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CHARACTERIZATION OF A TWO-WEIGHTED VECTOR-VALUED INEQUALITY FOR FRACTIONAL MAXIMAL OPERATORS

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ABSTRACT. We give a characterization of the weights $u(\cdot)$ and $v(\cdot)$ for which the fractional maximal operator M_s is bounded from the weighted Lebesgue spaces $L^p(l^r, vdx)$ into $L^q(l^r, udx)$ whenever $0 \leq s < n$, $1 < p, r < \infty$, and $1 \leq q < \infty$.

1. INTRODUCTION

The fractional maximal operator M_s of order s, $0 \leq s < n$, is defined by

$$(M_s f)(x) = \sup\left\{|Q|^{\frac{s}{n}-1}\int_Q |f(y)|\,dy;\;\; Q \text{ cube with } Q \ni x\right\}.$$

The cubes considered always have sides parallel to the coordinate axes. Here $M = M_0$ is the well-known Hardy–Littlewood maximal operator.

Our main purpose is to characterize the weights $u(\cdot)$ and $v(\cdot)$ for which there is $C > 0$ such that

$$\left\|\Big(\sum_k (M_s f_k)^r(\cdot)\Big)^{\frac{1}{r}}\right\|_{L^q_u} \leq C\left\|\Big(\sum_k f_k^r(\cdot)\Big)^{\frac{1}{r}}\right\|_{L^p_v} \tag{1.1}$$

for all sequences of nonnegative functions $(f_k(\cdot))_k$, with $1 < p, r < \infty$ and $1 \leq q < \infty$. The quantity $\|f(\cdot)\|_{L^t_w}$ is defined as $(\int_{\mathbb{R}^n} |f(x)|^t w(x)\,dx)^{\frac{1}{t}}$. For convenience, (1.1) will often be denoted by $M_s : L^p_v(l^r) \to L^q_u(l^r)$.

Such vector-valued inequalities are very important in many areas of analysis. For instance, they can be used to get weighted inequalities for other operators [1], and they are also involved in the study of some functional spaces [2].

Fefferman–Stein's well-known inequality states that $M : L^p_1(l^r) \to L^p_1(l^r)$ for $1 < p, r < \infty$. The one-weighted inequality $M : L^p_w(l^r) \to L^p_w(l^r)$ was

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solved by Andersen and John [3], and also independently by Kokilashvili [4]. In [5], a characterization of the two-weighted inequality $M : L^p_v(l^r) \to L^p_u(l^r)$ is obtained by applying the extrapolation theory of J. Garcia-Cuerva and Rubio de Francia and by adapting Sawyer's method for the two-weighted scalar inequality. For $s > 0$ it seems that only the boundedness $M_s : L^p_1(l^r) \to L^q_u(l^r)$, with $p < q$, was characterized by F. Ruiz and J. L. Torrea [6].

Our necessary and sufficient condition for $M : L^p_v(l^r) \to L^p_u(l^r)$ is completely different from the one given in [5], since the approach is not based on the extrapolation theory. We will use the atomic decomposition of tent spaces introduced by Coifman, Meyer, and Stein [7]. Such a technique has been introduced by the author in [8] to deal with

$$\big\|(M_s f)(\cdot)\big\|_{L^q_u} \le C\|f(\cdot)\|_{L^p_v} \quad \text{for all} \quad f(\cdot) \ge 0 \tag{1.2}$$

which is a scalar version and particular case of inequality (1.1). For $1 < p \le q < \infty$, Sawyer [9] proved that (1.2) holds if and only if for some $S > 0$

$$\big\|(M_s v^{-\frac{1}{p-1}}1\!\!1_Q)(\cdot)1\!\!1_Q(\cdot)\big\|_{L^q_u} \le S\big\|(v^{-\frac{1}{p-1}}1\!\!1_Q)(\cdot)\big\|_{L^p_v} \quad \text{for all cubes} \ \ Q. \tag{1.3}$$

Here $1\!\!1_Q(\cdot)$ denotes the characteristic function of the cube Q. In other words, (1.2) is true if this inequality is satisfied only for all particular functions of the form $f(\cdot) = v^{-\frac{1}{p-1}}(\cdot)1\!\!1_Q(\cdot)$. So it can be expected that $M_s : L^p_v(l^r) \to L^q_u(l^r)$ if and only if (1.1) is true for all sequences $f_k(\cdot) = v^{-\frac{1}{p-1}}(\cdot)1\!\!1_{Q_k}(\cdot)$. In this paper (see Theorem 1) we are able to derive such a boundedness if (1.1) is true for all $f_k(\cdot) = \lambda_k v^{-\frac{1}{p-1}}(\cdot)1\!\!1_{Q_k}(\cdot)$ with $\lambda_k > 0$.

Our results are presented in Section 2. The next Section 3 is devoted to their proofs except Theorem 1 and Proposition 8 whose proofs are given in Section 4. The proofs of the basic lemmas are given in Section 5.

2. Main Results

In this paper it is always assumed that $0 \le s < n$, $1 < p, r < \infty$, $1 \le q < \infty$ with $1/p - 1/q \le s/n$ and $u(\cdot)$, $v(\cdot)$, $\sigma(\cdot) = v^{-\frac{1}{p-1}}(\cdot)$ are weight functions.

Our main result, which can be seen as a generalization of Sawyer's theorem quoted above, is

Theorem 1. *Suppose that for some constant* $S > 0$

$$\begin{aligned}\Big\|\Big(\sum_k \big[\lambda_k(M_s v^{-\frac{1}{p-1}}1\!\!1_{Q_k})(\cdot)1\!\!1_{Q_k}(\cdot)\big]^r\Big)^{\frac{1}{r}}\Big\|_{L^q_u} &\le \\ \le S\Big\|\Big(\sum_k \big[\lambda_k(v^{-\frac{1}{p-1}}1\!\!1_{Q_k})(\cdot)\big]^r\Big)^{\frac{1}{r}}\Big\|_{L^p_v}&\end{aligned} \tag{2.1}$$

for all λ_k *and all cubes* Q_k*; then* $M_s : L^p_v(l^r) \to L^q_u(l^r)$*. Conversely, this boundedness implies condition* (2.1).

For the boundedness $M : L^p_v(l^r) \to L^p_u(l^r)$, Theorem 1 is only interesting when $r < p$. Indeed, by the interpolation theory for linearizable operators [1], it is known that the vector-valued inequality $M : L^p_v(l^r) \to L^p_u(l^r)$ becomes equivalent to the scalar inequality (1.2) (with $p = q$) when $p \le r$ (see [5] for details).

Next we apply Theorem 1 to treat some particular cases for which $p = q$. These cases correspond to significant and critical (for $s > 0$) situations in applications.

Corollary 2. *Assume that* $w(\cdot)$ *is a weight function with* $w^{1-p}(\cdot) \in L^1_{loc}(\mathbb{R}^n, dx)$*. Then* $M_s : L^p_{w^{1-p}}(l^r) \to L^p_w(l^r)$ *if and only if for some constant* $C > 0$

$$\int_Q w(y)\,dy \le C|Q|^{1-\frac{s}{n}} \quad \text{for all cubes} \quad Q. \tag{2.2}$$

Corollary 3. *Suppose that for some constant* $C > 0$

$$u(\cdot)(M_s v^{-\frac{1}{p-1}})^p(\cdot) \le C v^{-\frac{1}{p-1}}(\cdot);$$

then $M_s : L^p_v(l^r) \to L^p_u(l^r)$*. In particular, the boundedness*

$$M_s : L^p_w(l^r) \to L^p_{\frac{w^{1-p'}}{(M_s w^{1-p'})^p}}(l^r)$$

holds for any weight $w(\cdot)$ *with* $w^{1-p'}(\cdot) \in L^1_{loc}(\mathbb{R}^n, dx)$.

Proposition 4. *Let* $r < p$ *and* $1 < t < \infty$ *with* $0 \le s < \min\{\frac{n}{r}, \frac{n}{pt}\}$*. If for some constant* $C > 0$ $(M_{spt}u^t)^{\frac{1}{t}}(\cdot) \le Cv(\cdot)$*, then* $M_s : L^p_v(l^r) \to L^p_u(l^r)$*. In particular,* $M_s : L^p_{(M_{spt}w^t)^{\frac{1}{t}}}(l^r) \to L^p_w(l^r)$ *for any weight* $w(\cdot)$ *with* $w^t(\cdot) \in L^1_{loc}(\mathbb{R}^n, dx)$.

