GEORGIAN MATHEMATICAL JOURNAL: Vol. 6, No. 3, 1999, 201-212

ON UNCOUNTABLE UNIONS AND INTERSECTIONS OF MEASURABLE SETS

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ABSTRACT. We consider several natural situations where the union or intersection of an uncountable family of measurable (in various senses) sets with a "good" additional structure is again measurable or may fail to be measurable. We primarily deal with Lebesgue measurable sets and sets with the Baire property. In particular, uncountable unions of sets homeomorphic to a closed Euclidean simplex are considered in detail, and it is shown that the Lebesgue measure and the Baire property differ essentially in this aspect. Another difference between measure and category is illustrated in the case of some uncountable intersections of sets of full measure (comeager sets, respectively). We also discuss a topological form of the Vitali covering theorem, in connection with the Baire property of uncountable unions of certain sets.

0. INTRODUCTION

A mathematician working in probability theory, the theory of random processes or in various fields of modern analysis is quite frequently obliged to show that the union $\bigcup\{X_a : a \in A\}$ or the intersection $\bigcap\{X_a : a \in A\}$ of a given uncountable family $\{X_a : a \in A\}$ of measurable sets is measurable provided that the sets X_a have some additional "good" structure. Here the measurability is meant in the sense of the respective σ-algebra $\mathcal{S}$ of subsets of a given nonempty basic set E containing each set X_a $(a \in A)$. For instance, let E coincide with the real line R and let $\mathcal{S}$ be the σ-algebra of all Lebesgue measurable subsets of R. Let us consider the family $\mathcal{T}$ of all $\mathcal{S}$-measurable sets $X \subseteq R$ such that each point x from X is a density point of X. It is well known (see, e.g., [1], Chapter 22) that $\mathcal{T}$ forms a topology which is usually called the *density topology* on the real line. Thus the union of each subfamily of $\mathcal{T}$ belongs to $\mathcal{T}$, and hence is $\mathcal{S}$-measurable. In particular, we see that uncountable unions of $\mathcal{S}$-measurable sets belonging

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1991 *Mathematics Subject Classification.* 28A05, 28A12, 54E52.
Key words and phrases. Lebesgue measure, Jordan measure, Baire property, Vitali covering, Jordan curve, Suslin condition, simplex, meager set.

1072-947X/99/0500-0201$12.50/0

to $\mathcal{T}$ are $\mathcal{S}$-measurable, too. The same situation holds for the von Neumann-Maharam topology which is a generalization of the density topology and can be introduced for an arbitrary measure space $(E, \mathcal{S}, \mu)$ where E is a nonempty basic set, $\mathcal{S}$ is a σ-algebra of subsets of E and μ is a nonzero σ-finite complete measure defined on $\mathcal{S}$ (see [1], Chapter 22). Note that a number of interesting examples and questions concerning the measurability of uncountable unions of measurable sets are discussed in [2].

The main aim of this paper is to illustrate that the phenomenon producing measurable unions (or intersections) for uncountable families of "good" measurable sets depends on very delicate conditions. Another goal is to express some similarities and differences between measure and category in that aspect. In Section 1, we briefly consider the most classical situation dealing with an arbitrary family $\{S_t : t \in T\}$ of closed n-simplexes in the n-dimensional Euclidean space R^n. Applying some well-known properties of the standard Lebesgue and Jordan measures in R^n (see, e.g., [3]), we show that the set $\bigcup\{S_t : t \in T\}$ is Lebesgue measurable and has the Baire property. In Section 2, we are concerned with arbitrary families of sets that are homeomorphic to a closed n-simplex in R^n (such sets are called *topological n-simplexes*). It turns out that, in this situation, the measure and the category cases differ essentially. Namely, an arbitrary family of topological n-simplexes has the union with the Baire property. On the other hand, there exists a family of topological n-simplexes in R^n ($n \geq 2$) with the union non-measurable in the Lebesgue sense. Therefore, from this point of view, the Baire property behaves better than Lebesgue measure. Section 3 witnesses that in some situations it can happen conversely, that is, measure behaves better than the Baire property. Namely, in connection with a theorem of Goldstern [4] concerning certain uncountable intersections of coanalytic sets with the full probability measure, we show that an analogous result fails to be true in the category case. In Section 4, we introduce the notion of a topological Vitali space. It turns out that these spaces characterize the situation where, for a wide class of sets (called admissible), the union of an arbitrary subfamily is countably approximable and consequently has the Baire property.

