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REGULARITY IN MORREY SPACES OF STRONG SOLUTIONS TO NONDIVERGENCE ELLIPTIC EQUATIONS WITH VMO COEFFICIENTS

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Abstract. In this paper, by means of the theories of singular integrals and linear commutators,the authors establish the regularity in Morrey spaces of strong solutions to nondivergence elliptic equations with VMO coefficients.

1. Introduction

Let Ω be an open set of $\mathbb{R}^n$, $p \in (1,\infty)$, and $\lambda \in (0,n)$. For $f \in L^1_{\rm loc}(\Omega)$, let

$$\|f\|^p_{L^{p,\lambda}(\Omega)} = \sup_{\rho>0, x\in\Omega} \frac{1}{\rho^\lambda} \int\limits_{B_\rho(x)\cap\Omega} |f(y)|^p\, dy$$

and define $L^{p,\lambda}(\Omega)$ to be the set of measurable functions f such that $\|f\|_{L^{p,\lambda}(\Omega)} < \infty$, where, and in what follows, $B_\rho(x) = \{y \in \mathbb{R}^n : |x-y| < \rho\}$ for any $\rho > 0$. The space $L^{p,\lambda}(\Omega)$ is usually called the Morrey space.

Assuming $f \in L^{p,\lambda}(\Omega)$, the main purpose of this paper is to investigate the regularity in the Morrey space of the strong solution to the following Dirichlet problem on the second-order elliptic equation in nondivergence form:

$$\begin{cases} Lu \equiv \displaystyle\sum_{i,j=1}^n a_{ij}(x) u_{x_i x_j} = f & a.e. \ \text{ in } \ \Omega, \\ u = 0 & \text{on } \ \partial\Omega, \end{cases} \tag{1.1}$$

where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$; Ω is a bounded domain $C^{1,1}$ of $\mathbb{R}^n$; the coefficients $\{a_{ij}\}_{i,j=1}^n$ of L are symmetric and uniformly elliptic, i.e., for

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some $\nu \geq 1$ and every $\xi \in \mathbb{R}^n$,

$$a_{ij}(x) = a_{ji}(x) \quad \text{and} \quad \nu^{-1}|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \nu|\xi|^2 \tag{1.2}$$

with *a.e.* $x \in \Omega$. Moreover, we assume that $a_{ij} \in VMO(\Omega)$, the space of the functions of vanishing mean oscillation introduced by Sarason in [1].

Our method is based on integral representation formulas established in [2, 3] for the second derivatives of the solution u to (1.1), and on the theories of singular integrals and linear commutators in Morrey spaces. In fact, in §2, we will establish the boundedness in Morrey spaces for a large class of singular integrals and linear commutators. From this, we can deduce the interior estimates and boundary estimates for the solution to (1.1); therefore, by a standard procedure, we can obtain its whole estimates in Morrey spaces (see [3] and [4]). This will be done in §3.

It is worth pointing out that part of the interior estimates for the solution to (1.1) have been obtained in [5]. Here, we obtain the whole interior estimates in a different way, which seems much simpler than the corresponding ones in [5].

2. Singular Integrals and Linear Commutators

First, we have the following general theorem for the boundedness in Morrey spaces of sublinear operators.

Theorem 2.1. *Let $p \in (1, \infty)$ and $\lambda \in (0, n)$. If a sublinear operator T is bounded on $L^p(\mathbb{R}^n)$ and for any $f \in L^1(\mathbb{R}^n)$ with compact support and $x \notin \operatorname{supp} f$,*

$$|Tf(x)| \leq c \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n}\, dy, \tag{2.1}$$

then T is also bounded on $L^{p,\lambda}(\mathbb{R}^n)$.

Proof. Fix $x \in \mathbb{R}^n$ and $r > 0$. Write

$$f(y) = f(y)\chi_{B_{2r}(x)}(y) + \sum_{k=1}^{\infty} f(y)\chi_{B_{2^{k+1}r}(x)\setminus B_{2^k r}(x)}(y) \equiv \sum_{k=0}^{\infty} f_k(y). \tag{2.2}$$

Thus, by the $L^p(\mathbb{R}^n)$-boundedness of T and (2.1), we have

$$\left(\int_{B_r(x)} |Tf(z)|^p\, dz\right)^{1/p} \leq \sum_{k=0}^{\infty} \left(\int_{B_r(x)} |Tf_k(z)|^p\, dz\right)^{1/p} \leq$$

$$\leq c_p\|f_0\|_{L^p(\mathbb{R}^n)} + c\sum_{k=1}^{\infty}\left\{\int\limits_{B_r(x)}\left(\int\limits_{B_{2^{k+1}r}(x)\setminus B_{2^k r}(x)}\frac{|f(y)|}{|z-y|^n}\,dy\right)^p dz\right\}^{1/p} \leq$$

$$\leq c_p r^{\lambda/p}\|f\|_{L^{p,\lambda}(\mathbb{R}^n)} + c_p\sum_{k=1}^{\infty}\frac{r^{n/p}}{(2^k r)^{n/p}}\left(\int\limits_{B_{2^{k+1}r}(x)}|f(y)|^p\,dy\right)^{1/p} \leq$$

$$\leq c_p r^{\lambda/p}\|f\|_{L^{p,\lambda}(\mathbb{R}^n)} + c_p r^{\lambda/p}\|f\|_{L^{p,\lambda}(\mathbb{R}^n)}\sum_{k=1}^{\infty}\frac{1}{2^{k(n-\lambda)/p}} \leq$$

$$\leq c_p r^{\lambda/p}\|f\|_{L^{p,\lambda}(\mathbb{R}^n)}.$$

Therefore, $\|Tf\|_{L^{p,\lambda}(\mathbb{R}^n)} \leq c\|f\|_{L^{p,\lambda}(\mathbb{R}^n)}$. □

Condition (2.1) can be satisfied by many operators such as Bochner–Riesz operators at the critical index, Ricci–Stein's oscillatory singular integral, C. Fefferman's singular multiplier, and the following Calderón–Zygmund operators.

Definition 2.1. Let $k:\ \mathbb{R}^n\setminus\{0\}\to\mathbb{R}$. We say that $k(x)$ is a constant Calderón–Zygmund kernel (constant $C-Z$ kernel) if
(i) $k\in C^\infty(\mathbb{R}^n\setminus\{0\})$;
(ii) k is homogeneous of degree $-n$;
(iii) $\int_\Sigma k(x)\,d\sigma = 0$, where, and in what follows, $\Sigma=\{x\in\mathbb{R}^n : |x|=1\}$.

