$$

Since R is isomorphic to a subdirect sum of subdirectly irreducible rings Ri $(i \in I)$, each Ri satisfies $(*)$, Lemma 7, and (8). Now we take the ring Ri,

$i \in I$, and assume H is the heart of Ri (i.e., the intersection of all non-zero ideals of Ri). Then $H \neq (0)$ and $Hd = 0$ for any central zero divisor d.

Let $a \in N'(R_i)$. Then by (8) we have

$$n[a^n, y]a^{r+n-1}(1 - a^{h(n-1)(r+n-1)}) = 0 \quad \text{for all} \quad y \in R_i.$$

If $n[a^n, y]a^{r+n-1} \neq 0$, then $a^{h(n-1)(r+n-1)}$ and $1 - a^{h(n-1)(r+n-1)}$ are central zero divisors. Hence $(0) = H(1 - a^{h(n-1)(r+n-1)}) = H$. But $H \neq (0)$, which leads to a contradiction. Hence $n[a^n, y]a^{r+n-1} = 0$ for all $y \in Ri$, $i \in I$. From (7) and the above condition we get

$$0 = \pm y^{s+t}[a^n, y] = n[a^n, y]a^{n(r+n-1)}.$$

Again by Lemma 2, we get $[a^n, y] = 0$ for all $y \in R_i$. Hence

$$\pm[a, y]y^{s+t} = [a^n, y]a^r = 0 \quad \text{and} \quad [a, y] = 0.$$

Let $z \in N(R_i)$. Then by $(*)$ we have

$$(z^{r+n} - z)[x^n, y]x^r = [(zx)^n, y](zx)^r \mp y^s[(zx), y^m]y^t = 0.$$

Lemmas 1, 2, and 7 together with $Q(n)$ give

$$(z^{r+n} - z)[x, y] = 0 \quad \text{for all} \quad x, y \in R_i. \tag{9}$$

Now, as a special case , using (3) we get $(x^{h(r+n)} - x^h)[x, y] = 0$. If $[x, y] = 0$ for all x, y in R_i, then R satisfies $[x, y] = 0$ for all x, y in R and R is commutative. Further, if $[x, y] \neq 0$ for each x, y in R_i, then $x^{h(r+n-1)+1} - x \in N'(R_i)$ and so $x^{h(r+n-1)+1} - x \in N'(R)$. But $[a, y] = 0$ is satisfied by R. So $[x^{h(r+n-1)+1} - x, y] = 0$ for each x, y in R. Hence R is commutative by Theorem A.

Case (2). Let $m > 1$. Then by $(*)$ and together with Lemma 7, we get

$$[x^n, y]x^r = \pm m[x, y]y^{s+t+m-1} \quad \text{for all} \quad x, y \in R. \tag{10}$$

Replacing y by y^m in (10), we get

$$[x^n, y^m]x^r = \pm[x, y^m]y^{m(s+t+m-1)} \quad \text{for all} \quad x, y \in R.$$

Thus by Lemma 1 we obtain $my^{m-1}[x^n, y]x^r = \pm m[x, y^m]y^{m(s+t+m-1)}$.

Applying $(*)$ and Lemma 3, this becomes

$$m[x, y^m]y^{s+t+m-s}(1 - y^{h(m-1)(s+t+m-1)}) = 0 \quad \text{for all} \quad x, y \in R. \tag{11}$$

By Lemmas 7 and 1, $(*)$ becomes

$$n[x, y]x^{r+n-1} = \pm[x, y]y^{s+t+m-1} \quad \text{for all} \quad x, y \in R. \tag{12}$$

Suppose that $a \in N'(R_i)$. Then using (11) and the same argument as in case (1), we write $m[x, a^m]a^{s+t+m-1}(1 - a^{h(m-1)(s+t+m-1)}) = 0$. We can prove that

$$m[x, a^m]a^{s+t+m-1} = 0 \quad \text{for all} \quad x \in R_i. \tag{13}$$

Combining (12) and (13) we get $n[x, a^m]x^{r+n-1} = \pm m[x, a^m]a^{m(s+t+m-1)} = 0$ for all $x \in R_i$. Again using Lemma 1, this yields $n[x, a^m] = 0$. So $nm[x, a]a^{m-1} = 0$. So we shall show that