Our notation and terminology concerning set theory, general topology, measure and category are fairly standard. In particular, we denote:

ω – the set of all natural numbers, i.e., $\omega = \{0, 1, 2, \ldots, n, \ldots\}$;

μ – a measure on a given basic space (as a rule, we assume that μ is nonzero, σ-finite and complete);

$dom(\mu)$ – the family of all μ-measurable sets;

ν_n – the classical Jordan measure in R^n;

λ_n – the classical Lebesgue measure in R^n.

Let E be an arbitrary topological space. If $X \subseteq E$, then $\mathrm{cl}(X)$, $\mathrm{int}(X)$ and $\mathrm{bd}(X)$ stand for the closure, the interior and the boundary of X, respectively. We denote:

$NWD(E)$ – the ideal of all nowhere dense subsets of E;

$K(E)$ – the σ-ideal of all meager (i.e., first category) subsets of E;

$Br(E)$ – the σ-algebra of all subsets of E with the Baire property.

Recall that a set $X \subseteq E$ has the *Baire property* iff X can be represented as the symmetric difference $X = U \triangle P$ where U is open and P is meager in E. Moreover, if $X \subseteq E$ can be expressed in the form $X = U \triangle P$ where U is open and P is nowhere dense in E, then X is called an *open set modulo a nowhere dense set* (cf. [5], §8, V) or, briefly, an *open set modulo* $NWD(E)$.

We shall say that a subset X of a topological space E is *admissible* if

$$X \subseteq \mathrm{cl}(\mathrm{int}(X)).$$

Clearly, every open set is admissible. If X is a regular closed set (i.e., $X = \mathrm{cl}(\mathrm{int}(X))$), then X is admissible, as well. In addition, if X is admissible, then from the relations

$$\mathrm{int}(X) \subseteq X \subseteq \mathrm{cl}(\mathrm{int}(X)), \ \mathrm{cl}(\mathrm{int}(X)) \setminus \mathrm{int}(X) \in NWD(E)$$

it follows that X is open modulo a nowhere dense set (in general, the converse is false).

In our further considerations, the notion of a Vitali covering plays an essential role. So, we introduce this notion for a general topological space and for an arbitrary subset of that space. We say that a family $\mathcal{V}$ of subsets of a topological space E forms a *Vitali covering* of a given set $X \subseteq E$ if, for each point $x \in X$ and for each neighbourhood U of x, there exists a set $V \in \mathcal{V}$ such that $x \in V \subseteq U$. Consequently, if $\mathcal{V}$ is a Vitali covering of X, then the family of sets $\{V \cap X : V \in \mathcal{V}\}$ forms a net for the subspace X of E. As a rule, the Vitali coverings considered below are assumed to consist of admissible sets. We also recall that a topological space E satisfies the *Suslin condition* (the *countable chain condition*) if each disjoint family of nonempty open sets in E is countable.

1. Unions of n-Simplexes

By an *n-simplex* we mean a closed nondegenerate simplex in the Euclidean space R^n. If a real $\alpha > 0$ is fixed, we say that a bounded set $X \subseteq R^n$ is *α–regular* if $\lambda_n(X) \geq \alpha\lambda_n(V(X))$ where $V(X)$ denotes a closed ball with the minimal diameter, for which we have the inclusion $X \subseteq V(X)$ (clearly, in this definition a ball can be replaced by a cube). Note that every n-simplex is an α–regular set for some $\alpha > 0$.

Theorem 1.1. *Let $\{S_t : t \in T\}$ be an arbitrary family of n-simplexes in R^n. Then the set $X = \bigcup\{S_t : t \in T\}$ is Lebesgue measurable and possesses the Baire property.*

Proof. Obviously, the set X can be expressed in the form

$$X = \bigcup_{m \in \omega,\ m>0} \bigcup_{t \in T_m} S_t$$

where

$$T_m = \{t \in T : S_t \ is\ 1/m{-}regular\}.$$

It suffices to show that, for a fixed integer $m > 0$, the set

$$X_m = \bigcup\{S_t : t \in T_m\}$$

is Lebesgue measurable. For any $t \in T_m$, $x \in S_t$ and $c \in]0,1[$, let $S_t^c(x)$ denote the image of S_t under the homothetic transformation

$$y \to x + c(y - x)\ (y \in R^n).$$

Observe that the family of n-simplexes

$$\mathcal{F}_m = \{S_t^c(x) : t \in T_m \ \&\ x \in S_t \ \&\ c \in]0,1[\}$$

forms a Vitali covering of X_m. Additionally, $\mathcal{F}_m$ consists of simplexes which are $1/m$–regular sets. Thus, by the generalized Vitali theorem (see [6], Chapter 4, Theorem 3.1), there exists a countable disjoint family $\mathcal{F}_m^* \subseteq \mathcal{F}_m$ such that $\lambda_n(X_m \setminus \bigcup \mathcal{F}_m^*) = 0$. Evidently, $\bigcup \mathcal{F}_m^*$ is Lebesgue measurable. Also, we have $\bigcup \mathcal{F}_m^* \subseteq X_m$ by the definition of $\mathcal{F}_m$. Hence X_m is Lebesgue measurable.

