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PROPERTIES OF THE SALAGEAN OPERATOR

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Abstract. The object of the present paper is to show the properties of the Salagean operator for analytic functions in the open unit disk. The main results obtained here extend and improve the earlier results obtained by several authors.

Introduction. Let A be the class of functions $f(z)$ of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ which are analytic in the open unit disk $U = \{z : |z| < 1\}$. For $f(z)$ belonging to A, Salagean [1] has introduced the following operator called the Salagean operator:

$$D^0 f(z) = f(z), \quad D^1 f(z) = Df(z) = zf'(z),$$
$$D^n f(z) = D(D^{n-1} f(z)) \quad (n \in N = \{1, 2, 3, \dots\}).$$

Note that $D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k$, $n \in N_0 = \{0\} \cup N$.

A function $f(z) \in A$ is said to belong to the class $S_n(\alpha)$ if it satisfies

$$\operatorname{Re}\left\{\frac{D^{n+1} f(z)}{D^n f(z)}\right\} > \alpha \quad (z \in U)$$

for some α $(0 \le \alpha < 1)$ and $n \in N_0$.

For the class $S_n(\alpha)$, Owa, Shen, and Obradović [2] showed that if $f(z) \in A$ satisfies

$$\left|\frac{D^{n+1} f(z)}{D^n f(z)} - 1\right|^{1-\beta} \left|\frac{D^{n+2} f(z)}{D^{n+1} f(z)} - 1\right|^{\beta} <$$
$$< (1-\alpha)^{1-2\beta}\left(1 - \frac{3}{2}\alpha + \alpha^2\right)^{\beta} \quad (z \in U)$$

for some α $(0 \le \alpha \le 1/2)$ and β $(0 \le \beta \le 1)$, then $f(z) \in S_n(\alpha)$.

Also, Uralegaddi [3] proved that if

$$f(z) = z + a_{m+1} z^{m+1} + a_{m+2} z^{m+2} + \cdots \in S_n(0)$$

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for some $m \in N$ and $n \in N$, then $\operatorname{Re}\left\{\frac{D^n f(z)}{z}\right\}^{1/(n+1)} > \frac{1}{2}$ $(z \in U)$.

In the present paper, we first show that the above conclusions hold under a much weaker hypothesis. We then sharpen the above-mentioned result of Uralegaddi and some related results. Our main tool is the well-known Jack's lemma [4] (see also [5]):

Lemma. *Let $w(z)$ be a nonconstant analytic function in U with $w(0) = 0$. If $|w(z)|$ attains its maximum value at a point $z_0 \in U$ on the circle $|z| = r < 1$, then $z_0 w'(z_0) = m w(z_0)$, where m is real and $m \geq 1$.*

Results and Proofs. Now we derive

Theorem 1. *If $f(z) \in A$ satisfies*

$$\left|\frac{D^{n+1}f(z)}{D^n f(z)} - 1\right|^{\gamma} \left|\frac{D^{n+2}f(z)}{D^{n+1}f(z)} - 1\right|^{\beta} < M(\alpha, \beta, \gamma) \quad (z \in U) \tag{1}$$

for some α $(0 \leq \alpha < 1)$, β $(\beta \geq 0)$, and $\gamma \geq 0$ with $\beta + \gamma > 0$, then $f(z) \in S_n(\alpha)$, where $n \in N_0$ and

$$M(\alpha, \beta, \gamma) = \begin{cases} (1-\alpha)^{\gamma}\left(\frac{3}{2} - \alpha\right)^{\beta} & (0 \leq \alpha \leq 1/2), \\ 2^{\beta}(1-\alpha)^{\beta+\gamma} & (1/2 \leq \alpha < 1). \end{cases}$$

