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SOLUTION OF SOME WEIGHT PROBLEMS FOR THE RIEMANN–LIOUVILLE AND WEYL OPERATORS

A. MESKHI

ABSTRACT. The necessary and sufficient conditions are found for the weight function v, which provide the boundedness and compactness of the Riemann–Liouville operator R_α from L^p to L^q_v. The criteria are also established for the weight function w, which guarantee the boundedness and compactness of the Weyl operator W_α from L^p_w to L^q.

In this paper, the necessary and sufficient conditions are found for the weight function v (w), which provide the boundedness and compactness of the Riemann-Liouville transform $R_\alpha f(x) = \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt$ (of the Weyl transform $W_\alpha f(x) = \int_x^\infty \frac{f(t)}{(t-x)^{1-\alpha}} dt$) from L^p to L^q_v (from L^p_w to L^q) when $1 < p,\ q < \infty$, $\frac{1}{p} < \alpha < 1$ or $\alpha > 1$ ($\frac{q-1}{q} < \alpha < 1$ or $\alpha > 1$).

A complete description of the weight pairs (v, w) providing the boundedness of the operators R_α and W_α from L^p_w to L^q_v when $1 < p < q < \infty$ and $0 < \alpha < 1$ is given in [1]. For $1 < p \le q < \infty$ and $\alpha > 1$ a similar problem has been solved by many authors (see, e.g., [2, 3]).

The necessary and sufficient conditions for pairs of weights, which provide the boundedness of the above-mentioned operators when $1 < q < p < \infty$ and $\alpha > 1$, are obtained in [4].

For $1 < q \le p < \infty$ and $0 < \alpha < 1$, the two-weight problem for the operators R_α and W_α remains unsolved and in this context the results presented here are interesting.

Let v and w be positive almost everywhere, locally integrable functions defined on $\mathbb{R}_+$.

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Denote by L_v^p $(1 < p < \infty)$ a class of all Lebesgue-measurable functions defined on $\mathbb{R}_+$ for which

$$\|f\|_{L_v^p} = \Big(\int_0^\infty |f(x)|^p v(x)\,dx\Big)^{\frac{1}{p}} < \infty.$$

First, let us recall some familiar results.

Theorem A ([5–10]). *Let $1 \le p \le q < \infty$. The inequality*

$$\Big(\int_0^\infty \Big|\int_0^x f(t)dt\Big|^q v(x)dx\Big)^{\frac{1}{q}} \le c\Big(\int_0^\infty |f(x)|^p w(x)dx\Big)^{\frac{1}{p}}, \tag{1}$$

where the positive constant c does not depend on f, is fulfilled iff

$$D = \sup_{t>0}\Big(\int_t^\infty v(x)dx\Big)^{\frac{1}{q}}\Big(\int_0^t w^{1-p'}(x)dx\Big)^{\frac{1}{p'}} < \infty \quad \Big(p' = \frac{p}{p-1}\Big).$$

Moreover, if c is the best constant in (1)*, then $c \approx D$ (the symbol $\approx$ here denotes a two-sided inequality).*

Theorem B ([10]). *Let $1 \le q < p < \infty$. Then inequality* (1) *holds iff*

$$D_1 = \bigg(\int_0^\infty \bigg[\Big(\int_t^\infty v(x)dx\Big)\Big(\int_0^t w^{1-p'}(x)dx\Big)^{q-1}\bigg]^{\frac{p}{p-q}} w^{1-p'}(t)dt\bigg)^{\frac{p-q}{pq}} < \infty.$$

Moreover, if c is the best constant in (1)*, then $c \approx D_1$.*

We also need Kolmogorov's theorem formulated as follows (see, e.g., [11]):

Theorem C. *Let $1 < p, q < \infty$ and $K : L^p \to L_v^q$ be an integral operator of the form $Kf(x) = \int_0^\infty k(x,y)f(y)dy$. If*

$$\|\ \|k(x,\cdot)\|_{L^{p'}}\|_{L_v^q} < \infty,$$

then the operator K is compact.