Our argument in the category case also uses measure-theoretical tools. Namely, we apply the Jordan measure ν_n in R^n and the respective inner measure $(\nu_n)_*$. By the classical criterion, a bounded set $Y \subseteq R^n$ is Jordan measurable iff $\nu_n(\mathrm{bd}(Y)) = 0$ where $\mathrm{bd}(Y)$ stands for the boundary of Y. Since $\mathrm{bd}(Y)$ is compact, the equality $\nu_n(\mathrm{bd}(Y)) = 0$ implies that $\mathrm{bd}(Y)$ is nowhere dense. Consequently, every Jordan measurable set Y can be expressed as the union of the open set $\mathrm{int}(Y)$ and the nowhere dense set $Y \cap \mathrm{bd}(Y)$. In particular, we see that Y possesses the Baire property.

In view of the above remark, it suffices to express our set X as a countable union of Jordan measurable sets. For an arbitrary integer $m > 0$, we denote

$$B_m = \{x \in R^n : ||x|| \le m\}, \quad \mathcal{G}_m = \{S_t : t \in T \ \&\ S_t \subseteq B_m\},$$

where $||x||$ stands for the usual Euclidean norm of $x \in R^n$. We obviously have

$$X = \bigcup_{m \in \omega,\ m>0} \bigcup \mathcal{G}_m.$$

Now, let us consider the inner Jordan density of an arbitrary set $Z \subseteq R^n$ at a point $x \in R^n$, given by the formula

$$d_*(Z,x) = \inf\{((\nu_n)_*(Z \cap Q))/\nu_n(Q) : Q \in \mathcal{Q}(x)\},$$

where $\mathcal{Q}(x)$ stands for the family of all closed cubes with centre x and with diameters ≤ 1. According to the well-known result (see [3], Chapter 3), if there exists $\varepsilon > 0$ such that, for each point z of a bounded set $Z \subseteq R^n$, we have $d_*(Z, z) \geq \varepsilon$, then Z is Jordan measurable. Observe that

$$\mathcal{G}_m = \bigcup_{k \in \omega,\ k>0} \mathcal{G}_m^k$$

where

$$\mathcal{G}_m^k = \{S_t \in \mathcal{G}_m : (\forall x \in S_t)(d_*(S_t, x) \geq 1/k)\}.$$

Then the set $\bigcup \mathcal{G}_m^k$ is bounded and $d_*(\bigcup \mathcal{G}_m^k, x) \geq 1/k$ for each point $x \in \bigcup \mathcal{G}_m^k$. Consequently, $\bigcup \mathcal{G}_m^k$ is Jordan measurable. We thus see that the set X is expressible as a countable union of Jordan measurable sets. Hence it has the Baire property. $\square$

We want to finish this section with several simple remarks concerning the theorem just proved. First of all let us note that the union of a family of n-simplexes may have a rather bad descriptive structure. Indeed, suppose that $n \geq 2$ and consider some hyperplane Γ in the space R^n. Let Y be any subset of Γ. It is easy to construct a family of n-simplexes in R^n, such that the intersection of its union with the hyperplane Γ coincides with the set Y. Consequently, if Y is not λ_{n-1}-measurable or does not possess the Baire property in Γ, then the above-mentioned union is not an analytic (coanalytic) subset of R^n. In a similar way one can construct a family of n-simplexes whose union is not a projective subset of R^n.

It immediately follows from Theorem 1.1 that the union of an arbitrary family of convex bodies in R^n is λ_n-measurable (because any convex body in R^n can be represented as the union of a family of n-simplexes). Analogously, the union of an arbitrary family of convex bodies in R^n has the Baire property. The latter fact remains true for any family of convex bodies in a topological vector space (see, e.g., Theorem 2.1 below containing a more general result).

