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ON VITALI'S THEOREM FOR GROUPS OF MOTIONS OF EUCLIDEAN SPACE

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ABSTRACT. We give a characterization of all those groups of isometric transformations of a finite-dimensional Euclidean space, for which an analogue of the classical Vitali theorem [1] holds true. This characterization is formulated in purely geometrical terms.

A well-known result due to Vitali [1] states that there are subsets of the real line R, nonmeasurable in the Lebesgue sense. Moreover, the argument of Vitali yields simultaneously that if X is an arbitrary Lebesgue measurable subset of R with strictly positive measure, then there exists a subset of X nonmeasurable in the Lebesgue sense. This classical result was generalized in many directions (see, e.g., [2]–[12]). In the present paper, we consider some questions relevant to the above-mentioned result of Vitali, for various groups of motions of finite-dimensional Euclidean spaces. The main goal of the paper is to describe those groups of motions for which an analogue of Vitali's result remains true (we recall that, in Vitali's theorem, a basic group of transformations is the group of all translations of the real line).

Let E denote a finite-dimensional Euclidean space and let G be a subgroup of the group of all isometric transformations (i.e., motions) of E. Then the pair (E, G) can be regarded as a space equipped with a transformation group. We denote by K the open unit cube in E. Let μ be a measure given on E and let $\mathrm{dom}(\mu)$ denote the domain of μ. We shall say that μ is a G-measure on E if the following two conditions are fulfilled:

(1) $K \in \mathrm{dom}(\mu)$, $\mu(K) = 1$;

(2) μ is invariant with respect to G, i.e., $\mathrm{dom}(\mu)$ is a G-invariant σ-ring of subsets of E and, for each set $X \in \mathrm{dom}(\mu)$ and for each transformation $g \in G$, we have $\mu(g(X)) = \mu(X)$.

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For example, the standard Lebesgue measure on E (considered as a G-invariant measure) is a G-measure on E.

We say that a group G has the Vitali property if, for every G-measure μ on E and for each μ-measurable set $X \subset E$ with $\mu(X) > 0$, there exists at least one μ-nonmeasurable subset of X.

We say that a group G has the weak Vitali property if, for every G-measure μ on E, there exists at least one μ-nonmeasurable subset of E.

We are going to establish some necessary and sufficient conditions (formulated in terms of the pair (E, G)) under which a given group G has the Vitali property (the weak Vitali property, respectively).

First we wish to recall some auxiliary notions and facts.

Let F be a nonempty basic set and let Γ be a group of transformations of F. We recall that Γ acts freely on F if, for each point $x \in F$ and for any two distinct transformations $g \in \Gamma$ and $h \in \Gamma$, the relation $g(x) \neq h(x)$ is fulfilled.

In particular, the group of all translations of Euclidean space E acts freely on E.

We denote by ω the least infinite ordinal (cardinal) number and by $\mathbf{c}$ the cardinality continuum.

For any set X, the symbol $\mathrm{card}(X)$ denotes the cardinality of X.

Id_E is the identity transformation of a space E.

λ is the standard Lebesgue measure on a space E (note that the dimension of λ coincides with the dimension of E).

If L is an affine linear manifold in a space E, then $\dim(L)$ denotes the dimension of L.

If μ is an arbitrary measure on a space E, then μ^* and μ_* denote, respectively, the outer measure and the inner measure associated with μ.

Now, let us formulate several statements which we need in our further considerations.

Lemma 1. *Let E be an n-dimensional Euclidean space, G be a group of affine transformations of E with* $\mathrm{card}(G) < \mathbf{c}$, *and let Y be a subset of E such that*

(1) $\mathrm{card}(Y) = \mathbf{c}$;

(2) *any $n+1$ pairwise distinct points of Y do not belong to an affine hyperplane of E (in other words, Y is a set of points in general position).*

Then there exists a point $y \in Y$ for which the group G acts freely on the G-orbit $G(y)$.