Definition 2.2. Let Ω be an open subset of $\mathbb{R}^n$ and $k:\Omega\times\{\mathbb{R}^n\setminus\{0\}\}\to\mathbb{R}$. We say that $k(x)$ is a variable $C-Z$ kernel on Ω if
(i) $k(x,\cdot)$ is a $C-Z$ kernel for $a.e.\ x\in\Omega$;
(ii) $\max_{|j|\leq 2n}\|(\partial^j/\partial z^j)k(x,z)\|_{L^\infty(\Omega\times\Sigma)}\equiv M<\infty$.

Let k be a constant or a variable $C-Z$ kernel on Ω. We define the corresponding $C-Z$ operator by

$$Tf(x) = p.v.\int\limits_{\mathbb{R}^n} k(x-y)f(y)\,dy \quad\text{or}\quad Tf(x) = p.v.\int\limits_{\Omega} k(x,x-y)f(y)\,dy.$$

Obviously, in these cases it satisfies the conditions of Theorem 2.1; see Theorems 2.10 and 2.5 in [2]. Thus we have the following simple corollary.

Corollary 2.1. *Let $p\in(1,\infty)$ and $\lambda\in(0,n)$. If k is a constant or a variable $C-Z$ kernel on $\mathbb{R}^n$ and T is the corresponding operator, then there exists a constant $c=c(n,p,\lambda,k)$ or $c=c(n,p,\lambda,k,M)$ such that for all $f\in L^{p,\lambda}(\mathbb{R}^n)$, $\|Tf\|_{L^{p,\lambda}(\mathbb{R}^n)}\leq c(p,\lambda,k)\|f\|_{L^{p,\lambda}(\mathbb{R}^n)}$.*

From this corollary, by a proof similar to that of Theorem 2.11 in [2] (see also Theorem 2.2 in [5]), we obtain the following corollary.

Corollary 2.2. *Let $p \in (1,\infty)$, $\lambda \in (0,n)$, and Ω be an open subset of $\mathbb{R}^n$. If k is a variable $C-Z$ kernel on Ω and T is the corresponding operator, then there exists a constant $c = c(n,p,\lambda,k,M)$ such that for all $f \in L^{p,\lambda}(\Omega)$,*

$$\|Tf\|_{L^{p,\lambda}(\Omega)} \le c(p,\lambda,k)\|f\|_{L^{p,\lambda}(\Omega)}.$$

Now, let us consider the boundedness on Morrey spaces of the linear commutator $[a,T]$ defined by $[a,T]f = T(af)(x) - a(x)Tf(x)$. First, we recall the definitions of the spaces BMO and VMO. For the properties of these spaces, we refer to [1], [6], and [7].

Definition 2.3. Let Ω be an open subset of $\mathbb{R}^n$. We say that any $f \in L^1_{\rm loc}(\Omega)$ is in the space $BMO(\Omega)$ if

$$\sup_{\rho>0, x\in\Omega} \frac{1}{|B_\rho(x)\cap\Omega|} \int\limits_{B_\rho(x)\cap\Omega} |f(y) - f_{B_\rho(x)\cap\Omega}|\,dy \equiv \|f\|_* < \infty,$$

where $f_{B_\rho(x)\cap\Omega}$ is the average over $B_\rho(x)\cap\Omega$ of f.

Moreover, for any $f \in BMO(\Omega)$ and $r>0$, we set

$$\sup_{\rho\le r, x\in\Omega} \frac{1}{|B_\rho(x)\cap\Omega|} \int\limits_{B_\rho(x)\cap\Omega} |f(y) - f_{B_\rho(x)\cap\Omega}|\,dy \equiv \eta(r). \tag{2.3}$$

We say that any $f \in BMO(\Omega)$ is in the space $VMO(\Omega)$ if $\eta(r) \to 0$ as $r \to 0$ and refer to $\eta(r)$ as the VMO modulus of f.

Theorem 2.2. *Let $p \in (1,\infty)$, $\lambda \in (0,n)$ and $a \in BMO(\mathbb{R}^n)$. If a linear operator T satisfies* (2.1) *and $[a,T]$ is bounded on $L^p(\mathbb{R}^n)$, then $[a,T]$ is also bounded on $L^{p,\lambda}(\mathbb{R}^n)$.*

Proof. For any $x \in \mathbb{R}^n$ and $r > 0$, we write f as in (2.2). By the L^p-boundedness of $[a,T]$ and (2.1), we obtain

$$\begin{aligned}
\Bigg(\int\limits_{B_r(x)} \big|[a,T]f(z)\big|^p\,dz\Bigg)^{1/p} &\le \sum_{k=0}^\infty \Bigg(\int\limits_{B_r(x)} \big|[a,T]f_k(z)\big|^p\,dz\Bigg)^{1/p} \le \\
&\le c\|f_0\|_{L^p(\mathbb{R}^n)} + \\
+c\sum_{k=1}^\infty \Bigg\{\int\limits_{B_r(x)} \Bigg(\int\limits_{B_{2^{k+1}r}(x)\setminus B_{2^k r}(x)} &\frac{|a(y)-a(z)|\,|f(y)|}{|z-y|^n}\,dy\Bigg)^p dz\Bigg\}^{1/p} \le \\
&\le cr^{\lambda/p}\|f\|_{L^{p,\lambda}(\mathbb{R}^n)} + \\
+c\sum_{k=1}^\infty \frac{1}{(2^k r)^n}\Bigg\{\int\limits_{B_r(x)} \Bigg(\int\limits_{B_{2^{k+1}r}(x)} &|a(y)-a(z)|\,|f(y)|\,dy\Bigg)^p dz\Bigg\}^{1/p}.
\end{aligned}$$