$$n^2[x, a]x^{r+n-1} = n(n[x, a]x^{r+n-1}) = n([x^n, a]x^r)$$

By $(*)$ and Lemma 7 we get $n^2[x, a]x^{r+n-1} = \pm n(m[x, a]a^{s+t+m-1}) = 0$; replacing x by $x + 1$ and applying Lemma 2 we have $n^2[x, a] = 0$ for all $x \in R_i$, so that $[x^{n^2}, a] = n^2 x^{n^2-1}[x, a] = 0$. This implies that

$$[x^{n^2}, a] = 0 \quad \text{for all} \quad x \in R_i. \tag{14}$$

Next, let $z \in Z(R_i)$. By arguments similar to those we used in case (1) we have $(z^{r+n} - z)[x^n, y] = 0$ for all $x, y \in R_i$. Using (2), we get

$$(y^{h(r+n)} - y^h)[x^n, y] = 0 \quad \text{for all} \quad x, y \in R_i. \tag{15}$$

Let $y \in R_i$ and let $[x^n, y] = 0$. Then $[x^{n^2}, y^j - y] = 0$ for all positive integers j and $x \in R_i$. If $[x^n, y] = 0$, then $[x^{n^2}, y] = 0$. If $[x^n, y] \neq 0$, then (15) implies that $y^{h(r+n)} - y^h$ is a zero divisor. Hence $y^{h(r+n-1)+1} - y$ is also a zero divisor. But $[x^{n^2}, a] = 0$. Therefore

$$[x^{n^2}, y^{h(r+n-1)+1} - y] = 0 \quad \text{for all} \quad x, y \in R_i. \tag{16}$$

Since each R satisfies (16), the original ring R also satisfies (16). But R possesses $Q(n)$. So by Lemma 1 (16) gives $[x, y^{(r+n-1)+1} - y] = 0$. Hence R is commutative by Theorem A. $\square$

Remark 3. The property $Q(n)$ is essential in Theorem 2. To show this, we consider

Example 2. Let

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

be elements of the ring of all 3×3 matrices over Z_2, the ring of integers mod 2. If R is the ring generated by the matrices A, B, and S, then by the Dorroh construction with Z_2, we get a ring with unity. But R is noncommutative and satisfies $[x^2, y] = [x, y^2]$ for all x, y in R.

Remark 4. Also, if we neglect the restriction of unity in the hypothesis, R may be badly noncommutative. Indeed,

Example 3. Let D_k be the ring of $k \times k$ matrices over a division ring D and $A_k = \{a_{ij} \in D_k / a_{ij} = 0, i \leq j\}$. A_k is necessarily noncommutative for any positive integer $k > 2$. Now A_3 satisfies $(*)$ for all positive integers m, n.

Example 4. Let

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

be elements of the ring of all 3×3 matrices over Z_2. If R is the ring generated by the elements A_1, B_1, and S_1, then for each integer $m \geq 1$, the ring R satisfies the identity $[x^m, y] = [x, y^m]$ for all x, y in R, but R is noncommutative.

The following results are direct consequences of Theorem 2.

Corollary 1. *Let $n > 1$ and m be positive integers, and let r and t be any non-negative integers. Suppose that R is a ring with unity satisfying the polynomial identity $[x^n, y]x^r = [x, y^m]y^t$ for all $x, y \in R$. Further, if R has property $Q(n)$, then R is commutative.*

Corollary 2 ([3, Theorem 6]). *Let R be a ring with unity 1, and $n > 1$ be a fixed integer. If R^+ is n-torsion free and R satisfies the identity $x^n y - y x^n = x y^n - y^n x$ for all x, y in R, then R is commutative.*

Corollary 3 ([7, Theorem 2]). *Let $n \geq m \geq 1$ be fixed integers such that $mn > 1$, and let R be a ring with unity 1. Suppose that every commutator in R is m-torsion free. Further, if R satisfies the polynomial identity $[x^n, y] = [x, y^m]$ for all x, y in R. Then R is commutative.*

Now, the following theorem shows that the conclusion of Theorem 2 is still valid if the property $Q(n)$ is replaced by requiring m and n to be relatively prime positive integers.