The proof of this lemma is not difficult. Actually, it follows from the fact that every affine transformation of E is completely determined by its restriction to a subset of Y with cardinality $n+1$.

Lemma 1 implies a number of useful consequences. For example, one of such consequences is the next well-known statement.

Lemma 2. *Let G be a group of isometric transformations of a finite-dimensional Euclidean space E. Then the following two conditions are equivalent:*

(1) *G is discrete;*

(2) *for each point $e \in K$, the set $G(e)$ is locally finite in E.*

Lemma 3. *Let E be a finite-dimensional Euclidean space, G be a group of motions of E and K denote the open unit cube in E. Then the following three assertions are equivalent:*

(1) $\cup\{g(K) \ : \ g \in G\} = E$;

(2) *for every G-measure μ on E and for each bounded μ-measurable set $X \subset E$, we have $\mu(X) < +\infty$;*

(3) *for every G-measure μ on E and for each μ-measurable set $X \subset E$ with $\mu(X) > 0$, there exists a μ-measurable set $Y \subset X$ such that $0 < \mu(Y) < +\infty$.*

Proof. Suppose that assertion (1) is true. Let μ be an arbitrary G-measure on E and let X be an arbitrary bounded μ-measurable subset of E. Let $\mathrm{cl}(X)$ denote the closure of X. Obviously, the set $\mathrm{cl}(X)$ is compact and the family $\{g(K) \ : \ g \in G\}$ forms an open covering of $\mathrm{cl}(X)$. Consequently, there exists a finite subset H of G such that

$$X \subset \mathrm{cl}(X) \subset \cup\{h(K) \ : \ h \in H\}.$$

Hence we have

$$\mu(X) \leq \mu(\cup\{h(K) \ : \ h \in H\}) \leq \mathrm{card}(H) < +\infty$$

and thus assertion (2) is fulfilled.

Suppose again that assertion (1) is true and let X be an arbitrary μ-measurable subset of E with $\mu(X) > 0$. Clearly, we can write

$$E = \cup\{h(K) \ : \ h \in H\},$$

for some countable subset H of G. The latter equality yields

$$0 < \mu(X) = \mu(\cup\{h(K) \cap X \ : \ h \in H\}) \leq \sum_{h \in H} \mu(h(K) \cap X).$$

Consequently there exists an element $h \in H$ such that $\mu(h(K) \cap X) > 0$. Let us put $Y = h(K) \cap X$. Then we get

$$0 < \mu(Y) \leq \mu(h(K)) = \mu(K) = 1$$

and hence assertion (3) is fulfilled. In this way we have obtained the implications (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3).

Finally, suppose that assertion (1) is not true. Then there is a point $z \in E$ satisfying the relation

$$G(z) \cap (\cup\{g(K) \ : \ g \in G\}) = \varnothing.$$

Let us put $Z = \cup\{g(K) \ : \ g \in G\}$ and consider the family of sets

$$S = \{Z' \cup Z'' \ : \ Z' \subset Z,\ Z' \in \mathrm{dom}(\lambda),\ Z'' \subset G(z)\}.$$

It can easily be checked that S is a G-invariant σ-ring of subsets of E and $K \in S$. Now we define a functional ν on S in the following manner: $\nu(Z' \cup Z'') = \lambda(Z')$ if $Z'' = \varnothing$, and $\nu(Z' \cup Z'') = +\infty$ if $Z'' \neq \varnothing$. Then it is not difficult to see that this functional is a G-measure on E. Moreover, the one-element ν-measurable set $\{z\}$ with $\nu(\{z\}) = +\infty$ shows directly that the assertions (2) and (3) are false for ν. Thus we have established the implications (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1). $\square$

Lemma 4. *Let G be a group of motions of a finite-dimensional Euclidean space. Then the following two assertions are equivalent:*

(1) *for any point $e \in K$, the G-orbit $G(e)$ is not a locally finite subset of E;*