The latter boundedness is a substitute for Fefferman–Stein's well-known inequality

$$\int_{\mathbb{R}^n} (M_s f)^r(x) w(x)\,dx \le C \int_{\mathbb{R}^n} f^r(x)(M_{sr}w)(x)\,dx \quad \text{for all} \quad f(\cdot) \ge 0.$$

As proved in [5], it is possible to obtain a better estimate in the sense that $(M_{spt}w)^{\frac{1}{t}}(\cdot)$ can be replaced by a smaller operator. But we do not go in this direction.

In general, a condition like (2.1) is difficult to check for given weights $u(\cdot)$ and $v(\cdot)$. So our purpose is now to examine this condition more thoroughly for the case $r < p = q$. A general condition which implies (2.1) can be stated.

Proposition 5. *Let $r < p$ and $(\frac{p}{r})' = \frac{p}{p-r}$. Then condition* (2.1) *is satisfied whenever, for a fixed constant $C > 0$ and for each $g(\cdot) \in L^{(\frac{p}{r})'}(\mathbb{R}^n, dx)$, one can find $G(\cdot) \in L^{(\frac{p}{r})'}(\mathbb{R}^n, dx)$ such that*

$$\|G(\cdot)\|_{L^{(\frac{p}{r})'}(\mathbb{R}^n,dx)} \le \|g(\cdot)\|_{L^{(\frac{p}{r})'}(\mathbb{R}^n,dx)}$$

and

$$\int_Q (M_s\sigma 1\!\!1_Q)^r(x)u^{\frac{r}{p}}(x)g(x)\,dx \le C\int_Q \sigma^{\frac{r}{p}}(x)G(x)\,dx \quad \textit{for all cubes } Q. \tag{2.3}$$

As mentioned above, C. Pérez [5] has recently proved that $M : L^p_v(l^r) \to L^p_u(l^r)$ is equivalent to (2.3). For $s > 0$ it is also expected that (2.3) is equivalent to our condition (2.1), but this fact is not needed for the sequel. Proposition 5 was inspired by the results of [5].

Obviously, (2.1) implies Sawyer's condition (1.3). As for (2.3), it is strictly contained in (1.3), since, by duality,

$$\left(\int_Q (M_s\sigma 1\!\!1_Q)^p(x)u(x)\,dx\right)^{\frac{r}{p}} = \int_Q (M_s\sigma 1\!\!1_Q)^r(x)u^{\frac{r}{p}}(x)g(x)\,dx$$

and

$$\int_Q \sigma^{\frac{r}{p}}(x)G(x)\,dx \le \left(\int_Q \sigma(x)\,dx\right)^{\frac{r}{p}}$$

for some $g(\cdot) \in L^{(\frac{p}{r})'}(\mathbb{R}^n, dx)$ with the unit norm.

However, to get the boundedness $M_s : L^p_v(l^r) \to L^p_u(l^r)$ it would be desirable to have a sufficient condition whose form is closer to (1.3) rather than to (2.3) and (2.1).

Proposition 6. *Let $r < p$. Condition* (2.3) *or* (2.1) *is satisfied whenever for some $t > 1$ and $C > 0$*

$$\left(\frac{1}{|Q|}\int_Q (M_s\sigma 1\!\!1_Q)^{tp}(x)u^t(x)\,dx\right)^{\frac{1}{tp}} \le C\left(\frac{1}{|Q|}\int_Q \sigma^{\frac{r}{p}}(x)\,dx\right)^{\frac{1}{r}} \tag{2.4}$$

for all cubes Q.

Clearly, by the Hölder inequality, (2.4) implies Sawyer's condition (1.3). Conditions like (2.4) or (1.3) are not useful for computations. Indeed, the presence of the maximal operator M_s in the condition makes it difficult or sometimes impossible to determine by computation whether $M_s : L^p_v(l^r) \to L^p_u(l^r)$ for given weights $u(\cdot)$ and $v(\cdot)$. Therefore it would be more preferable to have a sufficient condition for this boundeness without using M_s. We shall give an example.

Proposition 7. *Let $r < p$. Condition* (2.3) *or* (2.1) *is satisfied if for some $t_2 > t_1 > 1$, and $C > 0$*

$$\left(\frac{1}{|Q|}\int_Q (|x|^s\sigma(x))^{p\delta}dx\right)^{\frac{1}{p\delta}}\left(\frac{1}{|Q|}\int_Q u^{t_2}(x)\,dx\right)^{\frac{1}{pt_2}} \le C\left(\frac{1}{|Q|}\int_Q \sigma^{\frac{r}{p}}(x)dx\right)^{\frac{1}{r}} \quad (2.5)$$

for all cubes Q; here $\frac{1}{\delta} = \frac{1}{t_1} - \frac{1}{t_2}$.

By Proposition 7 the Muckenhoupt condition

$$\left(\frac{1}{|Q|}\int_Q u(x)\,dx\right)^{\frac{1}{p}}\left(\frac{1}{|Q|}\int_Q \sigma(x)\,dx\right)^{\frac{1}{p'}} \le C \quad \text{for all cubes} \quad Q$$

characterizes the boundedness $M : L^p_v(l^r) \to L^p_u(l^r)$ whenever $u(\cdot)$, $\sigma^{\frac{r}{p}}(\cdot) \in RH_\infty$. Also, for $\sigma^{\frac{r}{p}}(\cdot) \in RH_\infty$ we have $M : L^p_v(l^r) \to L^p_u(l^r)$ if for some $C > 0$ and $t > 1$

$$\left(\frac{1}{|Q|}\int_Q u^t(x)\,dx\right)^{\frac{1}{tp}}\left(\frac{1}{|Q|}\int_Q \sigma(x)\,dx\right)^{\frac{1}{p'}} \le C \quad \text{for all cubes} \quad Q.$$

Recall that $w(\cdot) \in RH_\infty$ if $\sup_{x\in Q} w(x) \le C\frac{1}{|Q|}\int_Q w(y)\,dy$ for a fixed constant $C > 0$ and all cubes Q. Other sufficient conditions for $M_s : L^p_v(l^r) \to L^q_u(l^r)$ which do not involve the maximal operator M_s can be found in [10] (see also [5] for $p = q$ and $s = 0$). No more results in this direction are given here, since our main purpose is to prove the characterization in Theorem 1.

We end with a necessary and sufficient condition for $M_s : L^p_v(l^r) \to L^q_u(l^r)$ with special weights $v(\cdot)$. For this, recall that $w(\cdot) \in RD_\rho$, $\rho > 0$, if for some $c > 0$

$$\int_{Q'} w(y)\,dy \le c\Big(\frac{|Q'|}{|Q|}\Big)^{\rho}\int_Q w(y)\,dy \quad \text{for all cubes} \quad Q',\ Q \ \text{with} \quad Q' \subset Q.$$

Any doubling weight $w(\cdot)$ (and, in particular, any A_∞ weight) satisfies the reverse doubling condition RD_ρ.

Proposition 8. *Suppose that for some constant $A > 0$*

$$\left\|\Big(\sum_k \Big[\lambda_k|Q_k|^{\frac{s}{n}-1}\Big(\int_{Q_k} v^{-\frac{1}{p-1}}(y)\,dy\Big)1\!\!1_{Q_k}(\cdot)\Big]^r\Big)^{\frac{1}{r}}\right\|_{L^q_u} \le$$
$$\le A\left\|\Big(\sum_k \big[\lambda_k(v^{-\frac{1}{p-1}}1\!\!1_{Q_k})(\cdot)\big]^r\Big)^{\frac{1}{r}}\right\|_{L^p_v} \quad \text{for all cubes } Q_k \text{ and } \lambda_k > 0. \quad (2.6)$$

Then $M_s : L^p_v(l^r) \to L^q_u(l^r)$ whenever $v^{-\frac{1}{p-1}}(\cdot) \in RD_\rho$ with $1 - \frac{s}{n} \le \rho$. Conversely, the boundedness $M_s : L^p_v(l^r) \to L^q_u(l^r)$ implies condition (2.6).