For any $\sigma > 0$, we set

$$a_\sigma = \frac{1}{|B_\sigma(x)|} \int\limits_{B_\sigma(x)} a(y)\, dy.$$

By the John–Nirenberg's lemma on BMO functions and the fact that $|a_{2^{k+1}r} - a_r| \le c(n)(k+1)\|a\|_*$ (see [7]), we obtain

$$\begin{aligned}
&\Bigg\{ \int\limits_{B_r(x)} \Bigg(\int\limits_{B_{2^{k+1}r}(x)} |a(y) - a(z)|\, |f(y)|\, dy \Bigg)^p dz \Bigg\}^{1/p} \le \\
&\le \Bigg\{ \int\limits_{B_r(x)} |a_r - a(z)|^p\, dz \Bigg\}^{1/p} \int\limits_{B_{2^{k+1}r}(x)} |f(y)|\, dy + \\
&+ c r^{n/p} \int\limits_{B_{2^{k+1}r}(x)} |a(y) - a_r|\, |f(y)|\, dy \le \\
&\le c r^{n+\lambda/p} 2^{k\{n(1-1/p)+\lambda/p\}} \|a\|_* \|f\|_{L^{p,\lambda}(\mathbb{R}^n)} + \\
&+ c r^{n/p} \Bigg(\int\limits_{B_{2^{k+1}r}(x)} |a(y) - a_r|^{p'}\, dy \Bigg)^{1/p'} \Bigg(\int\limits_{B_{2^{k+1}r}(x)} |f(y)|^p\, dy \Bigg)^{1/p} \le \\
&\le c r^{n+\lambda/p} 2^{k\{n(1-1/p)+\lambda/p\}} \|a\|_* \|f\|_{L^{p,\lambda}(\mathbb{R}^n)} c r^{n+\lambda/p} 2^{k\{n(1-1/p)+\lambda/p\}} \times \\
&\times \Bigg\{ \Bigg(\frac{1}{|B_{2^{k+1}r}(x)|} \int\limits_{B_{2^{k+1}r}(x)} |a(y) - a_{2^{k+1}r}|^{p'}\, dy \Bigg)^{1/p'} + \\
&+ |a_{2^{k+1}r} - a_r| \Bigg\} \|f\|_{L^{p,\lambda}(\mathbb{R}^n)} \le \\
&\le c(k+1) r^{n+\lambda/p} 2^{k\{n(1-1/p)+\lambda/p\}} \|a\|_* \|f\|_{L^{p,\lambda}(\mathbb{R}^n)},
\end{aligned}$$

where, and in what follows, $1/p + 1/p' = 1$. Thus,

$$\begin{aligned}
&\Bigg(\int\limits_{B_r(x)} |[a,T]f(z)|^p\, dz \Bigg)^{1/p} \le \\
&\le c r^{\lambda/p} \|a\|_* \|f\|_{L^{p,\lambda}(\mathbb{R}^n)} \Bigg\{ 1 + \sum_{k=1}^{\infty} \frac{k+1}{2^{k(n-\lambda)/p}} \Bigg\} \le c r^{\lambda/p} \|a\|_* \|f\|_{L^{p,\lambda}(\mathbb{R}^n)}.
\end{aligned}$$

Therefore

$$\|[a,T]f\|_{L^{p,\lambda}(\mathbb{R}^n)} \le c\|a\|_* \|f\|_{L^{p,\lambda}(\mathbb{R}^n)}. \qquad \square$$

From this theorem and Theorems 2.7 and 2.10 in [2], we easily deduce the following corollary.

Corollary 2.3. *Let $p \in (1,\infty)$, $\lambda \in (0,n)$, and $a \in BMO(\mathbb{R}^n)$. If k is a constant or a variable $C-Z$ kernel on $\mathbb{R}^n$ and T the corresponding operator, then there exists a constant $c = c(n,p,\lambda,k)$ or $c = c(n,p,\lambda,k,M)$ such that for all $f \in L^{p,\lambda}(\mathbb{R}^n)$,*

$$\|[a,T]f\|_{L^{p,\lambda}(\mathbb{R}^n)} \le c\|a\|_*\|f\|_{L^{p,\lambda}(\mathbb{R}^n)}.$$

From this and the extension theorem of $BMO(\Omega)$-functions in [8, page 42 and 54], by a procedure similar to Theorem 2.11 in [2] and Theorem 2.2 in [5], we can obtain the following corollary.

Corollary 2.4. *Let $p \in (1,\infty)$ and $\lambda \in (0,n)$. Suppose Ω is an open subset of $\mathbb{R}^n$ and $a \in BMO(\mathbb{R}^n)$. If k is a variable $C-Z$ kernel on Ω and T the corresponding operator, then there exists a constant $c = c(n,p,\lambda,k,M)$ such that for all $f \in L^{p,\lambda}(\Omega)$,*

$$\|[a,T]f\|_{L^{p,\lambda}(\Omega)} \le c\|a\|_*\|f\|_{L^{p,\lambda}(\Omega)}.$$

We can also have the following local version of Corollary 2.4; see Theorem 2.13 in [1] for the proof.

Corollary 2.5. *Let $p \in (1,\infty)$ and $\lambda \in (0,n)$. Let Ω be an open subset of $\mathbb{R}^n$, $a \in VMO(\Omega)$, and η be as in (2.3). If k is a variable $C-Z$ kernel on Ω and T the corresponding operator, then for any $\varepsilon > 0$, there exists positive $\rho_0 = \rho_0(\varepsilon,\eta)$ such that for any ball B_r with the radius $r \in (0,\rho_0)$, $B_r \cap \Omega \equiv \Omega_r \neq \varnothing$ and all $f \in L^{p,\lambda}(\Omega_r)$,*

$$\|[a,T]f\|_{L^{p,\lambda}(\Omega_r)} \le c\varepsilon\|f\|_{L^{p,\lambda}(\Omega_r)},$$

where $c = c(a,p,\lambda,k)$ is independent of f and ε.

It is worth pointing out that Corollaries 2.4 and 2.5 have been obtained by Fazio and Ragusa in [5] in a different way. It seems that our method is much simpler than theirs.

Let $\mathbb{R}^n_+ = \{x = (x', x_n) : x' = (x_1,\ldots,x_{n-1}) \in \mathbb{R}^{n-1}, x_n > 0\}$. To establish the boundary estimates of the solutions to (1.1), we need to study the boundedness on $L^{p,\lambda}(\mathbb{R}^n_+)$ of some other integral operators. First, we have the following general theorem for sublinear operators.

Theorem 2.3. *Let $p \in (1,\infty)$, $\lambda \in (0,n)$, and $\widetilde{x} = (x', -x_n)$ for $x = (x', x_n) \in \mathbb{R}^n_+$. If a sublinear operator T is bounded on $L^p(\mathbb{R}^n_+)$ and for any $f \in L^1(\mathbb{R}^n_+)$ with compact support and $x \in \mathbb{R}^n_+$,*

$$|Tf(x)| \le c \int\limits_{\mathbb{R}^n_+} \frac{|f(y)|}{|\widetilde{x} - y|^n}\, dy, \tag{2.4}$$

then T is also bounded on $L^{p,\lambda}(\mathbb{R}^n_+)$.