(2) *for every G-measure μ on E and for each point $e \in K$, we have*

$$\{e\} \in \mathrm{dom}(\mu) \Rightarrow \mu(\{e\}) = 0.$$

Proof. (1) $\Rightarrow$ (2). Suppose that assertion (1) is true. Let e be an arbitrary point of K and let μ be a G-measure on E such that $\{e\} \in \mathrm{dom}(\mu)$. It is not difficult to deduce from assertion (1) that

$$\mathrm{card}(G(e) \cap K) \geq \omega.$$

If $\mu(\{e\}) > 0$, then (taking into account the G-invariance of μ) we immediately obtain that $\mu(K) = +\infty$, which is impossible. Consequently $\mu(\{e\}) = 0$ and the implication (1) $\Rightarrow$ (2) has been established.

(2) $\Rightarrow$ (1). Suppose that assertion (2) is true. Let $e \in K$. We must show that the orbit $G(e)$ is not locally finite in E. Assuming to the contrary that $G(e)$ is a locally finite subset of E, it is easy to define a G-measure ν on E such that

$$\{e\} \in \mathrm{dom}(\nu),\ \nu(\{e\}) > 0.$$

Indeed, we have the inequalities

$$0 < \mathrm{card}(G(e) \cap K) < \omega.$$

Let us denote $p = \mathrm{card}(G(e) \cap K)$. Further, for each subset X of E, let us put $\nu(X) = (1/p)\,\mathrm{card}(X \cap G(e))$ if $\mathrm{card}(X \cap G(e)) < \omega$ and $\nu(X) = +\infty$ if $\mathrm{card}(X \cap G(e)) = \omega$. Then ν is the required G-measure (defined on the family of all subsets of E). Thus we have established the implication (2) $\Rightarrow$ (1) and the proof of Lemma 4 is complete. $\square$

Lemma 5. *Let E be a finite-dimensional Euclidean space, e be a point of E and let L be an affine linear manifold in E containing e. Let $\{h_m : m < \omega\}$ be a countable family of motions of E, satisfying the relations:*

(1) *the family of points $\{h_m(e) : m < \omega\}$ is injective and*

$$\lim_{m \to +\infty} h_m(e) = e;$$

(2) *the family of manifolds $\{h_m(L) : m < \omega\}$ is finite.*

Denote by H the group of motions, generated by $\{h_m : m < \omega\}$. Then there exists a countable family $\{g_n : n < \omega\} \subset H$ such that:

(a) *$g_n(L) = L$ for all $n < \omega$;*

(b) *$\{g_n : n < \omega\}$ is convergent to some motion of E;*

(c) *the family $\{g_n|L : n < \omega\}$ is injective, where the symbol $g_n|L$ denotes, for each $n < \omega$, the restriction of g_n to L.*

We omit an easy purely geometrical proof of Lemma 5.

Now we can formulate and prove

Theorem 1. *Let G be a group of motions of a finite-dimensional Euclidean space E. Then G has the Vitali property if and only if the following two conditions are fulfilled:*

(1) *for any point $e \in K$, the G-orbit $G(e)$ is not locally finite in E;*

(2) *$\cup\{g(K) : g \in G\} = E$.*

Proof. Suppose that assertions (1) and (2) are satisfied and let us show that G has the Vitali property. Let μ be an arbitrary G-measure on E and let X be an arbitrary μ-measurable subset of E with $\mu(X) > 0$. We must find a μ-nonmeasurable subset of X. Taking into account Lemma 3, we may assume that X is a subset of K. Denote by $r = r(X)$ the least natural number such that there exists an affine linear manifold $L \subset E$ for which $\dim(L) = r$ and $\mu_*(L \cap X) > 0$. According to Lemma 4 we can assert that

$$r = \dim(L) > 0.$$

Consider the set $T = L \cap X$. If this set is nonmeasurable with respect to μ, then there is nothing to prove. So we may assume that $T \in \mathrm{dom}(\mu)$ and hence $\mu(T) > 0$. Fix a point $t \in T$. Applying condition (1) to t, we see that the orbit $G(t)$ is not locally finite in E. It can easily be deduced from this fact that there exists a countable family $\{h_m : m < \omega\} \subset G$ such that the corresponding family of points $\{h_m(t) : m < \omega\}$ is injective and

$$\lim_{m \to +\infty} h_m(t) = t.$$

We may also assume, without loss of generality, that $\{h_m : m < \omega\}$ converges to some motion of E.