Theorem 1. *Let $1 < p \le q < \infty$, $\frac{1}{p} < \alpha < 1$ or $\alpha > 1$. The inequality*

$$\|R_\alpha f\|_{L_v^q} \le A\|f\|_{L^p}, \tag{2}$$

where the positive constant A does not depend on f, is fulfilled iff

$$B = \sup_{t>0} B(t) = \sup_{t>0}\Big(\int_t^\infty \frac{v(x)}{x^{(1-\alpha)q}}dx\Big)^{\frac{1}{q}} t^{\frac{1}{p'}} < \infty. \tag{3}$$

Moreover, if A is the best constant in (2)*, then $A \approx B$.*

Proof. Sufficiency. Denoting $I_{1\alpha}f(x) = \int_0^{\frac{x}{2}} \frac{f(t)}{(x-t)^{1-\alpha}}dt$ and $I_{2\alpha}f(x) = \int_{\frac{x}{2}}^{x} \frac{f(t)}{(x-t)^{1-\alpha}}dt$ for $f \in L^p$ we write $R_\alpha f$ as $R_\alpha f(x) = I_{1\alpha}f(x) + I_{2\alpha}f(x)$. We obtain

$$\|R_\alpha f\|^q_{L^q_v} \le c_1 \int_0^\infty |I_{1\alpha}f(x)|^q v(x)dx + c_1 \int_0^\infty |I_{2\alpha}f(x)|^q v(x)dx = S_1 + S_2.$$

If $0 < t < \frac{x}{2}$, then $(x-t)^{\alpha-1} \le bx^{\alpha-1}$, where the positive constant b depends only on α. Consequently, using Theorem A with $w \equiv 1$, we have

$$S_1 \le c_2 \int_0^\infty \frac{v(x)}{x^{(1-\alpha)q}} \Big(\int_0^x |f(t)|dt \Big)^q dx \le c_3 B^q \|f\|^q_{L^p}.$$

Now we shall estimate S_2. Using the Hölder inequality and the condition $\frac{1}{p} < \alpha$, we obtain

$$S_2 = c_1 \int_0^\infty v(x) \Big| \int_{\frac{x}{2}}^x \frac{f(t)}{(x-t)^{1-\alpha}}dt \Big|^q dx \le$$

$$\le c_1 \int_0^\infty v(x) \Big(\int_{\frac{x}{2}}^x |f(t)|^p dt \Big)^{\frac{q}{p}} \Big(\int_{\frac{x}{2}}^x \frac{dt}{(x-t)^{(1-\alpha)p'}} \Big)^{\frac{q}{p'}} dx =$$

$$= c_4 \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} v(x) \cdot x^{(\alpha-1)q + \frac{q}{p'}} \Big(\int_{\frac{x}{2}}^x |f(t)|^p dt \Big)^{\frac{q}{p}} dx \le$$

$$\le c_4 \sum_{k\in\mathbb{Z}} \Big(\int_{2^{k-1}}^{2^{k+1}} |f(t)|^p dt \Big)^{\frac{q}{p}} \Big(\int_{2^k}^{2^{k+1}} v(x) \cdot x^{(\alpha-1)q + \frac{q}{p'}} dx \Big) \le$$

$$\le c_5 \sum_{k\in\mathbb{Z}} \Big(\int_{2^{k-1}}^{2^{k+1}} |f(t)|^p dt \Big)^{\frac{q}{p}} \Big(\int_{2^k}^{2^{k+1}} v(x) \cdot x^{(\alpha-1)q} dx \Big) \cdot 2^{\frac{kq}{p'}} \le$$

$$\le c_5 B^q \sum_{k\in\mathbb{Z}} \Big(\int_{2^{k-1}}^{2^{k+1}} |f(t)|^p dt \Big)^{\frac{q}{p}} \le c_6 B^q \|f\|^q_{L^p}$$

which proves the sufficiency.