Proof. Let $z \in \mathbb{R}^n_+$ and $\sigma > 0$. In what follows, we set $B^+_\sigma(z) = B_\sigma(z) \cap \mathbb{R}^n_+$. We consider two cases.

Case 1. $0 \le z_n < 2\sigma$. In this case, we write

$$f(y) = f(y)\chi_{B^+_{2^4\sigma}(z)}(y) + \sum_{\ell=4}^{\infty} f(y)\chi_{B^+_{2^{\ell+1}\sigma}(z)\setminus B^+_{2^\ell\sigma}(z)}(y) \equiv \sum_{\ell=3}^{\infty} f_\ell(y). \quad (2.5)$$

Therefore, by L^p-boundedness of T and (2.4), we obtain

$$\Bigg(\int\limits_{B^+_\sigma(z)} |Tf(x)|^p\,dx\Bigg)^{1/p} \le \sum_{\ell=3}^{\infty}\Bigg(\int\limits_{B^+_\sigma(z)} |Tf_\ell(x)|^p\,dx\Bigg)^{1/p} \le$$

$$\le c\|f_3\|_{L^p(\mathbb{R}^n_+)} + c\sum_{\ell=4}^{\infty}\Bigg\{\int\limits_{B^+_\sigma(z)}\Bigg(\int\limits_{B^+_{2^{\ell+1}\sigma}(z)\setminus B^+_{2^\ell\sigma}(z)} \frac{|f(y)|}{|\widetilde{x}-y|^n}\,dy\Bigg)^p dx\Bigg\}^{1/p} \le$$

$$\le c\sigma^{\lambda/p}\|f\|_{L^{p,\lambda}(\mathbb{R}^n_+)} + c\sum_{\ell=4}^{\infty}\frac{1}{(2^\ell\sigma)^n}\Bigg\{\int\limits_{B^+_\sigma(z)}\Bigg(\int\limits_{B^+_{2^{\ell+1}\sigma}(z)} |f(y)|\,dy\Bigg)^p dx\Bigg\}^{1/p} \le$$

$$\le c\sigma^{\lambda/p}\|f\|_{L^{p,\lambda}(\mathbb{R}^n_+)}\Bigg\{1 + \sum_{\ell=4}^{\infty}\frac{1}{2^{\ell(n-\lambda)/p}}\Bigg\} \le c\sigma^{\lambda/p}\|f\|_{L^{p,\lambda}(\mathbb{R}^n_+)},$$

which is the desirable estimate.

Case 2. There exists $\ell \in \mathbb{N}$ such that $2^\ell\sigma \le z_n < 2^{\ell+1}\sigma$. In this case, we write

$$f(y) = f(y)\chi_{B^+_{2^{\ell+4}\sigma}(z)}(y) + \sum_{k=1}^{\infty} f(y)\chi_{B^+_{2^{\ell+k+4}\sigma}(z)\setminus B^+_{2^{\ell+k+3}\sigma}(z)}(y) \equiv$$

$$\equiv \sum_{k=0}^{\infty} f_k(y). \quad (2.6)$$

From (2.4), it follows that

$$\Bigg(\int\limits_{B^+_\sigma(z)} |Tf(x)|^p\,dx\Bigg)^{1/p} \le c\Bigg\{\int\limits_{B^+_\sigma(z)}\Bigg(\int\limits_{B^+_{2^{\ell+4}\sigma}(z)} \frac{|f(y)|}{|\widetilde{x}-y|^n}\,dy\Bigg)^p dx\Bigg\}^{1/p} +$$

$$+c\sum_{k=1}^{\infty}\Bigg\{\int\limits_{B^+_\sigma(z)}\Bigg(\int\limits_{B^+_{2^{\ell+k+4}\sigma}(z)\setminus B^+_{2^{\ell+k+3}\sigma}(z)} \frac{|f(y)|}{|\widetilde{x}-y|^n}\,dy\Bigg)^p dx\Bigg\}^{1/p} \le$$

$$\leq \frac{c}{(2^\ell\sigma)^n}\Bigg\{\int\limits_{B_\sigma^+(z)}\Bigg(\int\limits_{B^+_{2^{\ell+4}\sigma}(z)}|f(y)|\,dy\Bigg)^p dx\Bigg\}^{1/p}+$$

$$+c\sum_{k=1}^{\infty}\frac{1}{(2^{\ell+k}\sigma)^n}\Bigg\{\int\limits_{B_\sigma^+(z)}\Bigg(\int\limits_{B^+_{2^{\ell+k+4}\sigma}(z)}|f(y)|\,dy\Bigg)^p dx\Bigg\}^{1/p}\leq$$

$$\leq c\sigma^{\lambda/p}\|f\|_{L^{p,\lambda}(\mathbb{R}^n_+)}\Bigg\{\frac{1}{2^{\ell(n-\lambda)/p}}+\sum_{k=1}^{\infty}\frac{1}{2^{(\ell+k)(n-\lambda)/p}}\Bigg\}\leq c\sigma^{\lambda/p}\|f\|_{L^{p,\lambda}(\mathbb{R}^n_+)},$$

by noting that $\ell\in\mathbb{N}$ and $\lambda<n$. From this, it is easy to deduce Theorem 2.3. $\square$

To state the following corollary, we need more notation. Let $a(x) = \{a_{in}(x)\}_{i=1}^n$ be as in (1.2) and define

$$T(x,y)\equiv x-\frac{2x_n}{a_{nn}(y)}a(y).$$

Then, as a simple corollary of the above Theorem 2.3 and Lemma 3.1 in [3], we have

Corollary 2.6. *Let $p\in(1,\infty)$, $\lambda\in(0,n)$, and Ω be an open subset of $\mathbb{R}^n_+$. If k is a variable $C-Z$ kernel on Ω, and for $x\in\Omega$ we define*

$$\widetilde{T}f(x)=\int\limits_{\Omega}k(x,T(x)-y)f(y)\,dy$$

with $T(x)=T(x,x)$, then there exists a constant $c=c(p,\lambda,\nu,k)$ such that for all $f\in L^{p,\lambda}(\Omega)$,

$$\|\widetilde{T}f\|_{L^{p,\lambda}(\Omega)}\leq c\|f\|_{L^{p,\lambda}(\Omega)}.$$

For the linear commutator on $\mathbb{R}^n_+$, we have

Theorem 2.4. *Let $p\in(1,\infty)$, $\lambda\in(0,n)$, and $a\in BMO(\mathbb{R}^n_+)$. If a linear operator $\widetilde{T}$ satisfies (2.4) and $[a,\widetilde{T}]$ is bounded on $L^p(\mathbb{R}^n_+)$, then $[a,\widetilde{T}]$ is also bounded on $L^{p,\lambda}(\mathbb{R}^n_+)$.*

Proof. Let $z\in\mathbb{R}^n_+$ and $\sigma>0$. Similarly to the proof of Theorem 2.3, we also consider two cases.