Therefore for $r < p$ and $\sigma(\cdot) = v^{-\frac{1}{p-1}}(\cdot) \in RD_\rho$ with $1 - \frac{s}{n} \leq \rho$ we have $M_s : L^p_v(l^r) \to L^p_u(l^r)$ whenever for some $t > 1$ and $A > 0$

$$|Q|^{\frac{s}{n}}\left(\frac{1}{|Q|}\int_Q \sigma(y)\,dy\right)\left(\frac{1}{|Q|}\int_Q u^t(y)\,dy\right)^{\frac{1}{pt}} \leq$$
$$\leq A\left(\frac{1}{|Q|}\int_Q \sigma^{\frac{r}{p}}(y)\,dy\right)^{\frac{1}{r}} \quad \textit{for all cubes} \quad Q. \tag{2.7}$$

Theorem 1 can be seen as an extension of Sawyer's result [9] for scalar functions to vector functions. Note that, whenever $s > 0$ and $1 < p < q < \infty$, for the scalar inequality (1.2) a better criterion than (1.3) has been established by Wheeden for the Euclidean space [11] and by Gogatishvili and Kokilashvili in a more general case, namely, for homogeneous type spaces [12]. For this range of s, p, and q it is interesting to find a criterion for $M_s : L^p_v(l^r) \to L^q_u(l^r)$ which will replace condition (2.1).

3. Proofs of Corollaries 2, 3, and Propositions 6, 7

To get $M_s : L^p_v(l^r) \to L^p_u(l^r)$, by Theorem 1 with $p = q$, it is sufficient to estimate

$$\mathcal{S} = \left\| \Big(\sum_k \left[\lambda_k (M_s \sigma 1\!\!1_{Q_k})(\cdot) 1\!\!1_{Q_k}(\cdot)\right]^r \Big)^{\frac{1}{r}} \right\|_{L^p_u} \tag{3.1}$$

by $C\|(\sum_k [\lambda_k(\sigma(\cdot)1\!\!1_{Q_k})(\cdot)]^r)^{\frac{1}{r}}\|_{L^p_v}$, for all sequences of nonnegative $(\lambda_k)_k$ and all sequences of cubes $(Q_k)_k$. Here $C > 0$ is a constant which depends only on s, n, p, q, $u(\cdot)$, and $v(\cdot)$.

Proof of Corollary 2. Suppose that $M_s : L^p_v(l^r) \to L^p_u(l^r)$. Take the sequence of functions $(f_k(\cdot))_k$ defined as $f_0(\cdot) = \sigma(\cdot)1\!\!1_Q(\cdot)$ and $f_k(\cdot) = 0$ for all $k \neq 0$. Since $v(\cdot) = w^{1-p}(\cdot)$ and $u(\cdot) = \sigma(\cdot) = w(\cdot)$, condition (2.2) follows immediately.

To prove the converse statement, with the above values of $u(\cdot)$ and $v(\cdot)$, condition (2.2) implies that $(M_s\sigma)(\cdot) \in L^\infty(\mathbb{R}^n, dx)$ and thus

$$\mathcal{S} \leq C\left\|\Big(\sum_k [\lambda_k 1\!\!1_{Q_k}(\cdot)]^r\Big)^{\frac{1}{r}}\right\|_{L^p_u} = C\left\|\Big(\sum_k [\lambda_k 1\!\!1_{Q_k}(\cdot)]^r\Big)^{\frac{1}{r}}\right\|_{L^p_\sigma} (\text{since } u(\cdot) = \sigma(\cdot))$$
$$= C\left\|\Big(\sum_k \left[\lambda_k \sigma(\cdot) 1\!\!1_{Q_k}(\cdot)\right]^r\Big)^{\frac{1}{r}}\right\|_{L^p_v} \quad (\text{since } \sigma(\cdot) = \sigma^p(\cdot)v(\cdot)). \qquad \square$$

Proof of Corollary 3. To get the first part of the corollary, we write (3.1) as

$$\mathcal{S} \leq \Big\| \Big(\sum_k \big[\lambda_k 1\!\!1_{Q_k}(\cdot)\big]^r \Big)^{\frac{1}{r}} \Big\|_{L^p_{u(\cdot)(M_s\sigma)^p(\cdot)}} \leq$$

$$\leq C \Big\| \Big(\sum_k \big[\lambda_k 1\!\!1_{Q_k}(\cdot)\big]^r \Big)^{\frac{1}{r}} \Big\|_{L^p_\sigma} = C \Big\| \Big(\sum_k \big[\lambda_k \sigma(\cdot) 1\!\!1_{Q_k}(\cdot)\big]^r \Big)^{\frac{1}{r}} \Big\|_{L^p_v}.$$

The proof of the second part is trivial, since in that case $v(\cdot) = w(\cdot)$ and $u(\cdot) = \sigma(\cdot)(M_s\sigma)^{-p}(\cdot)$. $\square$

Proof of Proposition 4. The second part of the proposition is an immediate consequence of the first one if we take $u(\cdot) = w(\cdot)$ and $v(\cdot) = (M_{spt}u^t)^{\frac{1}{t}}(\cdot)$.

The first part is proved by assuming $r < p$, $\mu = \frac{p}{r}$, $\mu' = \frac{p}{p-r}$ and again considering (3.1). By duality, there is a nonnegative function $g(\cdot) \in L^{\mu'}(\mathbb{R}^n, dx)$ with the unit norm such that

$$\begin{aligned} \mathcal{S}^r &= \int_{\mathbb{R}^n} \Big[\sum_k \lambda_k^r (M_s\sigma 1\!\!1_{Q_k})^r(x) 1\!\!1_{Q_k}(x) \Big] u^{\frac{r}{p}}(x) g(x)\, dx = \\ &= \sum_k \lambda_k^r \int_{Q_k} (M_s\sigma 1\!\!1_{Q_k})^r(x) u^{\frac{r}{p}}(x) g(x)\, dx. \end{aligned} \tag{3.2}$$

The key to the proof is the existence of a constant $C > 0$ for which

$$\int_Q (M_s\sigma 1\!\!1_Q)^r(x) u^{\frac{r}{p}}(x) g(x)\, dx \leq C \int_Q \sigma^r(x) (M_{spt}u^t)^{\frac{r}{tp}}(x) (Mg^{\nu'})^{\frac{1}{\nu'}}(x)\, dx \tag{3.3}$$

for all cubes Q. Here $M = M_0$ is the Hardy–Littlewood maximal operator, $\nu = t\frac{p}{r}$ (> 1), and $\nu' = \frac{\nu}{\nu-1}$. By inequality (3.3) the rest of the estimate of $\mathcal{S}^r$ is

$$\begin{aligned} \mathcal{S}^r &\leq C \sum_k \lambda_k^r \int_{Q_k} \sigma^r(x) (M_{spt}u^t)^{\frac{r}{tp}}(x) (Mg^{\nu'})^{\frac{1}{\nu'}}(x)\, dx = \quad (\text{by} \quad (3.3)) \\ &= C \int_{\mathbb{R}^n} \Big[\sum_k \lambda_k^r 1\!\!1_{Q_k}(x) \Big] \sigma^r(x) (M_{spt}u^t)^{\frac{r}{tp}}(x) (Mg^{\nu'})^{\frac{1}{\nu'}}(x)\, dx = \\ &= C \int_{\mathbb{R}^n} \Big[\sum_k \lambda_k^r 1\!\!1_{Q_k}(x) \Big] \Big[\sigma^p(x) (M_{spt}u^t)^{\frac{1}{t}}(x) \Big]^{\frac{r}{p}} (Mg^{\nu'})^{\frac{1}{\nu'}}(x)\, dx \leq \\ &\leq C C_1^{\frac{r}{p}} \int_{\mathbb{R}^n} \Big[\sum_k \lambda_k^r 1\!\!1_{Q_k}(x) \Big] \sigma^{\frac{r}{p}}(x) (Mg^{\nu'})^{\frac{1}{\nu'}}(x)\, dx \leq \\ &\qquad\qquad (\text{since} \quad (M_{spt}u^t)^{\frac{1}{t}}(\cdot) \leq C_1 v(\cdot)) \\ &\leq C C_1^{\frac{r}{p}} \Big(\int_{\mathbb{R}^n} \Big[\sum_k \lambda_k^r 1\!\!1_{Q_k}(x) \Big]^{\frac{p}{r}} \sigma(x)\, dx \Big)^{\frac{r}{p}} \Big(\int_{\mathbb{R}^n} (Mg^{\nu'})^{\frac{\mu'}{\nu'}}(x)\, dx \Big)^{\frac{1}{\mu'}} \leq \end{aligned}$$

$$\leq CC_1^{\frac{r}{p}}C_2\Big(\int_{\mathbb{R}^n}\Big[\sum_k \lambda_k^r \mathbb{1}_{Q_k}(x)\Big]^{\frac{p}{r}}\sigma(x)\,dx\Big)^{\frac{r}{p}}\Big(\int_{\mathbb{R}^n} g^{\mu'}(x)\,dx\Big)^{\frac{1}{\mu'}} \leq$$