Necessity. Let $f(x) = \chi_{(0,\frac{t}{2})}(x)$. Note that if $0 < y < \frac{t}{2}$ and $x > t$, then $(x-y)^{\alpha-1} \ge b_1 x^{\alpha-1}$, where the positive constant b_1 depends only on α.

We have

$$\|R_\alpha f\|_{L^q_v} \geq \Big(\int\limits_t^\infty v(x)\Big(\int\limits_0^{\frac{t}{2}} \frac{dy}{(x-y)^{1-\alpha}}\Big)^q dx\Big)^{\frac{1}{q}} \geq c_7\Big(\int\limits_t^\infty \frac{v(x)}{x^{(1-\alpha)q}}dx\Big)^{\frac{1}{q}} \cdot t.$$

On the other hand, $\|f\|_{L^p} = c_8 t^{\frac{1}{p}}$ and by virtue of inequality (2) we find that $B(t) \leq c_9 A$ for all $t > 0$. $\square$

A most complicated proof of a similar theorem is given in [12] for the case $p = q = 2$.

Remark 1. Condition (3) is equivalent to the condition

$$\widetilde{B} = \sup_{k\in\mathbb{Z}} \Big(\int\limits_{2^k}^{2^{k+1}} \frac{v(x)}{x^{(1-\alpha)q-\frac{q}{p'}}}dx\Big)^{\frac{1}{q}} < \infty. \tag{4}$$

Moreover, $B \approx \widetilde{B}$.

Indeed, the fact that (3) implies (4) follows from the proof of Theorem 1. Now let condition (4) be satisfied and $t \in (0,\infty)$. Then $t \in (2^m, 2^{m+1}]$ for some $m \in \mathbb{Z}$. We have

$$B(t)^q = \Big(\int\limits_t^\infty \frac{v(x)}{x^{(1-\alpha)q}}dx\Big)t^{\frac{q}{p'}} \leq \Big(\int\limits_{2^m}^\infty \frac{v(x)}{x^{(1-\alpha)q}}dx\Big)2^{\frac{(m+1)q}{p'}} =$$

$$= c_1 2^{\frac{mq}{p'}} \sum_{k=m}^\infty \Big(\int\limits_{2^k}^{2^{k+1}} \frac{v(x)}{x^{(1-\alpha)q}}dx\Big) \leq c_2 2^{\frac{mq}{p'}} \sum_{k=m}^\infty 2^{-\frac{kq}{p'}} \int\limits_{2^k}^{2^{k+1}} \frac{v(x)x^{\frac{q}{p'}}}{x^{(1-\alpha)q}}dx \leq$$

$$\leq c_2 \widetilde{B}^q 2^{\frac{mq}{p'}} \sum_{k=m}^\infty 2^{-\frac{kq}{p'}} \leq c_3 \widetilde{B}^q$$

and therefore $B \leq c_4 \widetilde{B} < \infty$.

By the duality argument and Theorem 1 we obtain

Theorem 2. *Let $1 < p \leq q < \infty$, $\frac{1}{q'} < \alpha < 1$ or $\alpha > 1$. For the inequality*

$$\|W_\alpha f\|_{L^q} \leq \overline{A}\|f\|_{L^p_w}, \tag{5}$$

where the positive constant $\overline{A}$ does not depend on f, to be valid it is necessary and sufficient that

$$\overline{B} = \sup_{t>0} \overline{B}(t) = \sup_{t>0} \Big(\int\limits_t^\infty \frac{w^{1-p'}(x)}{x^{(1-\alpha)p'}}dx\Big)^{\frac{1}{p'}} t^{\frac{1}{q}} < \infty. \tag{6}$$

Moreover, if $\overline{A}$ *is the best constant in inequality* (5), *then* $\overline{A} \approx \overline{B}$.

