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CROSSED SEMIMODULES AND SCHREIER INTERNAL CATEGORIES IN THE CATEGORY OF MONOIDS

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Abstract. We introduce the notion of a Schreier internal category in the category of monoids and prove that the category of Schreier internal categories in the category of monoids is equivalent to the category of crossed semimodules. This extends a well-known equivalence of categories between the category of internal categories in the category of groups and the category of crossed modules.

1. Introduction and Statement of Results

The description of internal categories in the category of groups as crossed modules is well known (see, e.g., [1]). There exist the same descriptions of internal categories in Rings, Lie algebras, etc. One can find different descriptions (not as crossed modules) of internal categories and groupoids in groups, Mal'tsev varieties of universal algebras, congruence modular varieties, and Mal'tsev categories in [2], [3], [4] and [5] respectively.

Let *Mon* denote the category of monoids. In this note we show that there exists an equivalence between the category of internal categories in *Mon* (note that monoids are not congruence modular) satisfying the so-called Schreier condition and the category of crossed semimodules. This answers a question posed by G. Janelidze (oral communication).

By a crossed semimodule we mean a pair

$$\Phi = \big(\mu : A \to X,\ \varphi : X \to \mathrm{End}(A)\big),$$

where A, X are monoids and μ, φ are homomorphisms of monoids satisfying

(CSM 1) $\mu(\varphi(x)(a)) + x = x + \mu(a)$, $x \in X$, $a \in A$,

(CSM 2) $\varphi(\mu(a))(a') + a = a + a'$, $a, a' \in A$.

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A crossed semimodule morphism is a commutative diagram

$$\begin{array}{rccc} & \Phi: \; A & \xrightarrow{\mu} & X \\ \tau = (\beta, \lambda) \downarrow & {\scriptstyle\beta}\downarrow & & {\scriptstyle\lambda}\downarrow \\ & \Phi': \; A' & \xrightarrow{\mu'} & X' \end{array}$$

with β and λ monoid homomorphisms, which satisfies

$$\beta(\varphi(x)(a)) = \varphi'(\lambda(x))(\beta(a))$$

for all $x \in X$ and $a \in A$.

Recall that an internal category in the category of monoids is a diagram

$$\begin{tikzcd} M: \; M_2 \arrow[r, shift left=2, "\pi_1"] \arrow[r, "m"] \arrow[r, shift right=2, "\pi_2"'] & M_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right=2, "d_1"'] & M_0 \arrow[l, bend right, "s_0"'] \end{tikzcd},$$

where M_0 is the monoid of objects, M_1 the monoid of morphisms, and d_0, d_1, s_0 are homomorphisms of monoids called the domain, the codomain, and the identity respectively; M_2 together with π_1 and π_2 forms the pullback

$$\begin{tikzcd} M_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & M_1 \arrow[d, "d_1"] \\ M_1 \arrow[r, "d_0"'] & M_0 \end{tikzcd} \quad ;$$

m is a homomorphism of monoids called the composition of M and usually written as $m(g, f) = g \circ f$; and the following conditions hold:

(i) $d_0 s_0 = d_1 s_0 = 1_{M_0}$,

(ii) $d_0 m = d_0 \pi_1$, $d_1 m = d_1 \pi_2$,

(iii) $m(m(h, g), f) = m(h, m(g, f))$,

(iv) $m(s_0 d_1(f), f) = f = m(f, s_0 d_0(f))$.

A morphism of internal categories in *Mon* is a commutative diagram

$$\begin{tikzcd}[column sep=huge]
M: \arrow[d, "\gamma=(\gamma_1,\gamma_0)"'] & M_2 \arrow[r, shift left=3, "\pi_1"] \arrow[r, "m"] \arrow[r, shift right=3, "\pi_2"'] \arrow[d, "\gamma_1\times\gamma_1"'] & M_1 \arrow[r, "d_0"] \arrow[r, shift right=3, "d_1"'] \arrow[d, "\gamma_1"'] & M_0 \arrow[l, shift right=3, "s_0"'] \arrow[d, "\gamma_0"] \\
M': & M_2' \arrow[r, shift left=3, "\pi_1'"] \arrow[r, "m'"] \arrow[r, shift right=3, "\pi_2'"'] & M_1' \arrow[r, "d_0'"] \arrow[r, shift right=3, "d_1'"'] & M_0' \arrow[l, shift right=3, "s_0'"']
\end{tikzcd},$$

where γ_0 and γ_1 are monoid homomorphisms.

An internal groupoid in *Mon* is an internal category M in *Mon* in which every morphism has an inverse, i.e., for every $f \in M_1$ there exists a unique $f^{-1} \in M_1$ such that $m(f, f^{-1}) = s_0 d_1(f)$ and $m(f^{-1}, f) = s_0 d_0(f)$.