(by the maximal inequality theorem and since $1<\mu'/\nu'$ or $\mu<\nu$)

$$\leq CC_1^{\frac{r}{p}}C_2\Big\|\Big(\sum_k \big[\lambda_k(\sigma(\cdot)\mathbb{1}_{Q_k})(\cdot)\big]^r\Big)^{\frac{1}{r}}\Big\|_{L^p_v}^r \quad (\text{recall that } \|g\|_{L^{\mu'}(\mathbb{R}^n,dx)}{=}1).$$

To get (3.3) it is sufficient to use

$$(M_{sr}u^{\frac{r}{p}}g)(\cdot) \leq (M_{spt}u^t)^{\frac{r}{tp}}(\cdot)(Mg^{\nu'})^{\frac{1}{\nu'}}(\cdot). \tag{3.4}$$

Indeed, by Fefferman–Stein's classical inequality and (3.4), we have

$$\int_Q (M_s\sigma\mathbb{1}_Q)^r(x)u^{\frac{r}{p}}(x)g(x)\,dx \leq C\int_Q \sigma^r(x)(M_{sr}u^{\frac{r}{p}}g)(x)\,dx \leq$$
$$\leq C\int_Q \sigma^r(x)(M_{spt}u^t)^{\frac{r}{tp}}(x)(Mg^{\nu'})^{\frac{1}{\nu'}}(x)\,dx.$$

Inequality (3.4) is an immediate consequence of the Hölder inequality, since

$$|Q|^{\frac{sr}{n}}\frac{1}{|Q|}\int_Q \phi(y)\psi(y)\,dy \leq \Big(|Q|^{\frac{sr\nu}{n}}\frac{1}{|Q|}\int_Q \phi^{\nu}(y)\,dy\Big)^{\frac{1}{\nu}}\Big(\frac{1}{|Q|}\int_Q \psi^{\nu'}(y)\,dy\Big)^{\frac{1}{\nu'}}$$

for $\phi(\cdot)=u^{\frac{r}{p}}(\cdot)$, $\psi(\cdot)=g(\cdot)$, $\nu=t\frac{p}{r}$, and $\nu'=\frac{\nu}{\nu-1}$. $\square$

Proof of Proposition 5. Suppose condition (2.3) is satisfied. To get (2.1), it is sufficient to estimate $\mathcal{S}^r$ for $r<p$. Similarly to (3.2), consider a nonnegative function $g(\cdot)\in L^{\mu'}(\mathbb{R}^n,dx)$ with the unit norm. Then

$$\mathcal{S}^r = \sum_k \lambda_k^r\int_{Q_k}(M_s\sigma\mathbb{1}_{Q_k})^r(x)u^{\frac{r}{p}}(x)g(x)\,dx \leq \quad (\text{by} \quad (3.2))$$
$$\leq C\sum_k \lambda_k^r\int_{Q_k}\sigma^{\frac{r}{p}}(x)G(x)\,dx = \quad (\text{by condition} \quad (2.3))$$
$$= C\int_{\mathbb{R}^n}\Big[\sum_k \lambda_k^r\mathbb{1}_{Q_k}(x)\Big]\sigma^{\frac{r}{p}}(x)G(x)\,dx \leq$$
$$\leq C\Big(\int_{\mathbb{R}^n}\Big[\sum_k \lambda_k^r\mathbb{1}_{Q_k}(x)\Big]^{\frac{p}{r}}\sigma(x)\,dx\Big)^{\frac{r}{p}}\Big(\int_{\mathbb{R}^n}G^{\mu'}(x)\,dx\Big)^{\frac{1}{\mu'}} \leq$$
$$\leq C\Big(\int_{\mathbb{R}^n}\Big[\sum_k \lambda_k^r\mathbb{1}_{Q_k}(x)\Big]^{\frac{p}{r}}\sigma(x)\,dx\Big)^{\frac{r}{p}}\Big(\int_{\mathbb{R}^n}g^{\mu'}(x)\,dx\Big)^{\frac{1}{\mu'}} \leq \quad (\text{by} \quad (2.3))$$
$$\leq C\Big\|\Big(\sum_k\big[\lambda_k(\sigma(\cdot)\mathbb{1}_{Q_k})(\cdot)\big]^r\Big)^{\frac{1}{r}}\Big\|_{L^p_v}^r \quad (\text{recall that } \|g\|_{L^{\mu'}(\mathbb{R}^n,dx)}=1). \quad \square$$

Proof of Proposition 6. Assume condition (2.4) is satisfied for some constant $C > 0$. To get (2.3), let $g(\cdot) \in L^{\mu'}(\mathbb{R}^n, dx)$. The above notation ($\mu = \frac{p}{r}$, $\nu = t\frac{p}{r}$, ..., etc.) is again used. Let $C_2 > 0$ be a constant associated with the boundedness of $M : L^{\frac{\mu'}{\nu'}}(\mathbb{R}^n, dx) \to L^{\frac{\mu'}{\nu'}}(\mathbb{R}^n, dx)$, and set $G(\cdot) = C_2^{-\frac{1}{\nu'}}(Mg^{\nu'})^{\frac{1}{\nu'}}(\cdot)$. Clearly, $\|G(\cdot)\|_{L^{\mu'}(\mathbb{R}^n,dx)} \leq \|g(\cdot)\|_{L^{\mu'}(\mathbb{R}^n,dx)}$ and we obtain (2.3), since

$$\frac{1}{|Q|}\int_Q (M_s\sigma 1\!\!1_Q)^r(x)u^{\frac{r}{p}}(x)g(x)\,dx \leq$$

$$\leq \left(\frac{1}{|Q|}\int_Q (M_s\sigma 1\!\!1_Q)^{tp}(x)u^t(x)\,dx\right)^{\frac{r}{tp}}\left(\frac{1}{|Q|}\int_Q g^{\nu'}(y)\,dy\right)^{\frac{1}{\nu'}} \leq$$

$$\leq C^r\left(\frac{1}{|Q|}\int_Q \sigma^{\frac{r}{p}}(x)\left[\frac{1}{|Q|}\int_Q g^{\nu'}(y)\,dy\right]^{\frac{1}{\nu'}}dx\right) \leq \quad \text{(by} \quad (2.4))$$

$$\leq C^r\left(\frac{1}{|Q|}\int_Q \sigma^{\frac{r}{p}}(x)(Mg^{\nu'})^{\frac{1}{\nu'}}(x)\,dx = C_2^{\frac{1}{\nu'}}C^r\frac{1}{|Q|}\int_Q \sigma^{\frac{r}{p}}(x)G(x)\,dx. \qquad \square$$

Proof of Proposition 7. Assume condition (2.5) is satisfied for some constant $C > 0$. To get (2.3), let $g(\cdot) \in L^{\mu'}(\mathbb{R}^n, dx)$ with $\mu = \frac{p}{r}$, and take $\nu = t_1\frac{p}{r}$ with $t_1 > 1$. Choose $G(\cdot)$ as in the proof of Proposition 6. Here we use the fact that $M_s : L^\alpha(\mathbb{R}^n, |x|^{s\alpha}dx) \to L^\alpha(\mathbb{R}^n, dx)$ for all $\alpha > 1$. Let t_2, δ with $t_1 < t_2$ and $\frac{1}{\delta} = \frac{1}{t_1} - \frac{1}{t_2}$. Then $1 = \frac{r}{p\delta} + \frac{r}{pt_2} + \frac{1}{\nu'}$, and by the Hölder inequality we get the conclusion, since

$$\frac{1}{|Q|}\int_Q (M_s\sigma 1\!\!1_Q)^r(x)u^{\frac{r}{p}}(x)g(x)\,dx \leq$$

$$\leq \left(\frac{1}{|Q|}\int_Q (M_s\sigma 1\!\!1_Q)^{p\delta}(x)\,dx\right)^{\frac{r}{p\delta}}\left(\frac{1}{|Q|}\int_Q u^{t_2}(x)\,dx\right)^{\frac{r}{pt_2}}\left(\frac{1}{|Q|}\int_Q g^{\nu'}(y)\,dy\right)^{\frac{1}{\nu'}} \leq$$

$$\leq c_1\left(\frac{1}{|Q|}\int_Q (|x|^s\sigma(x))^{p\delta}\,dx\right)^{\frac{r}{p\delta}}\left(\frac{1}{|Q|}\int_Q u^{t_2}(x)\,dx\right)^{\frac{r}{pt_2}}\left(\frac{1}{|Q|}\int_Q g^{\nu'}(y)\,dy\right)^{\frac{1}{\nu'}} \leq$$

$$\leq c_1C^{\frac{r}{p}}\left(\frac{1}{|Q|}\int_Q \sigma^{\frac{r}{p}}(x)\left[\frac{1}{|Q|}\int_Q g^{\nu'}(y)\,dy\right]^{\frac{1}{\nu'}}dx\right) \leq \quad \text{by} \quad (2.5)$$

$$\leq c_1C^{\frac{r}{p}}\left(\frac{1}{|Q|}\int_Q \sigma^{\frac{r}{p}}(x)(Mg^{\nu'})^{\frac{1}{\nu'}}(x)\,dx\right) = C_2^{\frac{1}{\nu'}}C^r\frac{1}{|Q|}\int_Q \sigma^{\frac{r}{p}}(x)G(x)\,dx. \qquad \square$$

4. Proof of Theorem 1 and Proposition 8

Since $M_s : L^p_v(l^r) \to L^q_u(l^r)$ immediately implies condition (2.1), our purpose in this section is to prove the converse. We shall state some basic lemmas and use them to derive the above boundedness.

