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ON SOME RINGS OF ARITHMETICAL FUNCTIONS

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ABSTRACT. In this paper we consider several constructions which from a given B-product $*_B$ lead to another one $\widetilde{*}_B$. We shall be interested in finding what algebraic properties of the ring $R_B = \langle C^{\mathbb{N}}, +, *_B \rangle$ are shared also by the ring $R_{\widetilde{B}} = \langle C^{\mathbb{N}}, +, \widetilde{*}_B \rangle$. In particular, for some constructions the rings R_B and $R_{\widetilde{B}}$ will be isomorphic and therefore have the same algebraic properties.

§ 1. INTRODUCTION

In [1] the author shows a new kind of convolution product called the B-product defined as follows. For every natural number n let B_n be the set of some pairs (r, s) of divisors of n.

For arithmetical functions f and g we define their B-product $f *_B$ as

$$(f *_B g)(n) = \sum_{(r,s)\in B_n} f(r)g(s) \quad \text{for} \quad n = 1, 2, 3, \dots . \tag{1}$$

This B-product generalizes simultaneously the A product of W. Narkiewicz [2] and the l.c.m. product and has a nonempty intersection with the ψ-product of D. H. Lehmer [3]. The τ-product of H. Scheid [4] is also a particular case of the B-product.

In [5] the author consideres a special kind of the B-product called the "multiplicative B-product". A B-product is multiplicative iff the following condition holds:

For every pair (m, n) of relatively prime natural numbers we have

$$(r, s) \in B_{mn} \quad \text{iff} \quad (r^{(m)}, s^{(m)}) \in B_m \quad \text{and} \quad (r^{(n)}, s^{(n)}) \in B_n, \tag{2}$$

where $k^{(n)}$ denotes the g.c.d. of k and n.

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In this paper we consider several constructions which, given a B-product $*_{B}$, lead to another one $\widetilde{*}_B$. We shall be interested in finding what algebraic properties of the ring $R_B = \langle C^{\mathbb{N}}, +, *_{B} \rangle$ are shared by the ring $R_{\widetilde{B}} = \langle C^{\mathbb{N}}, +, *_{B} \rangle$, where $C^{\mathbb{N}}$ denotes the set of all arithmetical functions. In particular, for some constructions the rings R_B and $R_{\widetilde{B}}$ will be isomorphic and therefore have the same algebraic properties.

§ 2. Twisted Products

Let R_B be a commutative and associative ring with a unit e. For a fixed invertible element $h \in R_B$, let $f^{(h)} = f *_{B} h$, where $f \in R_B$.

Evidently, $f \mapsto f^{(h)}$ is the one-to-one mapping of R_B onto itself preserving the set of invertible elements. It is also an isomorphism of the additive group of R_B.

We define the twisted product $*_{B}^{h}$ as follows:

$$f *_{B}^{h} g = f *_{B} g *_{B} h \quad \text{for} \quad f,\, g \in R_B.$$

In other words,

$$f^{(h)} *_{B} g^{(h)} = (f *_{B}^{h} g)(h).$$

This means that the rings $R_B^h = \langle C^{\mathbb{N}}, +, *_{B} \rangle$ are isomorphic. The isomorphism $R_B \to R_B^h$ is given by the twisting $f \mapsto f^{(h^{-1})}$, where h^{-1} is the inverse of h in R_B.

Therefore the ring R_B^h is also commutative and associative and $e^{(h^{-1})} = e *_{B} h^{-1} = h^{-1}$ is its unit element.

Let us remark that if the product $*_{B}$ is multiplicative and if the function h is multiplicative, then the product $*_{B}^{h}$ is multiplicative. In fact, if functions f and g are multiplicative, then the function $f *_{B}^{h} g = f *_{B} g *_{B} h$ is multiplicative, since $*_{B}$ preserves the multiplicativity.

In general, the twisted multiplication $*_{B}^{h}$ is not a B-product. We shall give below some conditions on $*_{B}$ and on h for $*_{B}^{h}$ to be a B-product.