Proof. (i) Let $0 \leq \alpha \leq 1/2$. Then we define the function $w(z)$ by

$$\frac{D^{n+1}f(z)}{D^n f(z)} = \frac{1 + (1-2\alpha)w(z)}{1 - w(z)} \quad (z \in U) \tag{2}$$

with $w(z) \neq 1$. It follows from (2) that

$$\left|\frac{D^{n+1}f(z)}{D^n f(z)} - 1\right|^{\gamma} \left|\frac{D^{n+2}f(z)}{D^{n+1}f(z)} - 1\right|^{\beta} = \left|\frac{2(1-\alpha)w(z)}{1-w(z)}\right|^{\beta+\gamma} \times$$
$$\times \left|1 + \frac{zw'(z)}{w(z)(1 + (1-2\alpha)w(z))}\right|^{\beta} < (1-\alpha)^{\gamma}\left(\frac{3}{2} - \alpha\right)^{\beta}$$

for all $z \in U$. Suppose that there exists a point $z_0 \in U$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1 \quad (w(z_0) \neq 1).$$

Then, by Lemma, we have $w(z_0) = e^{i\theta}$ and $z_0 w'(z_0) = m w(z_0)$ $(m \geq 1)$. Therefore we have

$$\left|\frac{D^{n+1}f(z_0)}{D^n f(z_0)} - 1\right|^{\gamma} \left|\frac{D^{n+2}f(z_0)}{D^{n+1}f(z_0)} - 1\right|^{\beta} = \left|2(1-\alpha)\frac{w(z_0)}{1-w(z_0)}\right|^{\beta+\gamma} \times$$
$$\times \left|1 + \frac{m}{1 + (1-2\alpha)w(z_0)}\right|^{\beta} = \frac{2^{\beta+\gamma}(1-\alpha)^{\beta+\gamma}}{|1-e^{i\theta}|^{\beta+\gamma}} \left|1 + \frac{m}{1 + (1-2\alpha)e^{i\theta}}\right|^{\beta} \geq$$
$$\geq (1+\alpha)^{\beta+\gamma}\left|1 + \frac{m}{2(1-\alpha)}\right|^{\beta} \geq (1-\alpha)^{\gamma}\left(\frac{3}{2} - \alpha\right)^{\beta}$$

which contradicts condition (1) for $0 \le \alpha \le 1/2$. This gives us $|w(z)| < 1$ for all $z \in U$, i.e., $f(z) = S_n(\alpha)$.

(ii) Let $1/2 \le \alpha < 1$. Define the function $w(z)$ by

$$\frac{D^{n+1} f(z)}{D^n f(z)} = \frac{\alpha}{\alpha - (1-\alpha) w(z)} \quad (z \in U).$$

Then condition (1) for $1/2 \le \alpha < 1$ leads us to

$$\frac{|zw'(z) + w(z)|^\beta |w(z)|^\gamma}{|\alpha - (1-\alpha) w(z)|^{\beta+\gamma}} \quad (z \in U). \tag{3}$$

If there exists a point $z_0 \in U$ such that $\max\limits_{|z| \le |z_0|} |w(z)| = |w(z_0)| = 1$, then by using the lemma we have

$$\frac{|z_0 w'(z_0) + w(z_0)|^\beta |w(z_0)|^\gamma}{|\alpha - (1-\alpha) w(z_0)|^{\beta+\gamma}} = \frac{(m+1)|^\beta}{|\alpha - (1-\alpha) w(z_0)|^{\beta+\gamma}} \ge 2^\beta,$$

which contradicts (2.7). Therefore we conclude that $f(z) \in S_n(\alpha)$. $\square$

Remark 1. Several cases of Theorem 1 with special values of the parameters n, α, β, and γ will improve some other interesting known results. For example, taking $n = \alpha = 0$ and $\beta = \gamma = 1$ in Theorem 1, we have

$$\left| \frac{z f''(z)}{f'(z)} \Big(\frac{z f''(z)}{f(z)} - 1 \Big) \right| < \frac{3}{2}, \qquad f(z) \in S^* = S_0(0),$$

which improves the result by Obradović [6, p. 229].