Case 1. $0\leq z_n<2\sigma$. In this case, we write f as in (2.5). We then have

$$\Bigg(\int\limits_{B_\sigma^+(z)}|[a,\widetilde{T}]f(x)|^p\,dx\Bigg)^{1/p}\leq\sum_{j=3}^{\infty}\Bigg(\int\limits_{B_\sigma^+(z)}|[a,\widetilde{T}]f_j(x)|^p\,dx\Bigg)^{1/p}\leq$$

$$\leq c\|a\|_*\|f_3\|_{L^p(\mathbb{R}^n_+)} +$$

$$+c\sum_{j=4}^{\infty}\left\{\int\limits_{B^+_\sigma(z)}\left(\int\limits_{B^+_{2^{j+1}\sigma}(z)\setminus B^+_{2^j\sigma}(z)}\frac{|a(y)-a(x)||f(y)|}{|\widetilde{x}-y|^n}\,dy\right)^p dx\right\}^{1/p} \leq$$

$$\leq c\sigma^{\lambda/p}\|a\|_*\|f\|_{L^{p,\lambda}(\mathbb{R}^n_+)} +$$

$$+c\sum_{j=4}^{\infty}\frac{1}{(2^j\sigma)^n}\left\{\int\limits_{B^+_\sigma(z)}\left(\int\limits_{B^+_{2^{j+1}\sigma}(z)}|a(y)-a(x)|\,|f(y)|\,dy\right)^p dx\right\}^{1/p} \leq$$

$$\leq c\sigma^{\lambda/p}\|a\|_*\|f\|_{L^{p,\lambda}(\mathbb{R}^n_+)} + c\sum_{j=4}^{\infty}\frac{1}{(2^j\sigma)^n}\left(\int\limits_{B^+_\sigma(z)}|a_\sigma-a(x)|^p\,dx\right)^{1/p}\times$$

$$\times\left(\int\limits_{B^+_{2^{j+1}\sigma}(z)}|f(y)|\,dy\right)+$$

$$+c\sum_{j=4}^{\infty}\frac{1}{(2^j\sigma)^n}\left\{\int\limits_{B^+_\sigma(z)}\left(\int\limits_{B^+_{2^{j+1}\sigma}(z)}|a(y)-a_\sigma|\,|f(y)|\,dy\right)^p dx\right\}^{1/p} \leq$$

$$\leq c\sigma^{\lambda/p}\|a\|_*\|f\|_{L^{p,\lambda}(\mathbb{R}^n_+)} + c\sigma^{\lambda/p}\|a\|_*\|f\|_{L^{p,\lambda}(\mathbb{R}^n_+)}\left(\sum_{j=4}^{\infty}\frac{1}{2^{j(n-\lambda)/p}}\right)+$$

$$+c\sum_{j=4}^{\infty}\frac{1}{(2^j\sigma)^n}(2^j\sigma)^{\lambda/p}\|f\|_{L^{p,\lambda}(\mathbb{R}^n_+)}\sigma^{n/p}\left(\int\limits_{B^+_{2^{j+1}\sigma}(z)}|a(y)-a_\sigma|^{p'}\,dy\right)^{1/p'} \leq$$

$$\leq c\sigma^{\lambda/p}\|a\|_*\|f\|_{L^{p,\lambda}(\mathbb{R}^n_+)} + c\sigma^{n/p}\|f\|_{L^{p,\lambda}(\mathbb{R}^n_+)}\left\{\sum_{j=4}^{\infty}\frac{1}{(2^j\sigma)^{(n-\lambda)/p}}(2^j\sigma)^{n/p'}\times\right.$$

$$\left.\times\left[\left(\frac{1}{|B^+_{2^{j+1}\sigma}(z)|}\int\limits_{B^+_{2^{j+1}\sigma}(z)}|a(y)-a_{2^{j+1}\sigma}|^{p'}\,dy\right)^{1/p'}+|a_{2^{j+1}\sigma}-a_\sigma|\right]\right\} \leq$$

$$\leq c\sigma^{\lambda/p}\|a\|_*\|f\|_{L^{p,\lambda}(\mathbb{R}^n_+)} +$$

$$+c\sigma^{\lambda/p}\|f\|_{L^{p,\lambda}(\mathbb{R}^n_+)}\sum_{j=4}^{\infty}\frac{1}{2^{j(n-\lambda)/p}}\big\{\|a\|_*+|a_{2^{j+1}\sigma}-a_\sigma|\big\} \leq$$

$$\leq c\sigma^{\lambda/p}\|a\|_*\|f\|_{L^{p,\lambda}(\mathbb{R}^n_+)}\left\{\sum_{j=4}^{\infty}\frac{j+1}{2^{j(n-\lambda)/p}}\right\} \leq c\sigma^{\lambda/p}\|a\|_*\|f\|_{L^{p,\lambda}(\mathbb{R}^n_+)},$$

where for any $r > 0$ we set

$$a_r = \frac{1}{|B_r^+(x)|} \int\limits_{B_r^+(x)} a(y)\,dy$$

and use the fact that $\lambda < n$ and $|a_{2^{j+1}\sigma} - a_\sigma| \le c(n)j\|a\|_*$. This estimation is the expected one.