We shall now consider the case $1 < q < p < \infty$. Applying the integration by parts, we obtain

Lemma 1. *Let* $1 < q < p < \infty$ *and* u *be a locally integrable function on* $\mathbb{R}_+$. *Then the equality*

$$\Big(\int_a^b u(x)dx\Big)^{\frac{p}{p-q}} = \frac{p}{p-q}\int_a^b \Big(\int_x^b u(t)dt\Big)^{\frac{q}{p-q}} u(x)dx$$

holds, where $0 \le a < b < \infty$.

Theorem 3. *Let* $1 < q < p < \infty$, $\frac{1}{p} < \alpha < 1$ *or* $\alpha > 1$. *The inequality*

$$\|R_\alpha f\|_{L^q_v} \le A_1 \|f\|_{L^p} \tag{7}$$

is fulfilled iff

$$B_1 = \bigg(\int_0^\infty \Big(\int_x^\infty \frac{v(t)}{t^{(1-\alpha)q}}dt\Big)^{\frac{p}{p-q}} x^{\frac{(q-1)p}{p-q}} dx\bigg)^{\frac{p-q}{pq}} < \infty. \tag{8}$$

Moreover, if A_1 *is the best constant in inequality* (7), *then* $A_1 \approx B_1$.

Proof. Sufficiency. In the notation introduced in the proof of Theorem 1 we have

$$\|R_\alpha f\|^q_{L^q_v} \le S_1 + S_2.$$

Using Theorem B with $w \equiv 1$ and the argument from the proof of Theorem 1, we obtain

$$S_1 \le c_2 B_1^q \|f\|^q_{L^p}.$$

Applying the Hölder inequality twice and the fact that $\frac{1}{p} < \alpha$, we have

$$S_2 \le c_1 \int_0^\infty \Big(\int_{\frac{x}{2}}^x |f(t)|^p dt\Big)^{\frac{q}{p}} \Big(\int_{\frac{x}{2}}^x \frac{dt}{(x-t)^{(1-\alpha)p'}}\Big)^{\frac{q}{p'}} v(x)dx =$$

$$= c_3 \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \Big(\int_{\frac{x}{2}}^x |f(t)|^p dt\Big)^{\frac{q}{p}} v(x) x^{(\alpha-1)q+\frac{q}{p'}} dx \le$$

$$\le c_3 \sum_{k\in\mathbb{Z}} \Big(\int_{2^{k-1}}^{2^{k+1}} |f(t)|^p dt\Big)^{\frac{q}{p}} \Big(\int_{2^k}^{2^{k+1}} v(x) x^{(\alpha-1)q+\frac{q}{p'}} dx\Big) \le$$

$$\leq c_3\Big(\sum_{k\in\mathbb{Z}}\int\limits_{2^k}^{2^{k+1}}|f(t)|^p dt\Big)^{\frac{q}{p}}\Big(\sum_{k\in\mathbb{Z}}\Big(\int\limits_{2^k}^{2^{k+1}}v(x)x^{(\alpha-1)q+\frac{q}{p'}}dx\Big)^{\frac{p}{p-q}}\Big)^{\frac{p-q}{p}}\leq$$

$$\leq c_4\|f\|^q_{L^p}\Big(\sum_{k\in\mathbb{Z}}\Big(\int\limits_{2^k}^{2^{k+1}}v(x)x^{(\alpha-1)q+\frac{q}{p'}}dx\Big)^{\frac{p}{p-q}}\Big)^{\frac{p-q}{p}}=c_4\|f\|^q_{L^p}\Big(\sum_{k\in\mathbb{Z}}S_{2k}\Big)^{\frac{p-q}{p}}.$$