Definition. A Schreier internal category in *Mon* is an internal category in *Mon* which satisfies the Schreier condition: for any $f \in M_1$ there exists a unique $g \in \operatorname{Ker}(d_0)$ such that

$$f = g + s_0 d_0(f).$$

We prove

Theorem. *The category of Schreier internal categories in the category of monoids is equivalent to the category of crossed semimodules.*

Restricting this equivalence gives

Corollary 1. *The category of Schreier internal groupoids in the category of monoids is equivalent to the category of crossed semimodules* $\Phi = (\mu : A \to X, \varphi : X \to \operatorname{End}(A))$ *such that* A *is a group.*

Note that Corollary 1 can be obtained as a special case of the theorem of [6].

Restricting this equivalence further, we obtain

Corollary 2 (R. Brown and C. B. Spencer [1]). . *The category of internal categories (= groupoids) in the category of groups is equivalent to the category of crossed modules.*

2. The Proofs

Let

$$x \xrightarrow{f} y \xrightarrow{f'} z$$

be a composable pair in a Schreier internal category M in *Mon*. Then there exist uniquely defined $g, h \in \mathrm{Ker}(d_0)$ such that

$$f = g + s_0 d_0(f) \quad \text{and} \quad f' = h + s_0(d_1(g) + d_0(f)).$$

Since m is a monoid homomorphism and (iv) holds, we obtain

$$\begin{aligned} m(f', f) &= m\big(h + s_0(d_1(g) + d_0(f)), g + s_0 d_0(f)\big) = \\ &= m(h, 0) + m\big(s_0(d_1(g) + d_0(f)), g + s_0 d_0(f)\big) = \\ &= m(h, s_0 d_0(h)) + m\big(s_0 d_1(g + s_0 d_0(f)), g + s_0 d_0(f)\big) = \\ &= h + g + s_0 d_0(f). \end{aligned}$$

So the composition m is uniquely defined,

$$m\big(h + s_0(d_1(g) + d_0(f)), g + s_0 d_0(f)\big) = h + g + s_0 d_0(f). \tag{1}$$

Proof of the theorem. Let

$$M: \quad M_2 \;\overset{\pi_1}{\underset{\pi_2}{\overset{m}{\rightrightarrows\!\!\to}}}\; M_1 \;\overset{s_0}{\underset{d_1}{\overset{d_0}{\leftarrow\!\!\rightrightarrows}}}\; M_0$$

be a Schreier internal category in *Mon*. By the Schreier condition, for any $x \in M_0$ and any $g \in \mathrm{Ker}(d_0)$ there exists a unique $\alpha(x, g) \in \mathrm{Ker}(d_0)$ such that

$$s_0(x) + g = \alpha(x, g) + s_0(x).$$

We have

$$\begin{aligned} &\alpha(x, g_1 + g_2) + s_0(x) = s_0(x) + g_1 + g_2 = \\ &= \alpha(x, g_1) + s_0(x) + g_2 = \alpha(x, g_1) + \alpha(x, g_2) + s_0(x), \\ &\alpha(x, 0) + s_0(x) = s_0(x), \\ &\alpha(x + y, g) + s_0(x + y) = s_0(x + y) + g = s_0(x) + s_0(y) + g = \\ &= s_0(x) + \alpha(y, g) + s_0(y) = \alpha(x, \alpha(y, g)) + s_0(x) + s_0(y) = \\ &= \alpha(x, \alpha(y, g)) + s_0(x + y), \\ &\alpha(0, g) + s(0) = s(0) + g. \end{aligned}$$

Hence, in view of the Schreier condition, we obtain

$$\alpha(x, g_1 + g_2) = \alpha(x, g_1) + \alpha(x, g_2), \quad \alpha(x, 0) = 0,$$

$$\alpha(x+y,g)=\alpha(x,\alpha(y,g)),\ \ \alpha(0,g)=g.$$

This means that we have a homomorphism of monoids

$$\psi:\ M_0\to \operatorname{End}\big(\operatorname{Ker}(d_0)\big),\quad \psi(x)(g)=\alpha(x,g),\ \ x\in M_0,\quad g\in \operatorname{Ker}(d_0).$$

Consider the pair of monoid homomorphisms

$$\Psi=\Big(d_1:\ \operatorname{Ker}(d_0)\to M_0,\ \psi: M_0\to \operatorname{End}\big(\operatorname{Ker}(d_0)\big)\Big).$$

It is clear that

$$d_1\big(\psi(x)(g)\big)+x=x+d_1(g),\quad x\in M_0,\quad g\in \operatorname{Ker}(d_0)$$

since $s_0(x)+g=\psi(x)(g)+s_0(x)$ and $d_1s_0=1_{M_0}$.
On the other hand,

$$s_0d_1(g)+g'=\psi(d_1(g))(g')+s_0d_1(g),\quad g,g'\in \operatorname{Ker}(d_0).$$

This and (1) give

$$\begin{aligned}&\psi(d_1(g))(g')+g=m\big(\psi(d_1(g))(g')+s_0d_1(g),g\big)=\\&=m\big(s_0d_1(g)+g',g\big)=m(s_0d_1(g),g)+m(g',0)=g+g',\end{aligned}$$

i.e.,

$$\psi(d_1(g))(g')+g=g+g',g,g'\in \operatorname{Ker}(d_0).$$

So Ψ is a crossed semimodule.