Now, consider the family of affine linear manifolds

$$\{h_m(L) \ : \ m < \omega\}.$$

Only two cases are possible.

1. This family is infinite. In this case we may suppose without loss of generality that

$$(\forall m < \omega)(\forall n < \omega)(m \neq n \Rightarrow h_m(L) \neq h_n(L)).$$

In particular, if $m \neq n$, we have

$$\dim(h_m(L) \cap h_n(L)) < \dim(L).$$

Recalling the definition of the natural number r, we easily deduce that

$$\mu(h_m(T) \cap h_n(T)) = 0 \ (m < \omega, \ n < \omega, \ m \neq n).$$

It immediately follows from the latter fact that

$$\mu(\cup\{h_m(T) \ : \ m < \omega\}) = +\infty$$

because $\mu(T) > 0$ and the countable family $\{h_m(T) \ : \ m < \omega\}$ is almost disjoint with respect to μ. On the other hand, since the sequence $\{h_m \ : \ m < \omega\}$ converges to some motion of E and the set T is bounded in E, the μ-measurable set

$$\cup\{h_m(T) \ : \ m < \omega\}$$

is bounded in E, too, and according to Lemma 3 the inequality

$$\mu(\cup\{h_m(T) \ : \ m < \omega\}) < +\infty$$

must be true. Thus we have obtained a contradiction which shows us that case 1 is impossible.

2. The family $\{h_m(L) \ : \ m < \omega\}$ is finite. In this case, applying Lemma 5, we see that there exists a countable family $\{g_n \ : \ n < \omega\} \subset G$ satisfying the following relations:

(a) $g_n(L) = L$ for all $n < \omega$;

(b) $\{g_n \ : \ n < \omega\}$ converges to some motion of E;

(c) the family $\{g_n|L \ : \ n < \omega\}$ is injective.

Further, denote by G' the subgroup of G generated by $\{g_n \ : \ n < \omega\}$. Obviously,

$$(\forall g \in G')(g(L) = L).$$

Let $\{Z_i \ : \ i \in I\}$ denote the family of all those G'-orbits in L which have a nonempty intersection with the set T. Let Y be a selector of the family of nonempty sets

$$\{Z_i \cap T \ : \ i \in I\}.$$

We are going to show that Y is nonmeasurable with respect to μ. Suppose otherwise, i.e., that $Y \in \mathrm{dom}(\mu)$. Then, taking into account the relations

$$\mu(T) > 0,\ T \subset \cup\{g(Y)\ :\ g \in G'\}$$

and the countability of the group G', we obtain that $\mu(Y) > 0$. It is not difficult to check that, for any two distinct natural numbers n and m, the set $g_n(Y) \cap g_m(Y)$ lies in an affine linear manifold whose dimension is strictly less than $r = \dim(L)$. According to the definition of r we have

$$\mu(g_n(Y) \cap g_m(Y)) = 0\ (n < \omega,\ m < \omega,\ n \neq m).$$

Finally, let us consider the set

$$\cup\{g_n(Y)\ :\ n < \omega\}.$$

This set is μ-measurable and bounded in E. Applying Lemma 3, we get

$$\mu(\cup\{g_n(Y)\ :\ n < \omega\}) < +\infty.$$

On the other hand, as mentioned above, the family $\{g_n(Y)\ :\ n < \omega\}$ is almost disjoint with respect to μ and $\mu(Y) > 0$. Consequently,

$$\mu(\cup\{g_n(Y)\ :\ n < \omega\}) = +\infty.$$

Thus we have obtained a contradiction which shows us that the Vitali property follows from the conjunction of conditions (1) and (2).