By Lemma 1, we find for S_{2k} that

$$S_{2k}\leq 2^{\frac{(k+1)qp}{p'(p-q)}}\Big(\int\limits_{2^k}^{2^{k+1}}v(x)x^{(\alpha-1)q}dx\Big)^{\frac{p}{p-q}}\leq$$

$$\leq c_5 2^{\frac{kqp}{p'(p-q)}}\int\limits_{2^k}^{2^{k+1}}\Big(\int\limits_{x}^{2^{k+1}}\frac{v(t)}{t^{(1-\alpha)q}}dt\Big)^{\frac{q}{p-q}}\frac{v(x)}{x^{(1-\alpha)q}}dx\leq$$

$$\leq c_5\int\limits_{2^k}^{2^{k+1}}\Big(\int\limits_{x}^{\infty}\frac{v(t)}{t^{(1-\alpha)q}}dt\Big)^{\frac{q}{p-q}}\frac{v(x)}{x^{(1-\alpha)q}}\cdot x^{\frac{q(p-1)}{p-q}}dx.$$

Using integration by parts we get

$$S_2\leq c_6\|f\|^q_{L^p}\Big(\int\limits_0^{\infty}\Big(\int\limits_x^{\infty}\frac{v(t)}{t^{(1-\alpha)q}}dt\Big)^{\frac{q}{p-q}}\frac{v(x)}{x^{(1-\alpha)q}}\cdot x^{\frac{q(p-1)}{p-q}}dx\Big)^{\frac{p-q}{p}}=$$

$$=c_7\|f\|^q_{L^p}\Big(\int\limits_0^{\infty}\Big(\int\limits_x^{\infty}\frac{v(t)}{t^{(1-\alpha)q}}dt\Big)^{\frac{p}{p-q}}x^{\frac{p(q-1)}{p-q}}dx\Big)^{\frac{p-q}{p}}=c_7\|f\|^q_{L^p}B_1^q$$

and finally we obtain inequality (7).

Necessity. Let $\frac{1}{p}<\alpha<1$ and $v_0(t)=v(t)\cdot\chi_{(a,b)}(t)$, $w_0(t)=\chi_{(a,b)}(t)$, where $0<a<b<\infty$, and let

$$f(x)=\Big(\int\limits_x^{\infty}\frac{v_0(t)}{t^{(1-\alpha)q}}dt\Big)^{\frac{1}{p-q}}\Big(\int\limits_0^x w_0(t)dt\Big)^{\frac{q-1}{p-q}}w_0(x).$$

Then we have

$$\|f\|_{L^p}=\Big(\int\limits_a^b\Big(\int\limits_x^{\infty}\frac{v_0(t)}{t^{(1-\alpha)q}}dt\Big)^{\frac{p}{p-q}}\Big(\int\limits_0^x w_0(t)dt\Big)^{\frac{(q-1)p}{p-q}}dx\Big)^{\frac{1}{p}}<\infty.$$

On the other hand,

$$\|R_\alpha f\|_{L^q_v} = \Big(\int_0^\infty v(x)\Big(\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}dt\Big)^q dx\Big)^{\frac{1}{q}} \geq$$

$$\geq c_8\Big(\int_0^\infty \frac{v(x)}{x^{(1-\alpha)q}}\Big(\int_0^x f(t)dt\Big)^q dx\Big)^{\frac{1}{q}} \geq c_9\Big(\int_0^\infty \frac{v(x)}{x^{(1-\alpha)q}} \times$$

$$\times\Big(\int_x^\infty \frac{v_0(y)}{y^{(1-\alpha)q}}dy\Big)^{\frac{q}{p-q}}\Big(\int_0^x\Big(\int_0^t w_0(y)dy\Big)^{\frac{q-1}{p-q}} w_0(t)dt\Big)^q dx\Big)^{\frac{1}{q}} \geq$$

$$\geq c_{10}\Big(\int_0^\infty \frac{v_0(x)}{x^{(1-\alpha)q}}\Big(\int_x^\infty \frac{v_0(y)}{y^{(1-\alpha)q}}dy\Big)^{\frac{q}{p-q}}\Big(\int_0^x w_0(y)dy\Big)^{\frac{(p-1)q}{p-q}} dx\Big)^{\frac{1}{q}} =$$