Theorem 2. *Let $f(z) = z + a_{m+1} z^{m+1} + a_{m+2} z^{m+2} + \cdots$ be analytic in U and satisfy*

$$\operatorname{Re}\left\{ \frac{D^{n+1} f(z)}{D^n f(z)} \right\} > \frac{2 - m(n+1)}{2} \quad (z \in U) \tag{4}$$

for some $m \in N$ and $n \in N_0$. Then

$$\operatorname{Re}\left\{ \frac{D^n f(z)}{z} \right\}^{1/(n+1)} > \frac{1}{2} \quad (z \in U). \tag{5}$$

Proof. Define the function $w(z)$ by

$$\left(\frac{D^n f(z)}{z} \right)^{1/(n+1)} = \frac{1}{1 + w(z)} \quad (z \in U). \tag{6}$$

Clearly, $w^{(j)}(0) = 0$ $(j = 0, 1, 2, \dots, m-1)$ and $w(z) \neq -1$. Making use of the logarithmic differentiation of both sides in (6), we have

$$\frac{D^{n+1} f(z)}{D^n f(z)} = 1 - (n+1) \frac{z w'(z)}{1 + w(z)}.$$

Suppose that $|w(z)| \not< 1$ for some $z \in U$. Then by Lemma there exists a point $z_0 \in U$ such that $z_0 w'(z_0) = kw(z_0)$ $(k \geq m)$ with $|w(z_0)| = 1$. This implies that

$$\operatorname{Re}\left\{\frac{D^{n+1}f(z_0)}{D^n f(z_0)}\right\} = 1 - (n+1)k \ \operatorname{Re}\left\{\frac{w(z_0)}{1+w(z_0)}\right\} \leq \frac{2-m(n+1)}{2}$$

which contradicts condition (4). Thus we have $|w(z)| < 1$ for all $z \in U$. This completes the proof of the theorem. □

Remark 2. It is worth mentioning that the lower bound $1/2$ in (5) is the best possible one. The function

$$f_0(z) = \sum_{k=1}^{\infty} \frac{\Gamma(k+n)}{\Gamma(n+1)\Gamma(k)\{(k-1)m+1\}^n}(-1)^{k-1} z^{(k-1)m+1} \tag{7}$$

or its rotation is extremal in the following sense:

$$\inf_{z\in U} \operatorname{Re}\left\{\frac{D^{n+1}f_0(z)}{D^n f_0(z)}\right\} = \frac{2-(n+1)m}{2}, \quad \inf_{z\in U} \operatorname{Re}\left\{\frac{D^n f_0(z)}{z}\right\} = \frac{1}{2}.$$

Indeed, for the function $f_0(z)$ defined by (7), we have

$$D^n f_0(z) = \sum_{k=1}^{\infty} \frac{\Gamma(k+n)}{\Gamma(n+1)\Gamma(k)}(-1)^{k-1} z^{(k-1)m+1} = \frac{z}{(1+z^m)^{n+1}},$$

which yields

$$\inf_{z\in U} \operatorname{Re}\left\{\frac{D^n f_0(z)}{z}\right\}^{1/(n+1)} = \inf_{z\in U} \operatorname{Re}\left\{\frac{1}{1+z^m}\right\} = \frac{1}{2},$$
$$\inf_{z\in U} \operatorname{Re}\left\{\frac{D^{n+1}f_0(z)}{D^n f_0(z)}\right\} = \inf_{z\in U} \operatorname{Re}\left\{1 - m(n+1)\frac{z^m}{1+z^m}\right\} = \frac{2-(n+1)m}{2}.$$