2. Unions of Topological n-Simplexes

By a *topological n-simplex* we mean a topological space homeomorphic to an n-simplex. Our purpose in this section is twofold. First, we are going to show that the union of an arbitrary family of topological n-simplexes in R^n has the Baire property. Next, we shall prove that, for $n \geq 2$, an analogous statement fails to be true if the possession of the Baire property is replaced by Lebesgue measurability. The first of these two results will be derived from the following general theorem:

Theorem 2.1. *Let E be an arbitrary topological space and let $\mathcal{F}$ be a family consisting of admissible subsets of E. Then $\bigcup \mathcal{F}$ is open modulo $NWD(E)$. In particular, $\bigcup \mathcal{F}$ has the Baire property.*

Proof. We put $V = \bigcup\{\mathrm{int}(F) : F \in \mathcal{F}\}$. Since $\mathcal{F}$ consists of admissible sets, we have, for each $F \in \mathcal{F}$,

$$F \subseteq \mathrm{cl}(\mathrm{int}(F)) \subseteq \mathrm{cl}(V).$$

Thus, we obtain $\bigcup \mathcal{F} \subseteq \mathrm{cl}(V)$ and, therefore,

$$V = \bigcup\{\mathrm{int}(F) : F \in \mathcal{F}\} \subseteq \bigcup \mathcal{F} \subseteq \mathrm{cl}(V).$$

Hence $\bigcup \mathcal{F} = V \cup Z$ for some set $Z \subseteq \mathrm{cl}(V) \setminus V$, which gives the assertion. $\square$

Corollary 2.1. *For each integer $n \geq 1$ and for each family of topological n-simplexes $\{S_t : t \in T\}$ in R^n, the set $\bigcup\{S_t : t \in T\}$ has the Baire property.*

Obviously, Corollary 2.1 yields another (purely topological) proof of the category part of Theorem 1.1.

The next statement shows that the measure case for unions of topological simplexes is completely different.

Theorem 2.2. *For each integer $n \geq 2$, there exists a family $\{Z_t : t \in T\}$ of topological n-simplexes in R^n such that the set $\bigcup\{Z_t : t \in T\}$ is not measurable in the Lebesgue sense.*

Proof. It is enough to consider the case $n = 2$ since, if a family $\{Z_t : t \in T\}$ satisfies the assertion of our theorem for $n = 2$, then the family $\{Z_t \times [0,1]^{n-2} : t \in T\}$ satisfies the assertion of the theorem for an arbitrary integer $n > 2$. So, we restrict our further consideration to the case $n = 2$. In the sequel, by a Jordan curve we mean a homeomorphic image of the unit circle. It is well known that there exists a Jordan curve L in R^2 possessing a positive two-dimensional Lebesgue measure, i.e., $\lambda_2(L) > 0$. The construction of L can be done directly. Another idea is to derive from the Denjoy-Riesz theorem (see, e.g., [5], §61, V, Theorem 5) that each compact zero-dimensional set C in R^2 is contained in a Jordan curve. Taking as C a Cantor-type set in R^2 with $\lambda_2(C) > 0$, we get the desired curve L. Now, by the Jordan Curve Theorem, $R^2 \setminus L$ has exactly two components: one bounded and one unbounded. The bounded component will be denoted by U. We shall use the Schönflies theorem (see [5], §61, II, Theorem 11) according to which, for any points $x \in L$ and $y \in U$, there is a simple arc l, with end-points x and y, such that $l \setminus \{x\} \subseteq U$. In fact, a bit sharper version is needed where the simple arc l is a quasi-polygonal curve, i.e., the set $l \setminus \{x\}$ consists of countably many linear segments which converge (in the Hausdorff metric) to $\{x\}$. (Cf., e.g., [7], Appendix to Chapter IX.)

Now, let us pick a Lebesgue nonmeasurable set $L^* \subseteq L$. (The existence of L^* is well known - see, e.g., [1], Chapter 5.) We shall define a family $\{Z_{x,y} : x \in L^*,\ y \in U\}$ of sets lying in the plane and satisfying the following conditions:

(1) $Z_{x,y}$ is homeomorphic to a closed nondegenerate triangle;

(2) $Z_{x,y} \setminus \{x\} \subseteq U$;

(3) x and y are boundary points of $Z_{x,y}$.