It remains to establish that if at least one of conditions (1) and (2) is not fulfilled, then our group G does not have the Vitali property.

Suppose first that condition (1) does not hold for G. Then there exists a point e of K for which the orbit $G(e)$ is a locally finite subset of E. Hence $0 < \mathrm{card}(K \cap G(e)) < \omega$. Let $p = \mathrm{card}(K \cap G(e))$. For each set $X \subset E$, we put $\nu(X) = (1/p)\,\mathrm{card}(X \cap G(e))$ if $\mathrm{card}(X \cap G(e)) < \omega$, and $\nu(X) = +\infty$ if $\mathrm{card}(X \cap G(e)) = \omega$. Then it is easy to see that the functional ν is a G-measure defined on the family of all subsets of E. Consequently G does not possess the Vitali property.

Suppose now that $\cup\{g(K)\ :\ g \in G\} \neq E$ and let z be an arbitrary point from the set $E \setminus \cup\{g(K)\ :\ g \in G\}$. Then

$$G(z) \cap (\cup\{g(K)\ :\ g \in G\}) = \varnothing.$$

As in the proof of Lemma 3, we denote $Z = \cup\{g(K)\ :\ g \in G\}$ and

$$S = \{Z' \cup Z''\ :\ Z' \subset Z,\ Z' \in \mathrm{dom}(\lambda),\ Z'' \subset G(z)\}.$$

Further, we define a functional ν on S in the same manner as in the proof of Lemma 3. Namely, we put $\nu(Z' \cup Z'') = \lambda(Z')$ if $Z'' = \varnothing$ and $\nu(Z' \cup Z'') = +\infty$ if $Z'' \neq \varnothing$. Then ν is a G-measure on E such that the set $\{z\}$ is ν-measurable, $\nu(\{z\}) > 0$ and $\{z\}$ does not contain ν-nonmeasurable subsets. Consequently the group G does not possess the Vitali property. $\square$

Slightly changing an argument presented above, we can also give a characterization of those groups of motions of a finite-dimensional Euclidean space, which have the weak Vitali property.

Theorem 2. *Let G be a subgroup of the group of all isometric transformations of a finite-dimensional Euclidean space E. Then the following two conditions are equivalent:*

(1) *for each point $e \in K$, the orbit $G(e)$ is not locally finite in E;*

(2) *G possesses the weak Vitali property.*

Proof. It suffices to establish only the implication (1) $\Rightarrow$ (2). Suppose that condition (1) is fulfilled and let μ be an arbitrary G-measure on E. We are going to show that the cube K contains a subset nonmeasurable with respect to μ. Denote by $r = r(\mu)$ the least natural number such that there exists an affine linear manifold L in E for which we have $\dim(L) = r$ and $\mu_*(L \cap K) > 0$. Let us put $T = L \cap K$. If T is nonmeasurable with respect to μ, then there is nothing to prove. So we can suppose that $T \in \mathrm{dom}(\mu)$ and, hence, $\mu(T) > 0$. Fix a point $t \in T$. According to (1), the orbit $G(t)$ is not locally finite in E. It follows from this fact that there exists a countable family $\{h_m \ : \ m < \omega\}$ of elements of G, satisfying the relations:

(a) the family of points $\{h_m(t) \ : \ m < \omega\}$ is injective;

(b) $\lim_{m \to +\infty} h_m(t) = t$;

(c) $\{h_m \ : \ m < \omega\}$ converges to some motion of E.