Theorem 3. *Suppose that $m > 1$, and $n > 1$ are relatively prime positive integers, and r, s, and t are nonnegative integers. Let R be a ring with unity satisfying $(*)$. Then R is commutative.*

Proof. Without loss of generality we may assume that R is a subdirectly irreducible ring. Suppose that $a \in N(R)$ and choose p and b as in Lemma 7. By the arguments of the proof of Lemma 7 we get $n[b, y] = 0$ and $m[b, y] = 0$, whence by Lemma 3 $[b, y] = 0$ for all y in R. But $a^{p-1} \in Z(R)$, i.e., $N(R) \subseteq Z(R)$, and by Lemma 7 we get $C(R) \subseteq N(R) \subseteq Z(R)$. The proof of (2) also works in the present situation. So there exists an integer h such that

$$x^h \in Z(R) \quad \text{for all} \quad x \in R. \tag{17}$$

Suppose that $c \in N'(R)$. Using the same steps as in the proof of Theorem 2 (see (14)), we obtain $[x^{n^2}, c] = 0$ and $[x^{m^2}, c] = 0$. Hence by Lemma 3 we get

$$[x, c] = 0 \quad \text{for all} \quad x \in R \quad \text{and} \quad c \in N'(R). \tag{18}$$

As is observed in the proof followed by (9) one can see that

$$n(z^{r+n} - z)[x, y] = 0 \quad \text{and} \quad m(z^{r+n} - z)[x, y] = 0, \quad \text{where} \quad z \in Z(R).$$

Again using Lemma 3, we get

$$(z^{r+n} - z)[x, y] = 0 \quad \text{for all} \quad x, y \in R \quad \text{and} \quad z \in Z(R). \tag{19}$$

Since $y^h \in Z(R)$, (17) yields $(y^{h(r+n)} - y^h)[x, y] = 0$ for all $x, y \in R$. Using the same arguments as in the proof of Theorem 2, we finally get $y^{h(r+n-1)+1} - y \in N'(R)$ so that (18) gives $y^{h(r+n-1)+1} - y \in Z(R)$ for all $y \in R$. Hence by Theorem A, R is commutative. □

As a consequence of Theorem 3 we obtain

Corollary 4. *Suppose that m and n are relatively prime positive integers, and let r and t be any nonnegative integers. Let R be a ring with unity satisfying $[x^n, y]x^r = [x, y^m]y^t$. Then R is commutative.*

Further, the following result deals with the commutativity of R for the case where $(*)$ is satisfied with $n = 1$. Thus we prove:

Theorem 4. *Suppose that R is a ring with unity, and $m, r, s,$ and t are fixed nonnegative integers such that $(m, r, s, t) \neq (1, 0, 0, 0)$. If R satisfies*

$$[x, y]x^r = \pm y^s[x, y^m]y^t \quad \textit{for all} \quad x, y \in R, \tag{**}$$

then R is commutative.

Proof. First, we consider the following cases:

Case 1. Let $m = 0$ in $(**)$. Then $[x, y]x^r = 0$. Replacing x by $x + 1$ and using Lemma 2, we obtain the commutativity of R.

Case 2. Let $m > 1$ in $(**)$. Then we choose the matrix for $x = e_{22}$ and $y = e_{12}$ fail to satisfy $(**)$. Thus $C, C(R) \subseteq Z(R)$ by Theorem.

Let $a \in N(R)$. Then there exists a positive integer p such that

$$a^h \in Z(R) \quad \text{for all} \quad h \geq p \quad \text{and} \quad p \quad \text{is minimal.} \tag{20}$$

If $p = 1$, then $a \in Z(R)$. Suppose that $p > 1$ and let $b = a^{p-1}$. Replacing b by y in $(**)$ we get $[x, b]x^r = \pm b^s[x, b^m]b^t$. Using (20) we have $[x, b]x^r = 0$ which by Lemma 2 becomes $[x, b] = 0$. Thus $a^{p-1} \in Z(R)$, which contradicts the minimality of p. Hence $p = 1$ and $N(R) \subseteq Z(R)$. Thus $C(R) \subseteq N(R) \subseteq Z(R)$ and the proof of Theorem 2 enables us to establish the commutativity of R.

Case 3. If $m = 1$ in $(**)$, we have

$$[x, y]x^r = \pm y^s[x, y]y^t \quad \text{for all} \quad x, y \in R. \tag{21}$$

Step (i). Assuming $r = 0$ in (21), we get

$$[x, y] = \pm y^s[x, y]y^t \quad \text{for all} \quad x, y \in R. \tag{22}$$

Then either $s > 0$ or $t > 0$. Trivially, we can see that $x = e_{22}$ and $y = e_{12}$ fail to satisfy (22). Hence $C(R) \subseteq N(R)$. Suppose that p and b are defined as in Case (2). Then (22) holds and becomes $[x, b] = \pm b^s[x, b]b^t = 0$ for all x in R, which is a contradiction. Hence $a \in Z(R)$ so that $N(R) \subseteq Z(R)$. Therefore

$$C(R) \subseteq N(R) \subseteq Z(R). \tag{23}$$

Using (23) and Lemma 1 we get $[x, y] = \pm[x, y]y^{r+s}$ for x, y in R. Hence R is commutative by Kezlan [16].