So, fix $x \in L^*$ and $y \in U$ and pick a point y' such that the segment $[y, y']$ is contained in U. By the above-mentioned sharp version of the Schönflies theorem, we choose quasi-polygonal curves $P_{x,y}$ (joining x and y) and $P_{x,y'}$ (joining x and y') such that $P_{x,y} \setminus \{x\} \subseteq U$, $P_{x,y'} \setminus \{x\} \subseteq U$. First, by a simple modification, we can ensure that $P_{x,y} \cap [y, y'] = \{y\}$, $P_{x,y'} \cap [y, y'] = \{y'\}$. Next, using the fact that the segments of $P_{x,y}$ and of $P_{x,y'}$ converge to the point $\{x\}$, we modify $P_{x,y}$ and $P_{x,y'}$ (step by step) to ensure that $P_{x,y} \cap P_{x,y'} = \{x\}$. If it is done, the set $L_{x,y} = P_{x,y} \cup [y, y'] \cup P_{x,y'}$ forms a Jordan curve. We define $Z_{x,y}$ as the closure of the bounded component of $R^2 \setminus L_{x,y}$. Finally, we put $t = (x, y)$, $Z_t = Z_{x,y}$, $T = L^* \times U$. Then the family $\{Z_t : t \in T\}$ satisfies the assertion of the theorem. Indeed, each set Z_t ($t \in T$) is homeomorphic to a closed nondegenerate triangle (this is a strong version of the Jordan theorem for the plane, which does not hold for an Euclidean space of higher dimension). Since the set $\bigcup\{Z_t : t \in T\} \setminus U = L^*$ is Lebesgue nonmeasurable, the set $\bigcup\{Z_t : t \in T\}$ is Lebesgue nonmeasurable, too. $\square$

3. Uncountable Intersections of Thick Sets

Here we consider the following problem concerning uncountable intersections of measurable sets. Let $\mathcal{S}$ be a σ-algebra of subsets of a given nonempty set E and let $\mathcal{I}$ be a σ-ideal of subsets of E such that $\mathcal{I} \subseteq \mathcal{S}$. In such a case, the triple $(E, \mathcal{S}, \mathcal{I})$ is called a *measurable space with a σ-ideal.* Additionally, let the pair $(\mathcal{S}, \mathcal{I})$ fulfil the countable chain condition (in short, ccc), which means that each disjoint subfamily of $\mathcal{S} \setminus \mathcal{I}$ is countable. One can ask the question what should be assumed, for an uncountable family $\{X_t : t \in T\}$ of subsets of E, such that $(\forall t \in T)(E \setminus X_t \in \mathcal{I})$, to get the relation $E \setminus \bigcap\{X_t : t \in T\} \in \mathcal{I}$. The main difficulty is to have $\bigcap\{X_t : t \in T\} \in \mathcal{S}$. It seems natural to introduce the set

$$W = \{(t, x) \in T \times E : x \in X_t\},$$

and thus the sets X_t are equal to the vertical sections $W_t = \{x \in E : (t, x) \in W\}$ of W. So, our problem can now be formulated as follows. Let W be a set of some "good" structure in the product set $T \times E$, with thick sections W_t, $t \in T$, (i.e., $E \setminus W_t \in \mathcal{I}$). How is it possible to obtain the relation $E \setminus \bigcap\{W_t : t \in T\} \in \mathcal{I}$?

In connection with the general question posed above, let us mention especially the result of Goldstern [4] which deals with a particular (but important for various applications) case. Namely, let E be a perfect Polish space with a probability Borel measure μ. The completion of μ is denoted by $\bar{\mu}$. Then $\mathcal{S}$ and $\mathcal{I}$ stand for the σ-algebra of $\bar{\mu}$-measurable sets and the σ-ideal of $\bar{\mu}$-null sets (respectively). It is well known that $(\mathcal{S}, \mathcal{I})$ fulfils the ccc. The role of T is played by the zero-dimensional perfect Polish product space ω^ω where ω is equipped with the discrete topology. A natural order in this product space is defined by $t \leq t' \Leftrightarrow (\forall n \in \omega)(t(n) \leq t'(n))$ for any t and t' from ω^ω.

M. Goldstern proved the following result (see [4], Lemma 6).

Theorem 3.1. *Assume that $W \subseteq \omega^\omega \times E$ is a coanalytic set satisfying the conditions:*

(a) $(\forall t \in \omega^\omega)(\bar{\mu}(W_t) = 1)$*;*

(b) $(\forall t, t' \in \omega^\omega)(t \leq t' \Rightarrow W_t \supseteq W_{t'})$*.*

Then $\bar{\mu}(\bigcap\{W_t : t \in \omega^\omega\}) = 1$.