We may assume, without loss of generality, that $\{h_m \ : \ m < \omega\}$ converges to Id_E. Then the family $\{h_m^{-1} \ : \ m < \omega\}$ converges to Id_E, too. Since K is an open subset of E, we can write

$$K \subset \cup\{\cap\{h_m^{-1}(K) \ : \ n < m < \omega\} \ : \ n < \omega\}.$$

Consequently there exists a natural number m_0 such that

$$\mu(T \cap (\cap\{h_m^{-1}(K) \ : \ m_0 < m < \omega\})) > 0.$$

Let us denote

$$Z = T \cap (\cap\{h_m^{-1}(K) \ : \ m_0 < m < \omega\}).$$

Then, for each integer $m > m_0$, we have $h_m(Z) \subset K$ and, therefore,

$$\cup\{h_m(Z) \ : \ m_0 < m < \omega\} \subset K, \ \mu(\cup\{h_m(Z) \ : \ m_0 < m < \omega\}) \leq 1.$$

Consider now the family of affine linear manifolds $\{h_m(L) \ : \ m_0 < m < \omega\}$ and suppose, for a while, that this family is infinite. Then we may assume that

$$h_m(L) \neq h_n(L) \ (m_0 < m < \omega, \ m_0 < n < \omega, \ m \neq n).$$

Recalling the definition of $r = \dim(L)$, we obtain

$$\mu(h_m(Z) \cap h_n(Z)) = 0$$

for any two distinct integers $m > m_0$ and $n > m_0$ (because the set $h_m(Z) \cap h_n(Z)$ lies in an affine linear manifold whose dimension is strictly less than r). Hence we get the equality

$$\mu(\cup\{h_m(Z) \ : \ m_0 < m < \omega\}) = +\infty,$$

which yields a contradiction. Thus the family $\{h_m(L) \ : \ m_0 < m < \omega\}$ is finite.

Further, applying Lemma 5, we can find a countable family $\{g_n \ : \ n < \omega\} \subset G$ satisfying the relations:

(a) $g_n(L) = L$ for all $n < \omega$;

(b) $\{g_n \ : \ n < \omega\}$ converges to some motion of E;

(c) the family $\{g_n|L \ : \ n < \omega\}$ is injective.

We may also assume, without loss of generality, that $\{g_n \ : \ n < \omega\}$ converges to Id_E. Now, since T is an open subset of L and $\{g_n^{-1} \ : \ n < \omega\}$ converges to Id_E, we can write

$$T \subset \cup\{\cap\{g_n^{-1}(T) \ : \ m < n < \omega\} \ : \ m < \omega\}.$$

Consequently there exists a natural number n_0 such that

$$\mu((\cap\{g_n^{-1}(T) \ : \ n_0 < n < \omega\}) \cap T) > 0.$$

Let us put

$$T' = (\cap\{g_n^{-1}(T) \ : \ n_0 < n < \omega\}) \cap T$$

and let G' denote the subgroup of G generated by the family $\{g_n \ : \ n_0 < n < \omega\}$. Obviously,

$$(\forall g \in G')(g(L) = L).$$

Further, let $\{Z_i \ : \ i \in I\}$ be the family of all those G'-orbits in L which have a nonempty intersection with T'. Let Y be a selector of the family $\{Z_i \cap T' \ : \ i \in I\}$. We assert that Y does not belong to $\mathrm{dom}(\mu)$. Suppose otherwise: $Y \in \mathrm{dom}(\mu)$. Then, taking into account the inclusion

$$T' \subset \cup\{g(Y) \ : \ g \in G'\}$$

and the inequality $\mu(T') > 0$, we get $\mu(Y) > 0$. On the other hand, for any two distinct natural numbers n and m, we have the equality

$$\mu(g_n(Y) \cap g_m(Y)) = 0$$

because the set $g_n(Y) \cap g_m(Y)$ lies in an affine linear manifold whose dimension is strictly less than $r = \dim(L)$. In particular, the family $\{g_n(Y) \ : \ n_0 < n < \omega\}$ is almost disjoint with respect to μ and, hence,

$$\mu(\cup\{g_n(Y) \ : \ n_0 < n < \omega\}) = +\infty.$$

But, for each integer $n > n_0$, we can write

$$g_n(Y) \subset g_n(T') \subset T \subset K$$

and consequently

$$\cup\{g_n(Y)\ :\ n_0 < n < \omega\} \subset K,\ \mu(\cup\{g_n(Y)\ :\ n_0 < n < \omega\}) \le 1.$$

Thus we have obtained a contradiction which shows us that the set Y does not belong to $\mathrm{dom}(\mu)$. $\square$

We wish to finish the paper with some remarks concerning the results presented above.