Theorem 3. *If $f(z) = z + a_{m+1}z^{m+1} + a_{m+2}z^{m+2} + \cdots \in Sn(\alpha)$ for some α $(0 \leq \alpha < 1)$, $n \in N_0$, and $m \in N$, then for any β $(0 < \beta \leq m/(2(1-\alpha)))$ we have the sharp estimate*

$$\operatorname{Re}\left\{\frac{D^n f(z)}{z}\right\}^{\beta} > 2^{2\beta(\alpha-1)/m} \quad (z \in U). \tag{8}$$

Proof. Define the function $w(z)$ by

$$\frac{D^n f(z)}{z} = \frac{1}{(1+w(z))^{2(1-\alpha)/m}} \quad (z \in U).$$

Then, clearly, $w^{(j)}(0) = 0$ $(j = 0, 1, 2, \dots, m-1)$ and $w(z) \neq -1$. Using the same method as the proof of Theorem 2, one can show that $|w(z)| < 1$ for all $z \in U$. Thus

$$\begin{aligned}\operatorname{Re}\left\{\frac{D^n f(z)}{z}\right\}^{\beta} &= \operatorname{Re}\left\{\frac{1}{(1+w(z))^{2\beta(1-\alpha)/m}}\right\} \geq \\ &\geq \left\{\operatorname{Re}\left(\frac{1}{1+w(z)}\right)\right\}^{2\beta(1-\alpha)/m} > 2^{2\beta(\alpha-1)/m} \quad (z \in U),\end{aligned}$$

where we use the fact that $\operatorname{Re}\{z^{\lambda}\} \geq \{\operatorname{Re}(z)\}^{\lambda}$ for $\operatorname{Re}(z) > 0$ and $0 \leq \lambda \leq 1$.

That the rezult (8) is the best possible one can be seen by considering the function

$$f_0(z) = \sum_{k=1}^{\infty} \frac{\Gamma(k + 2(1-\alpha)/m - 1)}{\Gamma(2(1-\alpha)/m)\Gamma(k)\{(k-1)m+1\}^n}(-1)^{k-1} z^{k-1} z^{(k-1)m+1}.$$

Since $D^n F_0(z) = z(1+z^m)^{2(\alpha-1)/m}$ for $F_0(z)$, we have

$$\begin{aligned}&\inf_{z\in U} \operatorname{Re}\left\{\frac{D^{n+1}F_0(z)}{D^n F_0(z)}\right\} = \alpha, \\ &\inf_{z\in U} \operatorname{Re}\left\{\frac{D^n F_0(z)}{z}\right\}^{\beta} = \inf_{z\in U} \operatorname{Re}\left\{\frac{1}{(1+z^m)^{2\beta(1-\alpha)/m}}\right\} = 2^{2\beta(\alpha-1)/m}. \quad \square\end{aligned}$$

Remark 3. Theorem 3 sharpens the result by Owa et al. [7] when $m = 1$. Furthermore, if we take $\alpha = 0$ and $\beta = 1/(n+1)$ in Theorem 3, then we obtain

Corollary 1. *If $f(z) = z + a_{m+1}z^{m+1} + a_{m+2}z^{m+2} + \cdots \in S_n(0)$ for some $m \in N$ and $n \in N_0$ with $m(n+1) \geq 2$, then we have the estimate*

$$\operatorname{Re}\left\{\frac{D^n f(z)}{z}\right\}^{1/(n+1)} > \frac{1}{4^{1/m(n+1)}} \quad (z \in U).$$

This sharpens the above-mentioned result by Uralegaddi [3]. If we take $n = \alpha = 0$ and $\beta = m/2$ in Theorem 3, then we have

Corollary 2. *If $f(z) = z + a_{m+1}z^{m+1} + a_{m+2}z^{m+2} + \cdots \in S^*$ with $m \in N$, then*

$$\operatorname{Re}\left\{\frac{f(z)}{z}\right\}^{m/2} > \frac{1}{2} \quad (z \in U).$$

The result is sharp.

It should be pointed out that Corollary 2 shows that the result by Golusin [8, Theorem 5] (see also [5, Theorem 11]) is sharp.

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