We are going to show that the category analogue of Goldstern's theorem is false. Now, we have that $E = [0,1]$, the σ-algebra $\mathcal{S}$ consists of subsets of E with the Baire property and the σ-ideal $\mathcal{I}$ consists of all meager subsets of E. It is well known that the pair $(\mathcal{S}, \mathcal{I})$ fulfils the ccc.

Theorem 3.2. *There is an open set $W \subseteq \omega^\omega \times [0,1]$ such that*

(a) $(\forall t \in \omega^\omega)(W_t$ *is comeager in* $[0,1])$,

(b) $(\forall t, t' \in \omega^\omega)(t \leq t' \Rightarrow W_t \supseteq W_{t'})$,

(c) *the set* $\bigcap\{W_t : t \in \omega^\omega\}$ *is countable.*

Proof. Let $\{p_j\}_{j \in \omega}$ be a one-to-one sequence of all rationals in $[0,1]$ and let $B(p, r)$ stand for the open ball in $[0,1]$ with centre p and radius r. Let the set $W \subseteq \omega^\omega \times [0,1]$ be given by

$$(t, x) \in W \Leftrightarrow x \in \bigcup_{j \in \omega} B(p_j, \sum_{k=j}^{\infty} \frac{1}{2^{t(k)+k}}).$$

Then W is open since it can be written as

$$W = \bigcup_{j \in \omega} \bigcup_{m=j+1}^{\infty} f_{j,m}^{-1}[(-\infty, 0)],$$

where a function $f_{j,m} : \omega^\omega \times [0,1] \to R$ given by the formula

$$f_{j,m}(t, x) = |x - p_j| - \sum_{k=j}^{m} \frac{1}{2^{t(k)+k}} \ ((t, x) \in \omega^\omega \times [0,1])$$

is continuous. Condition (a) clearly follows from the fact that the set

$$W_t = \bigcup_{j\in\omega} B(p_j, \sum_{k=j}^{\infty} \frac{1}{2^{t(k)+k}})$$

is open and dense, for each $t \in \omega^\omega$ (thus it is comeager). To check (b), consider any $t, t' \in \omega^\omega$ with $t \le t'$. Then

$$\sum_{k=j}^{\infty} \frac{1}{2^{t'(k)+k}} \le \sum_{k=j}^{\infty} \frac{1}{2^{t(k)+k}},$$

which implies that $W_{t'} \subseteq W_t$. Finally, to obtain (c), let us show that

$$\bigcap\{W_t : t \in \omega^\omega\} = \{p_j : j \in \omega\}.$$

Obviously, we have $p_j \in \bigcap\{W_t : t \in \omega^\omega\}$ for every natural number j. Consider any point $x \in [0,1] \setminus \{p_j : j \in \omega\}$. We shall find $t \in \omega^\omega$ such that $x \notin W_t$. First, pick $t(0) \in \omega$ such that $|x - p_0| > 1/2^{t(0)}$. Suppose now that natural numbers $t(0),\ t(1),\ \dots\ ,\ t(n)$ are chosen so that

$$(*)\ |x - p_j| > \sum_{k=j}^{n} \frac{1}{2^{t(k)+k}} \text{ for } j = 0, \dots, n.$$

Then pick $t(n+1) \in \omega$ so that

$$|x - p_j| > \sum_{k=j}^{n} \frac{1}{2^{t(k)+k}} + \frac{1}{2^{t(n+1)+n+1}} \text{ for } j = 0, \dots, n,$$

$$|x - p_{n+1}| > \frac{1}{2^{t(n+1)+n+1}}.$$

We have thus defined the sequence $t \in \omega^\omega$ satisfying $(*)$ for each $n \in \omega$. If $n \to \infty$ in $(*)$, we obtain

$$|x - p_j| \ge \sum_{k=j}^{\infty} \frac{1}{2^{t(k)+k}}$$

for each $j \in \omega$. Hence $x \notin W_t$. $\square$

4. Vitali Spaces

In this section, we consider some strong version of the measurability of uncountable unions of measurable sets. Let $(E, \mathcal{S}, \mathcal{I})$ be again a measurable space with a σ-ideal. (Recall that two triples connected, respectively, with measure and category are classical examples of a measurable space with a

σ-ideal.) We shall say that a family $\mathcal{F} \subseteq \mathcal{S}$ is *countably approximable* if there exists a countable subfamily $\mathcal{G}$ of $\mathcal{F}$ such that

$$\bigcup \mathcal{F} \setminus \bigcup \mathcal{G} \in \mathcal{I}.$$

Plainly, if a family is countably approximable, then its union is in $\mathcal{S}$ but, in general, the converse is not true.