As in the scalar case [9], the main problem is reduced to obtaining the dyadic version of (1.1) denoted by $M_s^{dya} : L_v^p(l^r) \to L_u^q(l^r)$ and having the form

$$\Big\| \Big(\sum_k (M_s^{dya} f_k)^r(\cdot) \Big)^{\frac{1}{r}} \Big\|_{L_u^q} \leq C \Big\| \Big(\sum_k f_k^r(\cdot) \Big)^{\frac{1}{r}} \Big\|_{L_v^p} \tag{4.1}$$

for all sequences of nonnegative functions $(f_k(\cdot))_k$. Here $C = cS$ with $c > 0$ depending only on s, n, p, and q;

$$(M_s^{dya} f)(x) = \sup \Big\{ |Q|^{\frac{s}{n}-1} \int_Q |f(y)|\, dy;\ \ Q \text{ dyadic cube with } Q \ni x \Big\}.$$

To get $M_s : L_v^p(l^r) \to L_u^q(l^r)$ we need three basic lemmas.

Lemma 4.1. *Suppose that for some constant $S > 0$*

$$\Big\| \Big(\sum_k \big[\lambda_k \big(M_s^{dya} \widetilde{\sigma} 1\!\!1_{Q_k} \big)(\cdot) 1\!\!1_{Q_k}(\cdot) \big]^r \Big)^{\frac{1}{r}} \Big\|_{L^q_{\widetilde{u}}} \leq S \Big\| \Big(\sum_k \big[\lambda_k (\widetilde{\sigma} 1\!\!1_{Q_k})(\cdot) \big]^r \Big)^{\frac{1}{r}} \Big\|_{L^p_{\widetilde{v}}}$$

for all sequences of nonnegative $(\lambda_k)_k$ and all sequences of dyadic cubes $(Q_k)_k$, where $\widetilde{\sigma}(\cdot) = \widetilde{v}^{-\frac{1}{p-1}}(\cdot)$. Then $M_s^{dya} : L^p_{\widetilde{v}}(l^r) \to L^q_{\widetilde{u}}(l^r)$ or, more precisely,

$$\Big\| \Big(\sum_k (M_s^{dya} f_k)^r(\cdot) \Big)^{\frac{1}{r}} \Big\|_{L^q_{\widetilde{u}}} \leq cS \Big\| \Big(\sum_k f_k^r(\cdot) \Big)^{\frac{1}{r}} \Big\|_{L^p_{\widetilde{v}}}$$

where $c > 0$ is a constant which depends only on s, n, p, q, and r but not on the weights $\widetilde{v}(\cdot)$ and $\widetilde{u}(\cdot)$.

This leads to

Lemma 4.2. *Condition* (2.1) *implies that*

$$\Big\| \Big(\sum_k ({}^z M_s f_k)^r(\cdot) \Big)^{\frac{1}{r}} \Big\|_{L_u^q} \leq cS \Big\| \Big(\sum_k f_k^r(\cdot) \Big)^{\frac{1}{r}} \Big\|_{L_v^p} \tag{4.2}$$

for all $z \in \mathbb{R}^n$ and for all sequences of nonnegative functions $(f_k(\cdot))_k$, where

$$({}^z M_s f)(x) =$$

$$= \sup \Big\{ |Q|^{\frac{s}{n}-1} \int_Q |f(y)|\, dy;\ \ Q \ni x \ \text{ and } \ Q - z \ \text{ is a closed dyadic cubes} \Big\}.$$

The constant $c > 0$ depends only on s, n, p, q, and r but not on the weights $v(\cdot)$ and $u(\cdot)$. Here S is the constant appearing in condition (2.1).

This connection of this result with $M_s : L_v^p(l^r) \to L_u^q(l^r)$ is given by

Lemma 4.3. *For all $r > 1$ there is $c > 0$ such that*

$$\Big[\sum_k (M_s^{2^N} f_k)^r(\cdot)\Big]^{\frac{1}{r}} \le \frac{c}{2^{n(N+3)}} \int\limits_{[-2^{N+2},2^{N+2}]^n} \Big[\sum_k ({}^z M_s f_k)^r(\cdot)\Big]^{\frac{1}{r}} dz \quad (4.3)$$

for all $N \in \mathbb{Z}$ and all sequences of nonnegative functions $(f_k(\cdot))_k$. Here the truncated maximal operator M_s^R, $R > 0$, is defined by

$$(M_s^R f)(x) = \sup\Big\{ |Q|^{\frac{s}{n}-1} \int_Q |f(y)|\,dy;\ \ Q \ni x \ \ \textit{with}\ \ |Q|^{\frac{1}{n}} \le R \Big\}.$$

Now let us see how these lemmas imply $M_s : L_v^p(l^r) \to L_u^q(l^r)$. It is sufficient to get $M_s^R : L_v^p(l^r) \to L_u^q(l^r)$ uniformly for all $R > 0$, since the conclusion appears after using the monotone convergence theorem. Thus let $R > 0$ such that $2^{N-1} < R \le 2^N$ for some integer N. Then

$$\Big\|\Big(\sum_k (M_s^R f_k)^r(\cdot)\Big)^{\frac{1}{r}}\Big\|_{L_u^q} \le \Big\|\Big(\sum_k (M_s^{2^N} f_k)^r(\cdot)\Big)^{\frac{1}{r}}\Big\|_{L_u^q} \le$$

$$\le \frac{c}{2^{n(N+3)}} \Big\| \int\limits_{[-2^{N+2},2^{N+2}]^n} \Big[\sum_k ({}^z M_s f_k)^r(\cdot)\Big]^{\frac{1}{r}} dz \Big\|_{L_u^q} \le \quad (\text{by} \quad (4.3))$$

$$\le \frac{c}{2^{n(N+3)}} \int\limits_{[-2^{N+2},2^{N+2}]^n} \Big\|\Big(\sum_k ({}^z M_s f_k)^r(\cdot)\Big)^{\frac{1}{r}}\Big\|_{L_u^q} dz \le$$

(by the Minkowski inequality)

$$\le c' S \Big\|\Big(\sum_k f_k^r(\cdot)\Big)^{\frac{1}{r}}\Big\|_{L_u^q} \quad (\text{by inequality} \quad (4.2)).$$

The latter inequality is obtained after using (4.2) for which the constant cS does not depend on z. So the proof of $M_s : L_v^p(l^r) \to L_u^q(l^r)$ will be completed after proving these basic lemmas, which will be done in the next section.

Proof of Proposition 8. The first part of this result follows from Theorem 1, since under the assumption on $\sigma(\cdot)$

$$(M_s \sigma 1\!\!1_Q)(\cdot) 1\!\!1_Q(\cdot) \le c\Big[|Q|^{\frac{s}{n}-1}\Big(\int_Q \sigma(y)\,dy\Big)\Big] 1\!\!1_Q(\cdot)$$

for all cubes Q and for a fixed constant $c > 0$. The proof of this inequality can be found in [8] (Proposition 2, p.93).