Remark 1. Let (F, Γ) be a space equipped with a transformation group and let μ be a σ-finite Γ-invariant measure on F. We say that the group Γ acts freely on F with respect to μ if, for any two distinct transformations g and h from Γ, we have

$$\mu^*(\{x \in F\ :\ g(x) = h(x)\}) = 0.$$

For example, the group of all motions of a finite-dimensional Euclidean space E acts freely on E with respect to the Lebesgue measure λ.

Some generalizations of the Vitali classical theorem were discussed for various groups of transformations acting freely with respect to a given nonzero σ-finite invariant measure (see, e.g., [10] and [11]). The freeness of a group of transformations with respect to a given invariant measure seems to be rather natural. However we want to point out that, even in the case of two-dimensional Euclidean space E, an example of a group G of motions of this space can be constructed, such that

(a) G has the Vitali property;

(b) for some G-measure ν given on E, the group G does not act freely with respect to ν.

In order to present such an example, let us take $E = R^2 = R \times R$ and let us define a group G as follows. First of all we put

$$g_1 = (0, 1/2),\ G_2 = R \times \{0\}.$$

Let G_1 denote the group generated by g_1. Obviously, G_1 is a discrete group of translations of E. Denote also by f the symmetry of E with respect to the line $R \times \{0\}$. Finally, let G be the group of transformations of E, generated by $G_1 \cup G_2 \cup \{f\}$. Evidently, G is not discrete and

$$\cup\{g(K_2)\ :\ g \in G\} = E,$$

where

$$K_2 = \{(x, y)\ :\ 0 < x < 1,\ 0 < y < 1\}$$

is the open unit cube in $E = R^2$. Hence, according to Theorem 1, G possesses the Vitali property. On the other hand, let us denote

$$P = \cup\{g(R \times \{0\})\ :\ g \in G\}.$$

Then we have

$$P = G(P) = G_1(P) = \cup\{g(R \times \{0\}) \ : \ g \in G_1\}.$$

For each Borel subset X of E, let us put

$$\nu(X) = \sum_{g \in G_1} \lambda_1(X \cap g(R \times \{0\})),$$

where λ_1 is the one-dimensional Lebesgue measure. It can easily be checked that ν is a σ-finite G-measure singular with respect to the two-dimensional Lebesgue measure λ_2 on $E = R^2$. Also, for two distinct transformations Id_E and f from G, the set

$$\{z \in E \ : \ Id_E(z) = f(z)\} = R \times \{0\}$$

is ν-measurable and we have

$$\nu(\{z \in E \ : \ Id_E(z) = f(z)\}) = +\infty.$$

Thus G does not act freely on E with respect to ν.

Clearly, a similar example can be constructed for any Euclidean space E with $\dim(E) \geq 2$.

Remark 2. It is easy to see that the corresponding analogues of Theorems 1 and 2 hold true for the unit Euclidean sphere equipped with a group of its rotations around its centre.

Remark 3. Let G be a discrete group of motions of Euclidean space E with $\dim(E) \geq 1$. As shown above, there are G-measures on E defined for all subsets of E, so we cannot assert, in general, that for any G-measure there exist nonmeasurable sets in E. But, for certain G-measures, we can establish the existence of nonmeasurable subsets of E and even the existence of nonmeasurable G-selectors. Namely, let G be a discrete group of motions of E, containing at least two distinct elements. Then, starting with Lemma 1 and using an argument similar to the classical Bernstein construction (see, e.g., [9]), it can be proved that there exists a G-selector Z satisfying the equalities

$$\lambda_*(Z) = \lambda_*(E \setminus Z) = 0.$$

In other words, both of the sets Z and $E \setminus Z$ are λ-thick in E and, consequently, they are not measurable with respect to λ (considered as a G-measure on E).

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