Case 2. There exists $\ell \in \mathbb{N}$ such that $2^\ell \sigma \le z_n < 2^{\ell+1}\sigma$. In this case, we write f as in (2.6). We then have

$$\Big(\int\limits_{B_\sigma^+(z)} |[a,\widetilde{T}]f(x)|^p\,dx \Big)^{1/p} \le \sum_{k=0}^{\infty} \Big(\int\limits_{B_\sigma^+(z)} |[a,\widetilde{T}]f_k(x)|^p\,dx \Big)^{1/p} \le$$

$$\le c \Big\{ \int\limits_{B_\sigma^+(z)} \Big(\int\limits_{B^+_{2^{\ell+4}\sigma}(z)} \frac{|a(y)-a(x)||f(y)|}{|\widetilde{x}-y|^n}\,dy \Big)^p dx \Big\}^{1/p} +$$

$$+ c\sum_{k=1}^{\infty} \Big\{ \int\limits_{B_\sigma^+(z)} \Big(\int\limits_{B^+_{2^{k+\ell+4}\sigma}(z)\setminus B^+_{2^{k+\ell+3}\sigma}(z)} \frac{|a(y)-a(x)||f(y)|}{|\widetilde{x}-y|^n}\,dy \Big)^p dx \Big\}^{1/p} \le$$

$$\le \frac{c}{(2^\ell\sigma)^n} \Big\{ \int\limits_{B_\sigma^+(z)} \Big(\int\limits_{B^+_{2^{\ell+4}\sigma}(z)} |a(y)-a(x)||f(y)|\,dy \Big)^p dx \Big\}^{1/p} +$$

$$+ c\sum_{k=1}^{\infty} \frac{1}{(2^{k+\ell}\sigma)^n} \Big\{ \int\limits_{B_\sigma^+(z)} \Big(\int\limits_{B^+_{2^{k+\ell+4}\sigma}(z)} |a(y)-a(x)|\,|f(y)|\,dy \Big)^p dx \Big\}^{1/p} \le$$

$$\le c\sigma^{\lambda/p}\|a\|_*\|f\|_{L^{p,\lambda}(\mathbb{R}^n_+)} \Big\{ \sum_{k=1}^{\infty} \frac{1}{2^{(k+\ell)(n-\lambda)/p}} \Big\} \le c\sigma^{\lambda/p}\|a\|_*\|f\|_{L^{p,\lambda}(\mathbb{R}^n_+)},$$

since $\ell \in \mathbb{N}$ and $\lambda < n$. Here, in the next to the last inequality, we used a computation technique similar to Case 1. By the above estimate we easily finish the proof of Theorem 2.4. □

We can also obtain the local version of Theorem 2.4; see Theorem 2.13 in [2] for the proof.

Corollary 2.7. *Let $p \in (1,\infty)$, $\lambda \in (0,n)$, and for any $\sigma > 0$, $B_\sigma^+(x) = \{(x', x_n) \in \mathbb{R}^n : |x| < \sigma, x_n > 0\}$. Let $a \in VMO(\mathbb{R}^n_+)$ and η be its VMO modulus. If k is a variable $C-Z$ kernel on $\mathbb{R}^n_+$ and $\widetilde{T}$ is as in Corollary 2.6, then for any $\varepsilon > 0$, there exists a positive $\rho_0 = \rho_0(\varepsilon,\eta)$ such that for* *any* $r \in (0, \rho_0)$ *and* $f \in L^{p,\lambda}(B_r^+)$,

$$\|[a, \widetilde{T}]f\|_{L^{p,\lambda}(B_r^+)} \le c\varepsilon \|f\|_{L^{p,\lambda}(B_r^+)}$$

with $c = c(\nu, p, \lambda, k)$ *independent of* f, ε, *and* r.

3. Elliptic Equations with VMO Coefficients

In this section, we will establish the regularity of the solution to (1.1). First, we have the following definition.

Definition 3.1. Let $p \in (1, \infty)$, $\lambda \in (0, n)$, and Ω be an open subset of $\mathbb{R}^n$. $f \in L^1_{\rm loc}(\Omega)$ is said to belong to the Sobolev–Morrey space $W^2L^{p,\lambda}(\Omega)$ if and only if u and its distributional derivatives, u_{x_i}, $u_{x_ix_j}$ $(i, j = 1, \dots, n)$ are in $L^{p,\lambda}(\Omega)$. Moreover, let $\|u\|_{W^2L^{p,\lambda}(\Omega)} \equiv \|u\|_{L^{p,\lambda}(\Omega)} + \sum_{i=1}^n \|u_{x_i}\|_{L^{p,\lambda}(\Omega)} + \sum_{i,j=1}^n \|u_{x_ix_j}\|_{L^{p,\lambda}(\Omega)}$.

We also assume that $f \in W^2_{loc}L^{p,\lambda}(\Omega)$ if $f \in W^2L^{p,\lambda}(\Omega')$ for every $\Omega' \subset\subset \Omega$.

Now, let Ω be an open bounded subset of $\mathbb{R}^n$ with $n \ge 3$ and $\partial\Omega \in C^{1,1}$,

$$L \equiv \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}$$

with a_{ij}'s satisfying (1.2). We also assume that $a_{ij} \in VMO(\Omega)$. Since for each function in $VMO(\Omega)$ there is an extension to $\mathbb{R}^n$ with the VMO modulus controlled by its original one (see [8, page 42 and 54]), without loss of generality, we may assume that a_{ij}'s belong to $VMO(\mathbb{R}^n)$. Let $f \in L^{p,\lambda}(\Omega)$, $p \in (1, \infty)$ and $\lambda \in (0, n)$. We are interested in the following Cauchy problem:

$$\begin{cases} Lu = f \quad a.e. \ \text{ in } \ \Omega, \\ u \in W^2L^{p,\lambda}(\Omega) \cap W_0^{1,p}(\Omega). \end{cases} \tag{3.1}$$

Note that Ω is bounded; therefore $f \in L^{p,\lambda}(\Omega)$ implies $f \in L^p(\Omega)$. By the results in [3], we know that for $f \in L^{p,\lambda}(\Omega)$ with $p \in (1, \infty)$ and $\lambda \in (0, n)$, (3.1) has a unique solution $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ satisfying

$$\|u\|_{W^{2,p}(\Omega)} \le c\|f\|_{L^p(\Omega)}, \tag{3.2}$$

where c is independent of f. Our main interest here is to improve (3.2) into

$$\|u\|_{W^2L^{p,\lambda}(\Omega)} \le c\|f\|_{L^{p,\lambda}(\Omega)}. \tag{3.3}$$

By a standard procedure, the proof of (3.3) consists in establishing the interior and boundary estimates of the solution to (3.1); see Theorems 4.4, 4.1, and 4.2 in [3]. Indeed, by a similar proof to Theorem 4.2 in [2] and Theorem 3.3 in [5], we can prove the following interior estimate.