In connection with the problem concerning the $\mathcal{S}$-measurability for unions of uncountable subfamilies of $\mathcal{S}$, it is natural to ask about useful criteria for the countable approximability of such subfamilies provided that the elements of a subfamily have some additional "good" structure. In particular, we are going to consider this question for a triple $(E, Br(E), K(E))$, where E is a topological space, and for a family of admissible subsets of E. As we know (see Introduction), every admissible set is open modulo $NWD(E)$ and, consequently, belongs to $Br(E)$.

We say that a Vitali covering $\mathcal{V}$ of a given subset X of E is *admissible* if $\mathcal{V}$ consists of admissible sets. A space E is called a *Vitali space* if, for each set $X \subseteq E$ and for each admissible Vitali covering $\mathcal{V}$ of X, there exists a disjoint countable family $\mathcal{W} \subseteq \mathcal{V}$ with $X \setminus \bigcup \mathcal{W}$ belonging to $K(E)$.

We say that E *almost satisfies the Suslin condition* if there exists a set $Z \in K(E)$ such that the subspace $E \setminus Z$ of E satisfies the Suslin condition. Clearly, if a topological space satisfies the Suslin condition, then it almost satisfies the Suslin condition but it is easy to find examples of spaces which disprove the converse assertion.

We are going to give a full characterization of topological spaces for which any family of admissible sets is countably approximable. For this purpose, we need several auxiliary facts.

We begin with the following consequence of the classical Banach theorem on the unions of open first category sets.

Lemma 4.1. *Every topological space E can be expressed in the form $E = E_0 \cup E_1$ where E_0 and E_1 are disjoint sets, E_0 is an open Baire subspace of E and E_1 is a closed first category subspace of E.*

Proof. (Cf. [8], Theorem 2.4.) According to the Banach Category Theorem (see [1], Theorem 16.1), we have the equality $E = E' \cup E''$ where E' is the largest (with respect to inclusion) open meager set in E, and $E'' = E \setminus E'$. Let us put $E_1 = E' \cup \mathrm{bd}(E')$, $E_0 = E \setminus E_1$. Since $\mathrm{bd}(E') \in NWD(E)$, we have $E_1 \in K(E)$ and, moreover, E_1 is closed in E. Hence E_0 is open in E and, by the maximality of E', we can infer that E_0 is a Baire space. $\square$

The set E_0 in the above-mentioned expression $E = E_0 \cup E_1$ will be called a *Baire kernel* of the space E. Obviously, the Baire kernel of E is unique modulo the σ-ideal $K(E)$.

Lemma 4.2. *For a topological space E, the following assertions are equivalent:*

(1) *E almost satisfies the Suslin condition;*

(2) *the Baire kernel E_0 of E satisfies the Suslin condition;*

(3) *for each disjoint family $\mathcal{F}$ of open sets in E, there exists a countable family $\mathcal{G} \subseteq \mathcal{F}$ such that $\bigcup \mathcal{F} \setminus \bigcup \mathcal{G} \in K(E)$.*

The proof of this lemma easily follows from Lemma 4.1 and will be omitted.

Lemma 4.3. *Let Z be a subset of a given topological space E. Suppose also that $\mathcal{V}$ is an admissible Vitali covering of Z. Then there exists a disjoint family $\mathcal{W} \subseteq \mathcal{V}$ such that $Z \setminus \bigcup \mathcal{W}$ is a nowhere dense set.*

Proof. (Cf. [9], Proposition 1.) We apply Zorn's lemma to choose $\mathcal{W}$ as a maximal (with respect to inclusion) disjoint subfamily of $\mathcal{V}$. To get the assertion, we suppose to the contrary that $\mathrm{int}(\mathrm{cl}(Z \setminus \bigcup \mathcal{W})) \neq \varnothing$. Hence we can pick a nonempty open set $U \subseteq \mathrm{cl}(Z \setminus \bigcup \mathcal{W})$. The latter relation implies that Z is dense in U and $U \cap \mathrm{int}(W) = \varnothing$ for each $W \in \mathcal{W}$. Since all sets from $\mathcal{W}$ are admissible, we also have $U \cap W = \varnothing$ for each $W \in \mathcal{W}$. Pick $x \in U \cap Z$. Since $\mathcal{V}$ is a Vitali covering of Z, there exists $V \in \mathcal{V}$ such that $x \in V \subseteq U$. Then $\mathcal{W} \cup \{V\}$ forms a disjoint subfamily of $\mathcal{V}$ which contradicts the maximality of $\mathcal{W}$. □