Step (ii). If $s = 0$ in (21), we get

$$[x, y]x^r = \pm[x, y]y^t \quad \text{for all} \quad x, y \in R. \tag{24}$$

Let $t = 0$. Then $r > 0$ and (24) become $[x, y]x^r = \pm[x, y]$. Hence R is commutative [12]. Now, let $r = 0$ and $t > 0$. Then (24) gives $[x, y] = \pm[x, y]y^t$ for all $x, y \in R$. Again R is commutative by Kezlan [16].

Finally, if $r > 0$, $t > 0$, then $x = e_{22}$ and $y = e_{12}$ fail to satisfy (24). Hence by Theorem C, $C(R) \subseteq N(R)$. For any positive integer k we have

$$[x, y]x^{kr} = \pm[x, y]y^{kt} \quad \text{for all} \quad x, y \in R. \tag{25}$$

Let $a \subseteq N(R)$. Then for sufficiently large k, we get $[x, a]x^{kr} = 0$. Using Lemma 2 this gives $a \in Z(R)$ and thus $C(R) \subseteq N(R) \subseteq Z(R)$. Further, we choose $q = (p^{t+1} - p) > 0$ for $t > 0$, p is a prime. We can prove that

$$x^q \in Z(R) \quad \text{for all} \quad x \in R. \tag{26}$$

Using (25) and (26), we get $[x^{qr+1}, y] = \pm[x, y^{qt+1}]$. In view of Proposition 3 (ii) of [10], there exists a positive integer l such that $[x, y^{(qt+1)l}] = 0$ for each x, y in R. But $(qt + l)^l = gh + 1$. So (25) becomes $[x, y]y^{hq} = 0$ and by Lemma 2 R is commutative.

Step (iii). Setting $t = 0$ in (21), we get

$$[x, y] = \pm y^s [x, y] \quad \text{for all} \quad x, y \in R. \tag{27}$$

Now we have $r > 0$ and $s > 0$. Without loss of generality we may assume that $s > 0$. So, trivially, we can see that $x = e_{22}$ and $y = e_{12}$ fail to satisfy (27). Hence by Theorem C, $C(R) \subseteq N(R)$. By the same arguments as in Step (ii) we can show the commutativity of R.

Step (iv). Suppose that $r > 0$, $s > 0$, and $t > 0$ in (21), and suppose that $x = e_{22}$, $y = e_{12}$ fail to satisfy (21). So $C(R) \subseteq N(R)$. p and b are defined in the same manner as in Case (2). So $[x, b]x^r = \pm b^s [x, b] b^t = 0$. Using Lemma 2, we get $[x, b] = 0$, which contradicts the minimality of p. Hence $N(R) \subseteq Z(R)$ so that $C(R) \subseteq N(R) \subseteq Z(R)$.

Since $C(R) \subseteq Z(R)$, we can write $[x, b]x^r = \pm [x, b] y^{s+t}$.

Using the same argument as in step (ii), we can get the commutativity of R. □

Theorem 5. *Suppose that $n > 0$ and m (resp. $m > 0$ and n) are two fixed non-negative integers. Suppose that a ring with unity satisfies the polynomial identity $[x^n \pm y^m, yx] = 0$ for all x, y in R. Further, if R has the property $Q(n)$, then R is commutative.*

Proof. By hypothesis, we $[x^n, y]x = \pm y[x, y^r]$. Hence R is commutative by Theorem 2. □

Corollary 5. *t Let $m > 1$ and $n > 1$ be relatively prime integers and R be a ring with unity satisfying $[x^n \pm y^m, yx] = 0$ for all x, y in R. Then R is commutative.*

Recently, Harmanci [4] proved that if $n > 1$ is a fixed integer and R is a ring with unity 1 which satisfies the identities $[x^n, y] = [x, y^n]$ and $[x^{n+1}, y] = [x, y^{n+1}]$ for each $x, y \in R$, then R must be commutative. In [17], Bell generalized this result. The following theorem further extends the result of Bell.