The second part is a consequence of Proposition 6 and the above inequality. □

5. Proofs of Basic Lemmas 4.1, 4.2, 4.3

Proof of Lemma 4.3. Inequality (4.3) is based on

$$(M_s^{2^N} g)(\cdot) \le \frac{c}{2^{n(N+3)}} \int\limits_{[-2^{N+2},2^{N+2}]^n} ({}^z M_s g)(\cdot)\, dz \quad \text{for all} \quad g(\cdot) \ge 0, \tag{5.1}$$

where $c > 0$ does not depend on N and $g(\cdot)$. This inequality was proved in [9] (Lemma 2, p. 8). Consider a sequence of nonnegative scalars $(\alpha_k)_k$ with $(\sum_k \alpha_k^{r'})^{\frac{1}{r'}} = 1$ and such that $(\sum_k (M_s^{2^N} f_k)^r(x))^{\frac{1}{r}} = \sum_k (M_s^{2^N} f_k)(x)\alpha_k$. Then inequality (4.3) appears after using (5.1), since

$$\Big(\sum_k (M_s^{2^N} f_k)^r(x)\Big)^{\frac{1}{r}} \le \frac{c}{2^{n(N+3)}} \int\limits_{[-2^{N+2},2^{N+2}]^n} \Big[\sum_k ({}^z M_s f_k)(x)\alpha_k\Big]\, dz \le$$

$$\le \Big(\sum_k \alpha_k^{r'}\Big)^{\frac{1}{r'}} \frac{c}{2^{n(N+3)}} \int\limits_{[-2^{N+2},2^{N+2}]^n} \Big[\sum_k ({}^z M_s f_k)^r(x)\Big]^{\frac{1}{r}}\, dz =$$

$$= \frac{c}{2^{n(N+3)}} \int\limits_{[-2^{N+2},2^{N+2}]^n} \Big[\sum_k ({}^z M_s f_k)^r(x)\Big]^{\frac{1}{r}}\, dz. \qquad \square$$

Proof of Lemma 4.2. The key to deriving inequality (4.2) is that for some constant $c > 0$

$$\Big\| \Big(\sum_k (M_s^{dya} g_k)^r(x)\Big)^{\frac{1}{r}} \Big\|_{L^q(u(x-z)dx)} \le cS \Big\| \Big(\sum_k g_k^r(x)\Big)^{\frac{1}{r}} \Big\|_{L^p(v(x-z)dx)} \tag{5.2}$$

for all $z \in \mathbb{R}^n$ and all sequences of nonnegative functions $(g_k(\cdot))_k$. Indeed, with $f_{k,z}(y) = f_k(y-z)$ we have that

$$\Big\| \Big(\sum_k ({}^z M_s f_k)^r(x)\Big)^{\frac{1}{r}} \Big\|_{L^q(u(x)dx)} = \Big\| \Big(\sum_k (M_s^{dya} f_{k,z})^r(x+z)\Big)^{\frac{1}{r}} \Big\|_{L^q(u(x)dx)} =$$

$$= \Big\| \Big(\sum_k (M_s^{dya} f_{k,z})^r(x)\Big)^{\frac{1}{r}} \Big\|_{L^q(u(x-z)dx)} \le$$

$$\le cS \Big\| \Big(\sum_k f_{k,z}^r(x)\Big)^{\frac{1}{r}} \Big\|_{L^p(v(x-z)dx)} = \quad \text{(by} \quad (5.2))$$

$$= cS \Big\| \Big(\sum_k f_k^r(x-z)\Big)^{\frac{1}{r}} \Big\|_{L^p(v(x-z)dx)} = cS \Big\| \Big(\sum_k f_k^r(x)\Big)^{\frac{1}{r}} \Big\|_{L^p(v(x)dx)}.$$

Now to get (5.2), by Lemma 4.1 with $\widetilde{u}(\cdot) = u(\cdot - z)$ and $\widetilde{v}(\cdot) = v(\cdot - z)$, it is sufficient to check that

$$\Big\| \Big(\sum_k \big[\lambda_k (M_s^{dya} \sigma_z \mathbb{1}_{Q_k})(x) \mathbb{1}_{Q_k}(x) \big]^r \Big)^{\frac{1}{r}} \Big\|_{L^q(u(x-z)dx)} \leq$$
$$\leq S \Big\| \Big(\sum_k \big[\lambda_k (\sigma_z \mathbb{1}_{Q_k})(x) \big]^r \Big)^{\frac{1}{r}} \Big\|_{L^p(v(x-z)dx)} \tag{5.3}$$

and all dyadic cubes $(Q_k)_k$; here $\sigma_z(y) = \sigma(y-z) = v^{-\frac{1}{p-1}}(y-z)$. To obtain this test condition, observe that

$$\Big\| \Big(\sum_k \big[\lambda_k (\sigma_z \mathbb{1}_{Q_k})(x) \big]^r \Big)^{\frac{1}{r}} \Big\|_{L^p(v(x-z)dx)} =$$
$$= \Big\| \Big(\sum_k \big[\lambda_k (\sigma \mathbb{1}_{Q_k - z})(x) \big]^r \Big)^{\frac{1}{r}} \Big\|_{L^p(v(x)dx)}$$

and

$$\Big\| \Big(\sum_k \big[\lambda_k (M_s^{dya} \sigma_z \mathbb{1}_{Q_k})(x) \mathbb{1}_{Q_k}(x) \big]^r \Big)^{\frac{1}{r}} \Big\|_{L^q(u(x-z)dx)} =$$
$$= \Big\| \Big(\sum_k \big[\lambda_k (M_s^{dya} \sigma_z \mathbb{1}_{Q_k})(x+z) \mathbb{1}_{Q_k - z}(x) \big]^r \Big)^{\frac{1}{r}} \Big\|_{L^q(u(x)dx)} \leq$$
$$\leq \Big\| \Big(\sum_k \big[\lambda_k (M_s \sigma \mathbb{1}_{Q_k - z})(x) \mathbb{1}_{Q_k - z}(x) \big]^r \Big)^{\frac{1}{r}} \Big\|_{L^q(u(x)dx)}.$$

With these observations, condition (2.1) (for all cubes) clearly implies the test condition (5.3) (for all dyadic cubes). □

Proof of Lemma 4.1. It is sufficient to find a constant $c > 0$ such that

$$\Big\| \Big(\sum_k (M_s^{dya,R} f_k)^r(\cdot) \Big)^{\frac{1}{r}} \mathbb{1}_{]0,R[^n}(\cdot) \Big\|_{L_u^q} \leq cS \Big\| \Big(\sum_k f_k^r(\cdot) \Big)^{\frac{1}{r}} \Big\|_{L_v^p} \tag{5.4}$$

for all $R > 0$ and all nonnegative functions $(f_k(\cdot))_k$. Here $(M_s^{dya,R} f)(x) = \sup\{|Q|^{\frac{s}{n}-1} \int_Q |f(y)|\,dy;\ Q$ is a dyadic cube with $x \in Q \subset]0,R[^n\}$. Inequality (5.4) is based on

Lemma 5.1. *There is a constant $c > 0$ such that for all $(f_k(\cdot))_k \in L_v^p(l^r)$ one can find nonegative scalars $(\lambda_{jk})_{k,j}$, and dyadic cubes $(Q_{jk})_{k,j}$ satisfying*

$$(M_s^{dya,R} f_k)^r(\cdot) \mathbb{1}_{]0,R[^n}(\cdot) \leq \sum_j \lambda_{jk}^r |Q_{jk}|_\sigma^{-\frac{r}{p}} \big(M_s^{dya} \sigma \mathbb{1}_{Q_{jk}} \big)^r(\cdot) \mathbb{1}_{Q_{jk}}(\cdot) \tag{5.5}$$

for all $r > 0$ and

$$\Big\|\Big(\sum_k\sum_j \lambda_{jk}^r |Q_{jk}|_\sigma^{-\frac{r}{p}} 1\!\!1_{Q_{jk}}(\cdot)\Big)^{\frac{1}{r}}\Big\|_{L^p_\sigma} \le c\Big\|\Big(\sum_k f_k^r(\cdot)\Big)^{\frac{1}{r}}\Big\|_{L^p_v}. \tag{5.6}$$