$$= c_{11}\Big(\int_0^\infty\Big(\int_x^\infty \frac{v_0(t)}{t^{(1-\alpha)q}}dt\Big)^{\frac{p}{p-q}}\Big(\int_0^x w_0(t)dt\Big)^{\frac{(q-1)p}{p-q}} w_0(x)dx\Big)^{\frac{1}{q}} =$$

$$= c_{11}\Big(\int_a^b\Big(\int_x^\infty \frac{v_0(t)}{t^{(1-\alpha)q}}dt\Big)^{\frac{p}{p-q}}\Big(\int_0^x w_0(t)dt\Big)^{\frac{(q-1)p}{p-q}} dx\Big)^{\frac{1}{q}}.$$

From inequality (7) we have

$$\Big(\int_a^b\Big(\int_x^\infty \frac{v_0(t)}{t^{(1-\alpha)q}}dt\Big)^{\frac{p}{p-q}}\Big(\int_0^x w_0(t)dt\Big)^{\frac{(q-1)p}{p-q}} dx\Big)^{\frac{q-p}{pq}} \leq c_{12}A_1,$$

where c_{12} does not depend on a and b. By Fatou's lemma we finally obtain condition (8). The case $\alpha > 1$ is proved similarly. □

By the duality argument and Theorem 3 we have

Theorem 4. *Let $1 < q < p < \infty$, $\frac{1}{q'} < \alpha < 1$ or $\alpha > 1$. The inequality*

$$\|W_\alpha f\|_{L^q} \leq \overline{A}_1\|f\|_{L^p_w}, \tag{9}$$

where the positive constanr $\overline{A}_1$ does not depend f, holds iff

$$\overline{B}_1 = \Big(\int_0^\infty\Big(\int_x^\infty \frac{w^{1-p'}(t)}{t^{(1-\alpha)p'}}dt\Big)^{\frac{q(p-1)}{p-q}} x^{\frac{q}{p-q}}dx\Big)^{\frac{p-q}{pq}} < \infty. \tag{10}$$

Moreover, if $\overline{A}_1$ is the best constant in inequality (9), then $\overline{A}_1 \approx \overline{B}_1$.

Let us now investigate the compactness of the operators R_α and W_α.

Theorem 5. *Let $1 < p \leq q < \infty$, $\frac{1}{p} < \alpha < 1$ or $\alpha > 1$. The operator R_α is compact from L^p to L^q_v iff condition* (3) *and the condition*

$$\lim_{t\to 0} B(t) = \lim_{t\to\infty} B(t) = 0$$

is satisfied.

Proof. Sufficiency. Let $0 < a < b < \infty$. We write $R_\alpha f$ as

$$R_\alpha f = \chi_{[0,a]} R_\alpha(f \cdot \chi_{(0,a)}) + \chi_{(a,b)} R_\alpha(f \cdot \chi_{(0,b)}) + \chi_{[b,\infty)} R_\alpha(f \cdot \chi_{(0,\frac{b}{2})}) + \\ + \chi_{[b,\infty)} R_\alpha(f \cdot \chi_{(\frac{b}{2},\infty)}) = P_{1\alpha} f + P_{2\alpha} f + P_{2\alpha} f + P_{4\alpha} f.$$

For $P_{2\alpha} f$ we have $P_{2\alpha} f(x) = \chi_{(a,b)}(x) \int_0^\infty k_1(x,y) f(y) dy$, with $k_1(x,y) = (x-y)^{\alpha-1}$ for $y < x$ and $k_1(x,y) = 0$ for $y \geq x$. Consequently

$$\int_a^b v(x)\Big(\int_0^\infty (k_1(x,y))^{p'} dy\Big)^{\frac{q}{p'}} dx \leq \Big(\int_a^b \frac{v(x)}{x^{(1-\alpha)q}} dx\Big) b^{\frac{q}{p'}} < \infty$$

and by Theorem C we conclude that $P_{2\alpha}$ is compact from L^p to L^q_v.