Theorem 3.1. *Let L satisfy the above assumption and $\eta=(\sum_{i,j=1}^n \eta_{ij}^2)^{1/2}$ where η_{ij} is the VMO modulus of a_{ij} in Ω. Let $\lambda \in (0,n)$, q, $p \in (1,\infty)$, $q \le p$, $f \in L^{p,\lambda}(\Omega)$, $u \in W^2L^{q,\lambda}(\Omega)\cap W_0^{1,q}(\Omega)$, and $Lu = f$ a.e. in Ω. Then $u \in W_{loc}^2 L^{p,\lambda}(\Omega)$. Moreover, given any $\Omega' \subset\subset \Omega$, there exists a constant $c = c(n,p,\lambda,\nu, dist(\Omega',\partial\Omega),\eta)$ such that*

$$\|u\|_{W^2L^{p,\lambda}(\Omega')} \le c\big\{\|u\|_{L^{p,\lambda}(\Omega)} + \|f\|_{L^{p,\lambda}(\Omega)}\big\}.$$

The case $q = p$ of Theorem 3.1 is just Theorem 3.3 in [5]. See [5] for the proof of Theorem 3.1, and we omit the details here. To finish the proof of (3.3), we still need to establish the following boundary estimate.

Theorem 3.2. *Let L, λ, q, p, and η be as in Theorem 3.1. Let $f \in L^{p,\lambda}(\Omega)$, $u \in W^2L^{q,\lambda}(\Omega) \cap W_0^{1,q}(\Omega)$, and $Lu = f$ a.e. in Ω. Then $u \in W^2L^{p,\lambda}(\Omega)$ and there exists a constant $c = c(n,p,\lambda,\nu,\partial\Omega,\eta)$ such that*

$$\|u\|_{W^2L^{p,\lambda}(\Omega)} \le c\big\{\|u\|_{L^{p,\lambda}(\Omega)} + \|f\|_{L^{p,\lambda}(\Omega)}\big\}.$$

To prove Theorem 3.2, we need to introduce more notation. We let $W_{\gamma_0}^{2,p}(B_\sigma^+)$ be the closure in $W^{2,p}$ of the subspace

$$C_{\gamma_0} = \Big\{u:\ \ u \text{ is the restriction to } B_\sigma^+ \\ \text{of a function in } C_0^\infty(B_\sigma^+) \text{ and } u(x',0)=0\Big\},$$

where, and in what follows, $B_\sigma = \{x \in \mathbb{R}^n : |x| < \sigma\}$ and $B_\sigma^+ = \{(x',x_n) \in \mathbb{R}^n : |x| < \sigma \text{ and } x_n > 0\}$ for any $\sigma > 0$. We also make the following assumption and refer to it as assumption (H).

$$\begin{cases} \text{Let } n\ge 3,\ \ b_{ij}\in VMO(\mathbb{R}^n),\ \ i,j=1,\dots,n, \\ b_{ij}(x)=b_{ji}(x),\ \ i,j=1,\dots,n,\ \ a.e. \text{ in } B_\sigma^+. \\ \text{There exists } \mu>0 \text{ such that for all } \xi\in\mathbb{R}^n, \\ \mu^{-1}|\xi|^2 \le \sum_{i,j=1}^n b_{ij}(x)\xi_i\xi_j \le \mu|\xi|^2,\ \ a.e. \text{ in } B_\sigma^+. \end{cases} \tag{H}$$

Let

$$\widetilde{L} \equiv \sum_{i,j=1}^n b_{ij}\frac{\partial^2}{\partial x_i\partial x_j}$$

and

$$\Gamma(x,t) = \frac{1}{(n-2)\omega_n(\mathrm{det} b_{ij})^{1/2}}\Big(\sum_{i,j=1}^n B_{ij}(x)t_it_j\Big)^{(2-n)/2},$$

$$\Gamma_i(x,t) = \frac{\partial}{\partial t_i}\Gamma(x,t),\quad \Gamma_{ij}(x,t) = \frac{\partial^2}{\partial t_i\partial t_j}\Gamma(x,t),$$

for *a.e.* $x \in B_\sigma^+$ and all $t \in \mathbb{R}^n \setminus \{0\}$, where B_{ij}'s are the entries of the inverse of the matrix $\{b_{ij}\}_{i,j=1,\dots,n}$. We also set $b(x) = \{b_{in}(x)\}_{i=1}^n$ and

$$T(x,y) = x - \frac{2x_n}{b_{nn}(y)} b(y).$$

By a covering and flattening argument, the proof of Theorem 3.2 can be reduced to establishing the estimate in $L^{p,\lambda}(B_r^+)$ of $u_{x_i x_j}$ with $u \in W_{\gamma_0}^{2,q}(B_r^+)$ and $\widetilde{L}u \in L^{p,\lambda}(B_r^+)$, where $1 < q \le p < \infty$ and $\lambda \in (0,n)$. To do so, we need the following key lemma established in [3, page 847].

Lemma 3.1. *Assume (H) and let $u \in W_{\gamma_0}^{2,p}(B_\sigma^+)$ with $p \in (1,\infty)$. Then*

$$u_{x_i x_j}(x) =$$
$$= p.v. \int_{B_\sigma^+} \Gamma_{ij}(x, x-y) \left\{ \sum_{h,k=1}^n [b_{hk}(x) - b_{hk}(y)] u_{x_h x_k}(y) + \widetilde{L}u(y) \right\} dy$$
$$+ \widetilde{L}u(x) \int_{|t|=1} \Gamma_i(x,y) t_j \, d\sigma_t + I_{ij}(x), \tag{3.4}$$

where for $i, j = 1, \dots, n-1$,

$$I_{ij}(x) =$$
$$= p.v. \int_{B_\sigma^+} \Gamma_{ij}(x, T(x) - y) \left\{ \sum_{h,k=1}^n [b_{hk}(x) - b_{hk}(y)] u_{x_h x_k}(y) + \widetilde{L}u(y) \right\} dy;$$

for $i = 1, \dots, n-1$,

$$I_{in}(x) = I_{ni}(x) = \int_{B_\sigma^+} \left(\sum_{\ell=1}^n B_\ell(x) \Gamma_{i\ell}(x, T(x) - y) \right) \{\cdots\} \, dy,$$

and

$$I_{nn}(x) = \int_{B_\sigma^+} \sum_{\ell,k=1}^n B_\ell(x) B_k(x) \Gamma_{\ell k}(x, T(x) - y) \{\cdots\} \, dy;$$

in the formulas above $T(x) = T(x;x)$, $B_i(x)$ is the i-th component of the vector $B(x) = T(e_n;x)$ with $e_n = (0,\dots,0,1)$, t_j is the j-th component of the outer normal to the sphere Σ, and in the curly brackets there is always the same expression as in the first case.