Theorem 6. *Suppose that $m > 1$ and $n > 1$ are fixed relatively prime integers, and let r, s, and t be fixed non-negative integers: R is a ring with unity satisfying both identities*

$$[x^n, y]x^r = \pm y^s [x, y^n] y^t \quad \text{and} \quad [x^m, y]x^r = \pm y^s [x, y^m] y^t. \tag{$***$}$$

Then R is commutative.

Proof. Suppose that b is as in the proof of Lemma 6. Using the proof of Theorem 1 and Theorem 2 of [4], we can show that $n[b, y] = 0$ and $m[b, y] = 0$ for all y in R. Applying Lemma 3, we get $[b, y] = 0$. By the same argument as in the proof of Lemma 6, we get $N(R) \subseteq Z(R)$. The matrices $x = e_{22}$ and $y = e_{12}$ fail to satisfy $(***)$. Thus by Theorem C, $C(R) \subseteq N(R)$.

And thus $C(R) \subseteq N(R) \subseteq C(R)$. Carrying out the argument of subdirectly irreducible rings for n and m, we obtain integers $\alpha > 1$ and $h > 1$ such that $[x^\alpha - x, y^{n^2}] = 0$ and $[x^h - x, y^{n^2}] = 0$ for all $x, y \in R$. Suppose that $g(x) = (x^\alpha - x)^h - (x^\alpha - x)$. Then $0 = [g(x), y^{n^2}] = n^2[g(x), y]y^{n^2-1}$ and $0 = [g(x), y^{m^2}] = m^2[g(x), y]y^{m^2-1}$. By Lemma 3 and Lemma 4 we get $[g(x), y]y^s = 0$ for all $x, y \in R$ and $s = \max\{m^2-1, n^2-1\}$. So $g(x) \in Z(R)$. But $g(x) = x^2h(x) - x$ with $h(x)$ having integral coefficients. Hence R is commutative by Theorem B. $\square$

As a consequence of Theorem 6 we get the result which is proved in [3].

Corollary 6. *Let $m > 1$ and $n > 1$ be relatively prime positive integers. If R is any ring with unity satisfying both identities $[x^m, y] = [x, y^m]$ and $[x^n, y] = [x, y^n]$ for all $x, y \in R$, then R is commutative.*

4. Extension for s-Unital Rings

We pause to recall a few preliminaries in order to make our paper self-contained as far as possible. A ring R is called right (resp. left) s-unital if $x \in xR$ (resp. $x \in Rx$) for all x in R, and R is called s-unital if for any finite subset F of R there exists an element e in R such that $xe = ex = x$(resp. $ex = x$ or $xe = x$) for all $x \in F$. The element is called a pseudo-identity of F (see [18]). The results proved in the earlier sections can be extended further to right s-unital rings.

The following Lemma is proved in [19].

Lemma 8. *Let R be a right (resp. left) s-unital ring. If for each pair of elements x, y of R there exist a positive integer $k = k(x, y)$ and an element $e^1 = e^1(x, y)$ of R such that $e'x^k = x^k$ and $e'y^k = y^k$ (resp $x^ke' = x^k$ and $y^ke' = y^k$), then R is s-unital.*

Theorem 7. *Suppose that $n > 1, m, r, s,$ and t are fixed non-negative integers, and let R be a right s-unital ring satisfying $(*)$. Further, if R has property $Q(n)$, then R is commutative.*

Proof. Let x and y be arbitrary elements of R. Suppose that R is a right s-unital ring. Then there exists an element $e \in R$ such that $xe = x$ and $ye = y$. Replacing x by e in $(*)$ we get $[e^n, y]e^r = \pm y^s[e, y^m]y^t$ for all $y \in R$. This implies that $y = e^ny$ for all $y \in R$. So $y \in Ry$. Hence in view of Lemma 8 R is an s-unital ring and by Proposition 1 of [8], we may assume that R has unity 1. Thus R is commutative by Theorem 2. $\square$

Corollary 7. *Let r and m be two fixed nonnegative integers. Suppose that R satisfies the polynomial identity $[x, y]x^r = [x, y^m]$ for all $x, y \in R$. Further,*

(i) *if R is a right s-unital ring, then R is commutative except* $(m, r) = (1, 0)$*;*

(ii) *if R is a left s-unital ring, then R is commutative except when* $(m, r) = (0, 1)$ *and* $r > 0$ *and* $m = 1$.

Remark 5. Let $t = 0$ in $(*)$. Then Theorem 7 and Corollary 7 are special cases of [20, Corollary 3] and [20, Theorem 5].