Theorem 2.1. *The twisted product $*_{B}^{h}$ is a B-product iff for every r, s, and n*

$$\sum_{\substack{d_1,\, d_2 \\ (r,s)\in B_{d_1} \\ (d_1,d_2)\in B_n}} h(d_2) = 0 \ \ \textit{or} \ \ 1. \tag{3}$$

Moreover if we denote by B_n^h the set corresponding to the B-product $*_{B}^{h}$, then $(r,s) \in B_n^h$ iff sum (3) *is equal to* 1.

Proof. $\Longrightarrow$ We have

$$(e_r *_B e_s)(n) = \begin{cases} 1 & \text{if} \quad (r,s) \in B_n, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, since by the assumption $*_B^h$ is a B-product, we have

$$(e_r *_B^h e_s)(n) = \begin{cases} 1 & \text{if} \quad (r,s) \in B_n^n, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand,

$$(e_r *_B^h e_s)(n) = (e_r *_B e_s *_B h)(n) = \sum_{\substack{d_1,\, d_2 \\ (d_1,d_2)\in B_n}} (e_r *_B e_S)(d_1) h(d_2) =$$

$$= \sum_{\substack{d_1,\, d_2 \\ (d_1,d_2)\in B_n \\ (r,s)\in B_{d_1}}} h(d_2). \tag{4}$$

Hence the result follows.

$\Longleftarrow$ Define B_n^h as the set of pairs (r,s) such that sum (3) is equal to 1. In view of (4), for any functions f and g, we have

$$(f *_B^h g)(n) = \Big[\Big(\sum_{r=1}^n f(r)e^r\Big) *_B^h *_B^h \Big(\sum_{s=1}^n g(s)e_s\Big)\Big](n) =$$

$$= \sum_{r,s=1}^n f(r)g(s)(e_r *_B^h e_s)(n) = \sum_{(r,s)\in B_n^h} f(r)g(s) = (f *_{B^h} g)(n).$$

If we start from the Dirichlet convolution $*$, then condition (3) of Theorem 2.1 takes the form

$$\sum_{\substack{d_1,\, d_2 \\ r_s = d_1 \\ d_1 d_2 = n}} h(d_2) = h\Big(\frac{n}{r_s}\Big) = 0 \text{ or } 1$$

for every r, s, and n.

Thus $h(m) = 0$ or 1 for every m. Moreover, since h is invertible, we conclude that $h(1) \neq 0$, i.e., $h(1) = 1$. $\square$

Corollary 2.2. *Let $h(1) = 1$ and $h(n) = 0$ or 1 for every n. Then the multiplication $*^h$ defined by*

$$f *^h g = f * g * h,$$

where $*$ is the Dirichlet convolution, is a B-product and

$$B_n^h = \Big\{(r,s):\ \ rs|n\ \ and\ \ h\Big(\frac{n}{rs}\Big) = 1\Big\}.$$

If, moreover, the function h is multiplicative, then the twisted product $*^h$ preserves the multiplicativity.

Rings $R = \langle C^{\mathbb{N}}, +, *\rangle$ and $R^h = \langle C^{\mathbb{N}}, +, *^h\rangle$ are isomorphic. The isomorphism is given by $f \mapsto f * h^{-1}$, where h^{-1} is the Dirichlet inverse of h. The function h^{-1} is the unit of the ring R^h.

Consequently R^h is a local ring without zero divisors.

A function f is invertible in R iff it is invertible in R^h iff $f(1) \neq 0$, and the inverse $f^{(-1)}$ of f in R^h is given by

$$f^{(-1)} = f^{-1} * h^{-2}.$$

§ 3. Strong Associativity

We introduce the important notion of a strong associativity. We say that a B-product is strongly associative iff for fixed d_1, d_2, d_3, n, the fulfilment of

$$(r, d_1) \in B_n \quad \text{and} \quad (d_2, d_3) \in B_r \tag{5}$$

for some r implies that $w = \frac{d_1 r}{d_2}$ (which is evidently a natural number) satisfies the condition

$$(d_2, w) \in B_n \quad \text{and} \quad (d_3, d_1) \in B_w \tag{6}$$

and, conversely, the fulfilment of (6) for w implies that $r = \frac{d_2 w}{d_1}$ (which is a natural number) satisfies (5).

From this definition and Theorem 2.1 of [1] it follows that every strongly associative B-product is associative. The converse does not hold in generaL. Nevertheless the following theorem is true.