Now, we can formulate and prove

Theorem 4.1. *For a topological space E, the following three assertions are equivalent:*

(1) *E almost satisfies the Suslin condition;*

(2) *E is a Vitali space;*

(3) *each family $\mathcal{F}$ of admissible subsets of E is countably approximable.*

Proof. (1) $\Rightarrow$ (2). Suppose that (1) is true. Let $Z \subseteq E$ and let $\mathcal{V}$ be an admissible Vitali covering of Z. By Lemma 4.3, there exists a disjoint family $\mathcal{W} \subseteq \mathcal{V}$ such that $Z \setminus \bigcup \mathcal{W} \in NWD(E)$. From Lemma 4.2 it follows that there exists a countable family $\mathcal{W}^* \subseteq \mathcal{W}$ such that $\bigcup \{\mathrm{int}(X) : X \in \mathcal{W}\} \setminus \bigcup \{\mathrm{int}(X) : X \in \mathcal{W}^*\} \in K(E)$. Hence we easily infer that $Z \setminus \bigcup \mathcal{W}^* \in K(E)$. Consequently, E is a Vitali space.

(2) $\Rightarrow$ (3). Suppose that (2) is true. Let $\mathcal{F}$ be a family of admissible subsets of E. We put $Z = \bigcup \{\mathrm{int}(X) : X \in \mathcal{F}\}$. Then $\bigcup \mathcal{F} \setminus Z \in NWD(E)$ (compare with the proof of Theorem 2.1). Let

$$\mathcal{V} = \{V \ : \ V \text{ is open in } E \ \& \ (\exists X \in \mathcal{F})(V \subseteq \mathrm{int}(X))\}.$$

Clearly, $\mathcal{V}$ forms an admissible Vitali covering of Z. Since E is a Vitali space, there exists a countable disjoint family $\mathcal{W} \subseteq \mathcal{V}$ such that $Z \setminus \bigcup \mathcal{W} \in K(E)$. For each $W \in \mathcal{W}$, choose $X_W \in \mathcal{F}$ such that $W \subseteq \mathrm{int}(X_W)$ and put $\mathcal{G} = \{X_W : W \in \mathcal{W}\}$. Then $\mathcal{G}$ is a countable subfamily of $\mathcal{F}$ and $\bigcup \mathcal{F} \setminus \bigcup \mathcal{G} \in K(E)$. We have thus proved that the family $\mathcal{F}$ is countably approximable.

(3) $\Rightarrow$ (1). Suppose that (3) is true. Let $\mathcal{F}$ be an arbitrary disjoint family of open sets in E. Then $\mathcal{F}$ consists of admissible sets. By assumption (3), there exists a countable family $\mathcal{G} \subseteq \mathcal{F}$ such that $\bigcup \mathcal{F} \setminus \bigcup \mathcal{G} \in K(E)$. This, by Lemma 4.2, immediately yields assertion (1). $\square$

Finally, we wish to note that the last theorem can be generalized (under certain natural assumptions, of course) to the situation of a measurable space with a σ-ideal. This generalization can be obtained by using some abstract versions of the Banach Category Theorem. One of such versions is contained, for instance, in the monograph by Morgan [10].

References

1. J. C. Oxtoby, Measure and category. *Springer-Verlag, New York,* 1980.

2. J. Kupka and K. Prikry, The measurability of uncountable unions. *Amer. Math. Monthly* **91**(1984), No. 2, 85–97.

3. H. Hadwiger, Vorlesungen über Inhalt, Oberfläche und Isoperimetrie. *Springer-Verlag, Berlin,* 1957.

4. M. Goldstern, An application of Shoenfield's absoluteness theorem to the theory of uniform distribution. *Monatsh. Math.* **116**(1993), 237–243.

5. K. Kuratowski, Topology, volumes 1,2. *Academic Press, New York,* 1966, 1968.

6. S. Saks, Theory of the integral. *G.E. Stechert, New York,* 1937.

7. J. Dieudonné, Foundations of modern analysis. *Academic Press, New York,* 1960.

8. J. Cichoń, A. Kharazishvili, and B. Węglorz, Subsets of the real line, part 1. *Łódź University Press, Łódź,* 1995.

9. M. Orlando, Vitali covers and category. *Exposition. Math.* **11**(1993), 163–167.

10. J. C. Morgan II, Point set theory. *Marcel Dekker, Inc., New York and Basel,* 1990.

(Received 09.04.1997)

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