Now, to finish the proof of Theorem 3.2, we only need to prove the following theorem.

Theorem 3.3. *Assume* (H). *Let* $\lambda \in (0,n)$, $q,p \in (1,\infty)$, *and* $q \le p$. *Set* $\widetilde{\eta} = (\sum_{i,j=1}^{n} \widetilde{\eta}_{ij}^2)^{1/2}$, *where* $\widetilde{\eta}_{ij}$ *is the* VMO *modulus of* b_{ij}, *and*

$$M \equiv \max_{i,j=1,\dots,n} \max_{|\alpha|\le 2n} \left\| \frac{\partial^\alpha}{\partial t^\alpha} \Gamma_{ij}(x,t) \right\|_{L^\infty(B_\sigma^+ \times \Sigma)}.$$

Then there exists a positive number $\rho_0 = \rho_0(n,q,p,M,\mu,\widetilde{\eta},\lambda)$, $\rho_0 < \sigma$, *such that for any* $r \in (0,\rho_0)$ *and any* $u \in W^{2,q}_{\gamma_0}(B_r^+)$ *with* $\widetilde{L}u \in L^{p,\lambda}(B_r^+)$, *we have* $u \in W^2L^{p,\lambda}(B_r^+)$. *Furthermore, there exists a constant* $c = c(n,p,\lambda,M,\mu,\widetilde{\eta})$ *such that*

$$\|u_{x_ix_j}\|_{L^{p,\lambda}(B_r^+)} \le c \|\widetilde{L}u\|_{L^{p,\lambda}(B_r^+)}. \tag{3.5}$$

Proof. Set for $i,j,h,k = 1,\dots,n$

$$S_{ijhk}(f)(x) = p.v. \int_{B_r^+} \Gamma_{ij}(x, x-y)(b_{hk}(x) - b_{hk}(y)) f(y)\, dy,$$

and for $i,j = 1,\dots,n-1$, $h,k = 1,\dots,n$

$$\widetilde{S}_{ijhk}(f)(x) = \int_{B_r^+} \Gamma_{ij}(x, T(x)-y)(b_{hk}(x) - b_{hk}(y)) f(y)\, dy,$$

for $i = 1,\dots,n-1$, $h,k = 1,\dots,n$

$$\widetilde{S}_{inhk}(f)(x) = \int_{B_r^+} \Big(\sum_{j=1}^{n} \Gamma_{ij}(x, T(x)-y) B_j(x) \Big) (b_{hk}(x) - b_{hk}(y)) f(y)\, dy,$$

and, finally, for $h,k = 1,\dots,n$

$$\widetilde{S}_{nnhk}(f)(x) =$$

$$= \int_{B_r^+} \Big(\sum_{i,j=1}^{n} \Gamma_{ij}(x, T(x)-y) B_i(x) B_j(x) \Big) (b_{hk}(x) - b_{hk}(y)) f(y)\, dy,$$

where $r \in (0,\sigma]$ and $f \in L^{s,\lambda}(B_r^+)$.

By Lemma 3.1 in [2] and Corollaries 2.5 and 2.7, we can fix ρ_0 so small that

$$\sum_{i,j,h,k} \|S_{ijhk} + \widetilde{S}_{ijhk}\| < 1,$$

where the norm of operators $S_{ijhk} + \widetilde{S}_{ijhk}$ is the norm in the space of linear operators from $L^{s,\lambda}(B_r^+)$ into itself if $r \in (0,\rho_0)$ and $s \in [q,p]$.

Consider $u \in W^{2,p}_{\gamma_0}(B^+_r)$ with $\widetilde{L}u \in L^{p,\lambda}(B^+_r)$, $r \in (0, \rho_0)$ and set

$$h_{ij} = p.v. \int\limits_{B^+_r} \Gamma_{ij}(x, x-y)\widetilde{L}u(y)\, dy + \widetilde{L}u(x) \int\limits_{|t|=1} \Gamma_i(x,t)t_j\, d\sigma_t + \widetilde{I}_{ij}(x),$$

where

$$\widetilde{I}_{ij}(x) = \begin{cases} \int\limits_{B^+_r} \Gamma_{ij}(x, T(x)-y)\widetilde{L}u(y)\, dy \quad \text{for } i,j = 1,\dots,n-1, \\ \int\limits_{B^+_r} \Big(\sum\limits_{\ell=1}^{n} \Gamma_{i\ell}(x, T(x)-y)B_\ell(x)\Big)\widetilde{L}u(y)\, dy \\ \qquad\qquad\qquad\qquad \text{for } i = 1,\dots,n-1, \quad j = n, \\ \int\limits_{B^+_r} \Big(\sum\limits_{\ell,m=1}^{n} \Gamma_{\ell m}(x, T(x)-y)B_\ell(x)B_m(x)\Big)\widetilde{L}u(y)\, dy \quad \text{for } i=j=n. \end{cases}$$

From Corollary 2.6, we easily deduce that $h_{ij} \in L^{p,\lambda}(B^+_r)$.

Consider $w \in [L^{p,\lambda}(B^+_r)]^{n^2}$ and define $Tw : [L^{p,\lambda}(B^+_r)]^{n^2} \to [L^{p,\lambda}(B^+_r)]^{n^2}$ by setting

$$Tw = ((Tw)_{ij})_{i,j=1,\dots,n} = \Big(\sum_{h,k=1}^{n} (S_{ijhk} + \widetilde{S}_{ijhk})(w_{ij}) + h_{ij}\Big)_{i,j=1,\dots,n}.$$

The operator T is a contraction on $[L^{p,\lambda}(B^+_r)]^{n^2}$ and thus has a unique fixed point $\widetilde{w}$. Since, by (3.4), $\{u_{x_ix_j}\}_{i,j=1,\dots,n}$ is also a fixed point in $[L^{q,\lambda}(B^+_r)]^{n^2}$ and $L^{p,\lambda}(B^+_r) \subseteq L^{q,\lambda}(B^+_r)$ if $q \le p$, the uniqueness of the fixed point implies that $u_{x_ix_j} = \widetilde{w}_{ij} \in L^{p,\lambda}(B^+_r)$ for $i,j = 1,\dots,n$. Then (3.5) can be deduced from (3.4), and Corollaries 2.2, 2.4, 2.6, and 2.7. $\square$

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