Theorem 3.1. *An associative τ-product is strongly associative iff*

$$\begin{gathered} d_2\tau(d_3, d_1) = \tau(d_2, d_3)d_1 \quad \textit{for all} \quad d_1,\ d_2,\ d_3 \quad \textit{satisfying} \\ \tau\big(\tau(d_2, d_3), d_1\big) \neq 0. \end{gathered} \tag{7}$$

Proof. $\Longleftarrow$ Let $r = \tau(d_2, d_3)$ and $n = \tau(r, d_1) = \tau(\tau(d_2, d_3), d_1) \neq 0$. Then (5) holds and by the strong associativity we have (6), i.e., $w = \tau(d_3, d_1)$, where $w = \frac{d_1 r}{d_2}$. Therefore we get (7).

$\Longrightarrow$ Suppose that (5) holds for some d_1, d_2, d_3, n. Then $r = \tau(d_2, d_3)$ and $n = \tau(r, d_1) = \tau(\tau(d_2, d_3), d_1)$. Since $n \neq 0$, we have (7) by the assumption. Hence

$$w = \frac{d_1 r}{d_2} = \tau(d_3, d_1), \quad \text{i.e.,} \quad (d_3, d_1) \in B_w.$$

Therefore

$$\tau(d_2, w) = \tau(d_2, \tau(d_3, d_1)) = \tau(\tau(d_2, d_3), d_1) = n,$$

i.e., $(d_2, w) \in B_n$. Thus (6) holds.

Similarly, we can prove that (6) implies (5). $\square$

§ 4. Unitary B-products

For a given B-product $*_B$ we define the corresponding unitary B-product denoted by $\circ_B$ as follows. Let

$$B_n^0 = \left\{(r,s): \ \ (r,s) \in B_n, \ \ (r,s) = 1, \ \ \left(rs, \frac{n}{rs}\right) = 1\right\}.$$

Then $(f \circ_B g)(n) = \sum\limits_{(r,s) \in B_n^0} f(r)g(s)$.

We shall investigate relations between the corresponding properties of B-products $*_B$ and $\circ_B$.

Theorem 4.1.

(i) *If* $*_B$ *is commutative, then* $\circ_B$ *is commutative.*

(ii) *If* $*_B$ *is strongly associative, then* $\circ_B$ *is associative.*

Proof. (i) is clear. To prove (ii) we have to show that for every d_1, d_2, d_3, and n

$$\sum_{\substack{r \\ (r,d_1) \in B_n^0 \\ (d_2,d_3) \in B_r^0}}{}' = \sum_{\substack{w \\ (d_2,w) \in B_n^0 \\ (d_3,d_1) \in B_w^0}}{}' \ . \tag{8}$$

Suppose that r satisfies $(r, d_1) \in B_n^0$, $(d_1, d_2) \in B_r^0$, i.e., $(r, d_1) \in B_n$, $(d_2, d_3) \in B_r$ and, moreover,

$$(r, d_1) = 1, \quad \left(rd_1, \frac{n}{rd_1}\right) = 1, \quad (d_2, d_3) = 1, \quad \left(d_2 d_3, \frac{r}{d_2 d_3}\right) = 1. \tag{$*$}$$

By the strong associativity of $*_B$ we find for $w = \frac{rd_1}{d_2}$ that $(d_2, w) \in B_n$, $(d_3, d_1) \in B_w$.

To prove that w satisfies

$$(d_2, w) \in B_n^0, \quad (d_3, d_1) \in B_w^0,$$

it is sufficient to show that

$$\begin{aligned} (d_2, w) = 1, \quad \left(d_2 w, \frac{n}{d_2 w}\right) = 1, \\ (d_3, d_1) = 1, \quad \left(d_3 d_1, \frac{w}{d_3 d_1}\right) = 1 \end{aligned} \tag{$**$}$$

in view of (6). The formulas $(**)$ follow from $(*)$. We have proved that to every summand of L, H, S of (8) there corresponds a summand in R, H, S. Similarly, one can give the inverse correspondence. $\square$

Theorem 4.2. *If e_1 is the unit in the ring R_B, then e_1 is also the unit for the corresponding unitary product $\circ_B$.*

Proof. In view of Corollary 2.6 of [1] e_1 is the unit in the ring R_B iff $(1,n)$, $(n,1) \in B_n$ for $n \geq 1$ and $(k,1) \notin B_n$, $(1,k) \notin B_n$ for $k \neq n$, $n > 1$.