Remark 6. In Corollary 7, for $m > 1, R$ is commutative by Theorem 6. However, for $m = 0$ (resp. $m = 1$ and $r > 0$) it is trivial to prove the commutativity of R.

Theorems such as Theorem 3, Theorem 4, Theorem 5, and Theorem 6 can also be proved for right s-unital rings by the same lines as above employing the necessary variations.

Remark 7. If we take $m = 0$ and $n \geq 1$ in $(*)$, then Theorem 7 need not be true for left s-unital rings. Also, when $m = 0$ and $t = 1$, Corollary 4 is not valid for s-unital rings. Indeed,

Example 5. Let $R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$ be a subring of all 2×2 matrices over $GF(2)$ which is a non-commutative left s-unital ring satisfying $(*)$.

Remark 8. If $m = 0$ and $n > 0$ in $(*)$, then Theorem 7 need not be true for left s-unital rings. Owing to this fact, Example 5 disproves Theorems 3, 4, 5, 6, and 7 for left s-unital case whenever both r and s are positive.

Acknowledgments

The authors would like to thank Professor Abdus Salam, the International Atomic Energy Agency, and UNESCO for hospitality at the International Center for Theoretical Physics, Trieste. We also would like to thank Professor M. S. Narasimhan of the Mathematics Section, ICTP. Further, we extend our thanks to the referee for his valuable comments.

References

1. H. A. S. Abujabal and V. Perić, Commutativity of s-unital rings through a Streb result. *Radovi Mat.* **7**(1991), 73–92.

2. H. A. S. Abujabal and M. A. Khan, On commutativity of one-sided s-unital rings. *Internat. J. Math. Math. Sci.* **15**(1992), 813–818.

3. H. E. Bell, On the power map and ring commutativity. *Canad. Math. Bull.* **21**(1978), 399–404.

4. A. Harmanci, Two elementary commutativity theorems for rings. *Acta Math. Acad. Sci. Hungar.* **29**(1977), 23–29.

5. P. M. Cohn, Algebra, vol. 2. *John Wiley and Sons Inc., London,* 1977.

6. W. Streb, Zur Struktur nichtkommutativer Ringe. *Math. J. Okayama Univ.* **31**(1989), 135–140.

7. H. Komatsu and H. Tominaga, Chacron's condition and commutativity theorems. *Math. J. Okayama Univ.* **31**(1989), 101–120.

8. Y. Hirano, Y. Kobayashi, and H. Tominaga, Some polynomial identities and commutativity of s-unital rings. *Math. J. Okayama Univ.* **24** (1982), 7–13.

9. N. Jacobson, Structure of rings. *Amer. Math. Soc. Colloq. Publ.,* 1964.

10. W. K. Nicholson and A. Yaqub, A commutativity theorem for rings and groups. *Canad. Math. Bull.* **22**(1979), 419–423.

11. M. A. Quadri and M. A. Khan, Commutativity theorem for left s-unital rings. *Bull. Inst. Math. Acad. Sinica* **15**(1987), 323–327.

12. H. Tominaga and A. Yaqub, A commutativity theorem for one-sided s-unital rings. *Math. J Okayama Univ.* **26**(1984), 125–128.

13. I. N. Herstein, A generalization of a theorem of Jacobson. *Amer. J. Math.* **73**(1951), 756–762.

14. I. N. Herstein, The structure of a certain class of rings. *Amer. J. Math.* **75**(1953), 864–871.

15. T. P. Kezlan, A note on commutativity of semiprime PI-rings. *Math. Japon.* **27**(1982), 267–268.

16. T. P. Kezlan, On identities which are equivalent with commutativity. *Math. Japon.* **29**(1984) 135–139.

17. H. E. Bell, On rings with commuting powers. *Math. Japon.* **24**(1979), 473–478.

18. H. Tominaga, On s-unital rings. *Math. J. Okayama* **18**(1976), 117–134.

19. E. Psomopoulos, H. Tominaga, and A. Yaqub, Some commutativity theorems for n-torsion free rings. *Math. J. Okayama Univ.* **23**(1981) 37–39.

20. H. Komatsu, A commutativity theorem for rings. *Math. J. Okayama Univ.* **26**(1984), 109–111.

(Received 27.10.1995, revised 6.09.1996)

Authors' address:
Department of Mathematics
Faculty of Science
King Abdul Aziz University
P. O. Box 31464, Jeddah 21497
Saudi Arabia