It is for the proof of such results that the atomic decomposition of tent spaces is useful. Assuming Lemma 5.1 to be true, inequality (5.4) can be derived as follows:

$$\begin{aligned}
&\Big\|\Big(\sum_k (M_s^{dya,R} f_k)^r(\cdot) 1\!\!1_{]0,R[^n}(\cdot)\Big)^{\frac{1}{r}}\Big\|^r_{L^q_u} = \Big\|\sum_k (M_s^{dya,R} f_k)^r(\cdot) 1\!\!1_{]0,R[^n}(\cdot)\Big\|_{L_u^{\frac{q}{r}}} \le \\
&\quad \le \Big\|\sum_k\sum_j \lambda_{jk}^r |Q_{jk}|_\sigma^{-\frac{r}{p}} (M_s^{dya}\sigma 1\!\!1_{Q_{jk}})^r(\cdot) 1\!\!1_{Q_{jk}}(\cdot)\Big\|_{L_u^{\frac{q}{r}}} \le \quad \text{(by (5.5))}\\
&\quad \le S^r\Big\|\Big(\sum_k\sum_j \lambda_{jk}^r |Q_{jk}|_\sigma^{-\frac{r}{p}} \sigma(\cdot) 1\!\!1_{Q_{jk}}(\cdot)\Big)^{\frac{1}{r}}\Big\|^r_{L^p_v} = \quad \text{(by condition (2.1))}\\
&\quad = S^r\Big\|\Big(\sum_k\sum_j \lambda_{jk}^r |Q_{jk}|_\sigma^{-\frac{r}{p}} 1\!\!1_{Q_{jk}}(\cdot)\Big)^{\frac{1}{r}}\Big\|^r_{L^p_\sigma} \le \quad \text{(since } \sigma^p(\cdot)v(\cdot) = \sigma(\cdot))\\
&\qquad\qquad \le cS^r\Big\|\Big(\sum_k f_k^r(\cdot)\Big)^{\frac{1}{r}}\Big\|^r_{L^p_\sigma} \quad \text{(by (5.6))}.
\end{aligned}$$

Here $c > 0$ does not depend on R.

We are going to introduce the notions of tent spaces needed for our purpose as in [8]. Our definitions will be slightly different from those introduced in Coifman, Meyer, and Stein in [7], since we use the dyadic versions of the spaces they defined.

Let $X =]0,\infty[^n - \{(2^{-l}k_j)_j\}$ $((k_j)_j \in \mathbb{N}^n)$, and $\widetilde{X} = X \times 2^{\mathbb{Z}}$. For each couple $(x, w) \in \widetilde{X}$ there is a unique (open) dyadic cube $Q = Q_{yw}$ containing y and having the side length $w = 2^{-l}$. Let

$$(y, w) \in \widetilde{\Gamma}(x) \quad \text{if and only if} \quad x \in Q_{yw}. \tag{5.7}$$

Also define

$$\widehat{\Omega} = \Big(\bigcup \big\{\widetilde{\Gamma}(x);\ \ x \in \Omega^c\big\}\Big)^c \tag{5.8}$$

for each measurable set $\Omega \subset]0,\infty[^n$. Thus

$$(y, w) \in \widehat{\Omega} \quad \text{if and only if} \quad Q_{yw} \subset \Omega. \tag{5.9}$$

Finally, define the functional $\mathcal{A}_\infty$ acting on each measurable function $\widetilde{f}$ on $\widetilde{X}$ by

$$(\mathcal{A}_\infty \widetilde{f})(x) = \sup\big\{|\widetilde{f}(y, w)|;\ \ (y, w) \in \widetilde{\Gamma}(x)\big\}. \tag{5.10}$$

The following *atomic decomposition Lemma* was proved in [8] (Lemma 2, p. 96).

Lemma 5.2. *Let $0 < p < \infty$. There is $C > 0$ such that for all functions $\widetilde{g}(y, w = 2^{-l})$ with a support contained in $\widehat{Q[0,R]}$ and $\|(\mathcal{A}_\infty \widetilde{g})(\cdot)\|_{L^p_v} < \infty$, one can find scalars $\lambda_j > 0$, dyadic cubes Q_j, and functions $\widetilde{a}_j(y,w)$ which satisfy the following conditions:*

$$\operatorname{supp} \widetilde{a}_j \ \text{ are disjoint and } \ |\widetilde{a}_j(y,w)| \le |Q_j|_\sigma^{-\frac{1}{p}} \widetilde{1\!\!1}_{\widehat{Q}_j}(y,w); \tag{5.11}$$

$$\widetilde{g}(y,w) = \sum_j \lambda_j \widetilde{a}_j(y,w) \quad a.e.; \tag{5.12}$$

$$\sum_j \lambda_j^r |Q_j|_\sigma^{-\frac{r}{p}} 1\!\!1_{Q_j}(\cdot) \le C^r (\mathcal{A}_\infty \widetilde{g})^r(\cdot) \quad \text{for all} \quad r > 0. \tag{5.13}$$

Here $Q[0,R]$ denotes the cube $]0,R[^n$, $R > 0$. Contrary to λ_j, Q_j, and $\widetilde{a}_j(y,w)$'s, the constant $C > 0$ does not depend on $R > 0$ and the function $\widetilde{g}(y,w)$.

A fundamental observation is that

$$\begin{aligned}
&(M_s^{dya,R} f)(x) = \\
&= \sup\left\{ |Q_{yw}|^{\frac{s}{n}-1} \int_{Q_{yw}} |f(y)|\,dy;\ \ Q_{yw} \ni x \ \text{ and } \ Q_{yw} \subset (]0,R[)^n \right\} = \\
&= \sup\{\widetilde{\omega}(y,w)\widetilde{g}(y,w);\ \ Q_{yw} \ni x \ \text{ and } \ Q_{yw} \subset (]0,R[)^n\},
\end{aligned}$$

where $\widetilde{\omega}(y,w) = |Q_{yw}|^{\frac{s}{n}-1} |Q_{yw}|_\sigma = |Q_{yw}|^{\frac{s}{n}-1} \int_{Q_{yw}} \sigma(z)\,dz$ and

$$\widetilde{g}(y,w) = \begin{cases} |Q_{yw}|_\sigma^{-1} \int_{Q_{yw}} g(z)\sigma(z)\,dz & \text{if} \quad (y,w) \in \widehat{Q[0,R]} \\ 0 \quad \text{else} \end{cases}$$

and $g(\cdot) = \sigma^{-1}(\cdot) f(\cdot)$. Moreover,

$$(\mathcal{A}_\infty \widetilde{g})(\cdot) \le (N_\sigma g)(\cdot) \tag{5.14}$$

with $(N_\sigma g)(x) = \sup\{|Q|_\sigma^{-1} \int_Q |g(y)|\sigma(y)\,dy;\ Q$ dyadic with $Q \ni x\}$. Consequently, for $1 < p < \infty$,

$$\begin{aligned}
\|(\mathcal{A}_\infty \widetilde{g})(\cdot)\|_{L^p_\sigma} &\le \|(N_\sigma g)(\cdot)\|_{L^p_\sigma} \le c(n,p)\|\sigma^{-1}(\cdot) f(\cdot)\|_{L^p_\sigma} = \\
&= c(n,p)\|f(\cdot)\|_{L^p_v}.
\end{aligned} \tag{5.15}$$

The latter inequality can be obtained by interpolation, that is, by proving with the aid of the classical arguments that $N_\sigma : L^1(\mathbb{R}^n, \sigma(x)dx) \to L^{1,\infty}(\mathbb{R}^n, \sigma(x)dx)$ and $N_\sigma : L^\infty(\mathbb{R}^n, \sigma(x)dx) \to L^\infty(\mathbb{R}^n, \sigma(x)dx)$.

Now we are ready to give

Proof of Lemma 5.1. This result is essentially based on Lemma 5.2. Indeed, if $(f_k(\cdot))_k$ is a sequence of nonnegative functions with $\|(\sum_k f_k^r(\cdot))^{\frac{1}{r}}\|_{L^p_v} < \infty$, then, in particular, $f_k(\cdot) \in L^p_v$ for each k. So by (5.15) the associated function $\widetilde{g}_k(y,w)$ defined as above satisfies $\|(\mathcal{A}_\infty \widetilde{g}_k)(\cdot)\|_{L^p_v} < \infty$. Hence by Lemma 5.2 one can find scalars $\lambda_{jk} > 0$, dyadic cubes Q_{jk}, and functions $\widetilde{a}_{jk}(y, w = 2^{-l})$ satisfying the following conditions:

$$\operatorname{supp} \widetilde{a}_{jk} \text{ are disjoint and } |\widetilde{a}_{jk}(y,w)| \le |Q_{jk}|_\sigma^{-\frac{1}{p}} \widetilde{1\!\!1}_{\widehat{Q}_{jk}}(y,w); \tag{5.16}$$

$$\widetilde{g}_k(y,w) = \sum_j \lambda_{jk} \widetilde{a}_{jk}(y,w) \text{ a.e.}; \tag{5.17}$$

$$\sum_j \lambda_{jk}^r |Q_{jk}|_\sigma^{-\frac{r}{p}} 1\!\!1_{Q_{jk}}(\cdot) \le C^r (\mathcal{A}_\infty \widetilde{g}_k)^r(\cdot) \text{ for all } r > 0. \quad \square \tag{5.18}$$