Let $\gamma=(\gamma_1,\gamma_0):M\to M'$ be a morphism of Schreier internal categories in *Mon*. The Schreier condition allows us to write

$$\begin{aligned}&\gamma_1(\psi(x)(g))+s_0'\gamma_0(x)=\gamma_1(\psi(x)(g))+\gamma_1s_0(x)=\\&=\gamma_1(\psi(x)(g)+s_0(x))=\gamma_1(s_0(x)+g)=s_0'\gamma_0(x)+\gamma_1(g)=\\&=\psi'(\gamma_0(x))(\gamma_1(g))+s_0'\gamma_0(x),\end{aligned}$$

i.e.,

$$\gamma_1(\psi(x)(g))+s_0'\gamma_0(x)=\psi'(\gamma_0(x))(\gamma_1(g))+s_0'\gamma_0(x).$$

From this, by the same condition, we obtain

$$\gamma_1(\psi(x)(g))=\psi'(\gamma_0(x))(\gamma_1(g)),\quad x\in M_0,\quad g\in \operatorname{Ker}(d_0).$$

This means that the commutative diagram

$$\begin{tikzcd}\operatorname{Ker}(d_0) \arrow[r, "d_1"] \arrow[d, "\gamma_1"'] & M_0 \arrow[d, "\gamma_0"'] \\ \operatorname{Ker}(d_0') \arrow[r, "d_1'"] & M_0'\end{tikzcd}$$

is a crossed semimodules morphism.

So we have the functor

$$S:\ \textit{Schreier internal categories in Mon} \to \textit{Crossed semimodules},$$

defined by $S(M)=\Psi$ and $S(\gamma_1,\gamma_0)=(\gamma_1|\operatorname{Ker}(d_0),\gamma_0):\Psi\to\Psi'$.

There is a standard way to show that S is an equivalence of categories. Let

$$\Phi=\Big(\mu:\ A\to X, \varphi:\ X\to \operatorname{End}(A)\Big)$$

be a crossed semomodule. Using the semidirect product $A\times_\varphi X$, we obtain the diagram

$$C:\ (A\times_\varphi X)_{d_0}\times_{d_1}(A\times_\varphi X) \overset{\pi_1}{\underset{\pi_2}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xrightarrow{m}}}}} A\times_\varphi X \overset{s_0}{\underset{d_1}{\underset{\longrightarrow}{\overset{\longleftarrow}{\xrightarrow{d_0}}}}} X\ ,$$

where $d_0(a,x)=x$, $d_1(a,x)=\mu(a)+x$, $s_0(x)=(0,x)$, $m((a',\mu(a)+x),(a,x))=(a'+a,x)$ and π_1,π_2 are the projections. It is plain to see that C is a Schreier internal category in *Mon*. On the other hand, any crossed semimodule morphism $\tau=(\beta,\lambda):\Phi\to\Phi'$ defines the morphism of Schreier internal categories $\gamma=(\gamma_1,\lambda):C\to C'$, where $\gamma_1(a,x)=(\beta(a),\lambda(x))$.

Thus we have the functor

$$T:\ \textit{Crossed semimodules} \ \to\ \textit{Schreier internal categories in Mon},$$

defined by $T(\Phi)=C$ and $T(\tau)=\gamma:C\to C'$.

It is straightforward to see that $1\cong ST$. Due to the Schreier condition, $1\cong TS$ holds, too. $\square$

Proof of Corollary 1. Suppose M is a Schreier internal groupoid in *Mon*. Consider the crossed semimodule

$$S(M)=\Big(d_1:\operatorname{Ker}(d_0)\to M_0,\ \psi:\ M_0\to \operatorname{End}\big(\operatorname{Ker}(d_0)\big)\Big).$$

For any $g\in\operatorname{Ker}(d_0)$ there exists a unique $h\in\operatorname{Ker}(d_0)$ such that

$$m\big(h+s_0d_1(g),g\big)=0 \quad\text{and}\quad m\big(g,h+s_0d_1(g)\big)=s_0d_1(g).$$

Using (1), we obtain

$$h+g=0 \quad\text{and}\quad g+h+s_0d_1(g)=s_0d_1(g).$$

Then the Schreier condition gives

$$h+g=0 \quad\text{and}\quad g+h=0,$$

i.e., $\operatorname{Ker}(d_0)$ is a group.

Conversely, suppose $\Phi = (\mu : A \to X, \varphi : X \to \mathrm{End}(A))$ is a semimodule such that A is a group. Then it is clear that for any morphism (a, x) of $T(\Phi)$ we have $(a, x)^{-1} = (-a, \mu(a) + x)$, i.e., $T(\Phi)$ is a Schreier internal groupoid in *Mon*. $\square$

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