In a similar manner we show that $P_{3\alpha}$ is compact too.

Using Theorem 1 for the operators $P_{1\alpha}$ and $P_{4\alpha}$, we obtain

$$\|P_{1\alpha}\| \leq c_1 \sup_{0<t<a} B(t) \quad \text{and} \quad \|P_{4\alpha}\| \leq c_2 \sup_{t>\frac{b}{2}} B(t).$$

Consequently

$$\|R_\alpha - P_{2\alpha} - P_{3\alpha}\| \leq \|P_{1\alpha}\| + \|P_{4\alpha}\| \leq c_1 \sup_{0<t<a} B(t) + c_2 \sup_{t>\frac{b}{2}} B(t) \to 0$$

as $a \to 0$ and $b \to \infty$.

Thus the operator R_α is compact, since it is a limit of compact operators. The sufficiency is proved.

Necessity. Note that the fact $B < \infty$ follows from Theorem 1. Thus we need to prove the remaining part. Let $f_t(x) = \chi_{(0,t)}(x) t^{-1/p}$. Then the sequence f_t is weakly convergent to 0. Indeed, assuming that $\varphi \in L^{p'}$, we obtain

$$\Big|\int_0^\infty f_t(x)\varphi(x) dx\Big| \leq \Big(\int_0^t |\varphi(x)|^{p'} dx\Big)^{\frac{1}{p'}} \to 0 \text{ as } t \to 0.$$

On the other hand, we have

$$\|R_\alpha f_t\|_{L^q_v} \geq \Big(\int_t^\infty v(x)\Big(\int_0^t \frac{f_t(y)}{(x-y)^{1-\alpha}} dy\Big)^q dx\Big)^{\frac{1}{q}} \geq$$

$$\geq c_3\Big(\int\limits_t^\infty \frac{v(x)}{x^{(1-\alpha)q}}dx\Big)^{\frac{1}{q}} t^{\frac{1}{p'}} = c_3 B(t).$$

Using the fact that a compact operator maps a weakly convergent sequence into a strongly convergent form, we find that $B(t) \to 0$ as $t \to 0$.

Keeping in mind that the operator W_α is compact from $L^{q'}_{v^{1-q'}}$ to $L^{p'}$ and arguing as above, we prove the remaining part of the theorem. □

In [12] a similar theorem is proved for the case $p = q = 2$.

Since the operator W_α is compact from L^p_w to L^q iff the operator R_α is compact from $L^{q'}$ to $L^{p'}_{w^{1-p'}}$, by Theorem 5 we obtain

Theorem 6. *Let $1 < q < p < \infty$, $\frac{1}{q'} < \alpha < 1$ or $\alpha > 1$. The operator W_α is compact from L^p_w to L^q iff condition* (6) *and the condition*

$$\lim_{t\to 0} \overline{B}(t) = \lim_{t\to\infty} \overline{B}(t) = 0$$

are fulfilled.

Theorem 7. *Let $1 < q < p < \infty$, $\frac{1}{p} < \alpha < 1$ or $\alpha > 1$. The operator R_α is compact from L^p to L^q_v iff condition* (8) *is satisfied.*

Proof. The sufficiency is proved as in proving Theorem 5 while the necessity follows from Theorem 3. □

By the duality argument we have

Theorem 8. *Let $1 < q < p < \infty$, $\frac{1}{q'} < \alpha < 1$ or $\alpha > 1$. The operator W_α is compact from L^p_w to L^q iff condition* (10) *is fulfilled.*

In [13, 14] the necessary and sufficient conditions are found for the operators R_α and W_α to be compact when $1 < p \leq q < \infty$ and $\alpha = 1$.

An analogous problem for $\alpha > 1$ was investigated in [15].

Remark 2. In Theorems 1 and 5 it suffices to consider v as a measurable almost everywhere, positive function. The same assumption can be made for w in Theorems 2 and 6.

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Author's address:
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., Tbilisi 380093
Georgia