Let us check inequality (5.5). Since

$$(M_s^{dya,R} f_k)^r(x) = \sup\{\widetilde{\omega}^r(y,w)\widetilde{g}_k{}^r(y,w);\ Q_{yw} \ni x \text{ and } Q_{yw} \subset (]0,R[)^n\},$$

for all (y,w) with $x \in Q_{yw} \subset Q[0,R]$ we have

$$\begin{aligned}
\widetilde{\omega}^r(y,w)\widetilde{g}_k^r(y,w) &\le \widetilde{\omega}^r(y,w) \sum_j \lambda_{jk}^r |\widetilde{a}_{jk}^r(y,w)| \le (\operatorname{supp} \widetilde{a}_j \text{ are disjoint}) \\
&\le \sum_j \lambda_{jk}^r |Q_{jk}|_\sigma^{-\frac{r}{p}} \left[|Q_{yw}|^{\frac{s}{n}-1} |Q_{yw}|_\sigma \widetilde{1\!\!1}_{\widehat{Q}_j}(y,w)\right]^r \le \\
&\qquad (\text{by the definition of } \widetilde{\omega}(y,w) \text{ and } (5.16)) \\
&\le \sum_j \lambda_{jk}^r |Q_{jk}|_\sigma^{-\frac{r}{p}} \left[|Q_{yw}|^{\frac{s}{n}-1} |Q_{yw} \cap Q_{jk}|_\sigma \widetilde{1\!\!1}_{\widehat{Q}_{jk}}(y,w)\right]^r = \\
&\qquad (\text{since by } (5.9): \ Q_{yw} \subset Q_{jk}) \\
&= \sum_j \lambda_{jk}^r |Q_{jk}|_\sigma^{-\frac{r}{p}} \left[\left(|Q_{yw}|^{\frac{s}{n}-1} \int_{Q_{yw}} \sigma(z) 1\!\!1_{Q_{jk}}(z)\, dz\right) \widetilde{1\!\!1}_{\widehat{Q}_{jk}}(y,w)\right]^r \le \\
&\le \sum_j \lambda_{jk}^r |Q_{jk}|_\sigma^{-\frac{r}{p}} (M_s^{dya} \sigma 1\!\!1_{Q_{jk}})^r(x) 1\!\!1_{Q_{jk}}(x) \quad \text{recall that } x \in Q_{yw} \subset Q_{jk}.
\end{aligned}$$

In checking inequality (5.6) the main key is

$$\left\| \Big(\sum_k (N_\sigma g_k)^r(\cdot)\Big)^{\frac{1}{r}} \right\|_{L^p_\sigma} \le c \left\| \Big(\sum_k g_k^r(\cdot)\Big)^{\frac{1}{r}} \right\|_{L^p_\sigma}, \tag{5.19}$$

where $c > 0$ does not depend on the sequence $(g_k(\cdot))_k$. Indeed, using (5.18), (5.14), and the above inequality, we obtain

$$\Big\| \Big(\sum_k \sum_j \lambda_{jk}^r |Q_{jk}|_\sigma^{-\frac{r}{p}} 1\!\!1_{Q_{jk}}(\cdot) \Big)^{\frac{1}{r}} \Big\|_{L^p_\sigma} = \Big\| \sum_k \sum_j \lambda_{jk}^r |Q_{jk}|_\sigma^{-\frac{r}{p}} 1\!\!1_{Q_{jk}}(\cdot) \Big\|_{L^{\frac{p}{r}}_\sigma} \le$$
$$\le C^r \Big\| \sum_k (\mathcal{A}_\infty \widetilde{g}_k)^r(\cdot) \Big\|_{L^{\frac{p}{r}}_\sigma} = C^r \Big\| \Big(\sum_k (\mathcal{A}_\infty \widetilde{g}_k)^r(\cdot) \Big)^{\frac{1}{r}} \Big\|^r_{L^p_\sigma} \le$$
$$\le C^r \Big\| \Big(\sum_k (N_\sigma g_k)^r(\cdot) \Big)^{\frac{1}{r}} \Big\|^r_{L^p_\sigma} \le (cC)^r \Big\| \Big(\sum_k g_k^r(\cdot) \Big)^{\frac{1}{r}} \Big\|^r_{L^p_\sigma} =$$
$$= (cC)^r \Big\| \Big(\sum_k f_k^r(\cdot) \Big)^{\frac{1}{r}} \Big\|^r_{L^p_v},$$

since $g_k(\cdot) = \sigma^{-1}(\cdot) f_k(\cdot)$ and $\sigma^{1-p}(\cdot) = v(\cdot)$.

The boundedness (5.19) can be obtained by using the same classical arguments [1] as for $M_0 : L^p_1(l^r) \to L^p_1(l^r)$. Indeed the main points are (5.15) and Fefferman–Stein's inequality

$$\int_{\mathbb{R}^n} (N_\sigma f)^r(x) g(x) \sigma(x)\, dx \le c \int_{\mathbb{R}^n} f^r(x) (N_\sigma g)(x) \sigma(x)\, dx, \quad 1 < r < \infty, \tag{5.20}$$

where $c > 0$ is a constant which does not depend on the nonnegative functions $f(\cdot)$, $g(\cdot)$. Similarly to (5.15), inequality (5.20) can be obtained by interpolation, since $N_\sigma : L^\infty[\mathbb{R}^n, (N_\sigma g)(x)\sigma(x)dx] \to L^\infty[\mathbb{R}^n, g(x)\sigma(x)dx]$ and $N_\sigma : L^1[\mathbb{R}^n, (N_\sigma g)(x)\sigma(x)dx] \to L^{1\infty}[\mathbb{R}^n, g(x)\sigma(x)dx]$. In this last boundedness no doubling condition $\sigma(\cdot)$ is needed, since we work with dyadic cubes. By (5.15) it is clear that $N_\sigma : L^p_\sigma(l^p) \to L^p_\sigma(l^p)$. And since $N_\sigma : L^p_\sigma(l^\infty) \to L^p_\sigma(l^\infty)$, by interpolation [1] we get inequality (5.19) for $p \le r < \infty$. The corresponding result for $1 < r < p$ can be obtained by duality using inequality (5.20) (see also [1] for details with the nonweighted case $M_0 : L^p_1(l^r) \to L^p_1(l^r)$).

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References

1. J. Garcia Cuerva and J. L. Rubio de Francia, Weighted norm inequalities and related topics. *North Holland Math. Stud.* 116, *Notes Mat.* 104, *North Holland Publishing Co., Amsterdam–N. Y.*, 1985.

2. H. P. Heiniga and R. Johnson, Weighted norm inequalities for L^r-valued integral operators and applications. *Math. Nachr.* **107**(1982), 161–174.

3. K. Andersen and R. John, Weighted inequalities for vector-valued maximal functions and singular integrals. *Studia Math.* **49**(1980), 19–31.

4. V. M. Kokilashvili, Maximal inequalities and multipliers in weighted Lizorkin–Triebel spaces. *Soviet. Math. Dokl.* **19(2)**(1978), 272–276.

5. C. Pérez, Sharp weighted estimates for the vector-valued maximal function. *Preprint Univ. Madrid*, 1995.

6. F. J Ruiz and J. L. Torrea, Weighted and vector-valued inequalities. *Trans. Amer. Math. Soc.* **295**(1986), 213–232.

7. R. R. Coifman, Y. Meyer, and E. M. Stein, Some new function spaces. *J. Funct. Anal.* **62**(1985), 304–335.

8. Y. Rakotondratsimba, Weighted inequalities for maximal operators via atomic decompositions of tent spaces. *SUT J. Math.* **2**(1996), 91–99.

9. E. Sawyer, A characterization of a two-weight norm inequality for maximal operators. *Studia Math.* **75**(1982), 1–11.

10. Y. Rakotondratsimba, Weighted vector-valued inequalities for fractional maximal operators. (*Preprint*).

11. R. Wheeden, A characterization of some weighted norm inequalities for fractional maximal functions. *Studia Math.* **107**(1993), 257–272.

12. A. Gogatishvili and V. Kokilashvili, Criteria of strong type two-weighted inequalities for fractional maximal functions. *Georgian Math J.* **3**(1996), No. 5, 423–446.

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