These conditions clearly imply that $(1,n)$, $(n,1) \in B_n^0$ and $(k,1) \notin B_n^0$ and $(1,k) \notin B_n^0$ for $k \neq n$, $n > 1$.

Therefore, using once more Corollary 2.6 of [1], we deduce that e_1 is the unit with respect to $\circ_B$. $\square$

Theorem 4.3. *If the B-product $*_B$ is multiplicative, then the corresponding unitary product $*_B$ is multiplicative.*

Proof. Let m and n be coprime natural numbers and suppose that

$$(r,s) \in B_{mn}^0. \tag{A}$$

Then $(r,s) \in B_{mn}^0$ and hence by the multiplicativity of $*_B$ we get $(r^{(m)}, s^{(m)}) \in B_m$ and, similarly, $(r^{(n)}, s^{(n)}) \in B_n$. Moreover, from $(r,s)=1$ and $(rs, \frac{mn}{rs})=1$ we conclude that

$$(r^{(m)}, s^{(m)}) = 1, \quad (r^{(n)}, s^{(n)}) = 1 \quad \text{and futher}$$
$$\left(r^{(m)}s^{(m)}, \frac{m}{r^{(m)}s^{(m)}}\right) = 1, \quad \left(r^{(n)}s^{(n)}, \frac{n}{r^{(n)}s^{(n)}}\right) = 1.$$

Thus

$$(r^{(m)}, s^{(m)}) \in B_m^0 \quad \text{and} \quad (r^{(n)}, s^{(n)}) \in B_n^0. \tag{B}$$

Similarly, one can prove that (B) implies (A). $\square$

§ 5. The Narkiewicz Product

For a given B-product $*_B$ we define a new B-product Δ_B as follows: Let

$$B_n^{\Delta} = \{(r,s) : \ (r,s) \in B_n, \ rs = n\}.$$

Then

$$(f\Delta_B g)(n) = \sum_{(r,s)\in B_n^{\Delta}} f(r)f(s) = \sum_{(r,\frac{n}{r})\in B_n} f(r)g\Big(\frac{n}{r}\Big).$$

Evidently, Δ_B is the Narkiewicz product. It will be called the Narkiewicz product corresponding to the B-product $*_B$. We shall investigate some important properties of the product Δ_B.

Theorem 5.1. *If the product* $*_{B}$ *is commutative, then* Δ_{B} *is also commutative.*

Proof is clear. □

Theorem 5.2. *If the* B*-product* $*_{B}$ *is strongly associative, then the corresponding Narkiewicz product* Δ_{B} *is associative.*

Proof. For fixed d_1, d_2, d_3, n we have

$$\sum_{\substack{r \\ (r,d_1)\in B_n^{\Delta} \\ (d_2,d_3)\in B_r^{\Delta}}}' 1 = \sum_{\substack{r \\ (r,d_1)\in B_n,\ rd_1=n \\ (d_2,d_3)\in B_r,\ d_2d_3=r}}' 1.$$

If $n \neq d_1d_2d_3$, then this sum is equal to 0. We assume that $n = d_1d_2d_3$. Then the sum is equal to

$$\sum_{\substack{(\frac{n}{d_1},d_1)\in B_n \\ (d_2,d_3)\in B_{\frac{n}{d_1}}}} 1$$

for $r = \frac{n}{d_1}$.

By the strong associativity of $*_{B}$ for $w = \frac{rd_1}{D_2}$ the above sum is equal to

$$\sum_{\substack{w \\ (d_2,w)\in B_n \\ (d_3,d_1)\in B_w}} 1 = \sum_{\substack{(d_2,\frac{n}{d_2})\in B_n \\ (d_3,d_1)\in B_{\frac{n}{d_2}}}} 1 = \sum_{\substack{(d_2\overline{w})\in B_n^{\Delta} \\ (d_3,d_1)\in B_{\overline{w}}^{\Delta}}} 1.$$

Therefore the product Δ_{B} is associative. □

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