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OSCILLATORY BEHAVIOUR OF SOLUTIONS OF TWO-DIMENSIONAL DIFFERENTIAL SYSTEMS WITH DEVIATED ARGUMENTS

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ABSTRACT. Sufficient conditions are established for the oscillation of proper solutions of the system

$$u_1'(t) = f_1\big(t, u_1(\tau_1(t)), \dots, u_1(\tau_m(t)), u_2(\sigma_1(t)), \dots, u_2(\sigma_m(t))\big),$$
$$u_2'(t) = f_2\big(t, u_1(\tau_1(t)), \dots, u_1(\tau_m(t)), u_2(\sigma_1(t)), \dots, u_2(\sigma_m(t))\big),$$

where $f_i : \mathbb{R}_+ \times \mathbb{R}^{2m} \to \mathbb{R}$ $(i = 1, 2)$ satisfy the local Carathéodory conditions and $\tau_i, \sigma_i : \mathbb{R}_+ \to \mathbb{R}_+$ $(i = 1, \dots, m)$ are continuous functions such that $\sigma_i(t) \le t$ for $t \in \mathbb{R}_+$, $\lim\limits_{t\to+\infty} \tau_i(t) = +\infty$, $\lim\limits_{t\to+\infty} \sigma_i(t) = +\infty$ $(i = 1, \dots, m)$.

§ 1. Statement of the Problem and Formulation of the Main Results

Consider the system

$$\begin{aligned} u_1'(t) &= f_1\big(t, u_1(\tau_1(t)), \dots, u_1(\tau_m(t)), u_2(\sigma_1(t)), \dots, u_2(\sigma_m(t))\big), \\ u_2'(t) &= f_2\big(t, u_1(\tau_1(t)), \dots, u_1(\tau_m(t)), u_2(\sigma_1(t)), \dots, u_2(\sigma_m(t))\big), \end{aligned} \tag{1.1}$$

where $f_i : \mathbb{R}_+ \times \mathbb{R}^{2m} \to \mathbb{R}$ $(i = 1, 2)$ satisfy the local Carathéodory conditions and $\tau_i, \sigma_i : \mathbb{R}_+ \to \mathbb{R}_+$ $(i = 1, \dots, m)$ are continuous functions such that $\sigma_i(t) \le t$ for $t \in \mathbb{R}_+$, $\lim\limits_{t\to+\infty} \tau_i(t) = +\infty$, $\lim\limits_{t\to+\infty} \sigma_i(t) = +\infty$ $(i = 1, \dots, m)$.

Definition 1.1. Let $t_0 \in \mathbb{R}_+$ and $a_0 = \inf\big[\min\{\tau_i(t), \sigma_i(t) : i = 1, \dots, m\} : t \ge t_0\big]$. A continuous vector-function (u_1, u_2) defined on $[a_0, +\infty[$ is said to be a *proper* solution of system (1.1) in $[t_0, +\infty[$ if it

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is absolutely continuous on each finite segment contained in $[t_0,+\infty[$, satisfies (1.1) almost everywhere on $[t_0,+\infty[$, and

$$\sup\big\{|u_1(s)|+|u_2(s)|:\ \ s\geq t\big\}>0 \quad \text{for} \quad t\geq t_0.$$

Definition 1.2. A proper solution (u_1,u_2) of system (1.1) is said to be *weakly oscillatory* if either u_1 or u_2 has a sequence of zeros tending to infinity. This solution is said to be *oscillatory* if both u_1 and u_2 have sequences of zeros tending to infinity. If there exists $t_*\in\mathbb{R}_+$ such that $u_1(t)u_2(t)\neq 0$ for $t\geq t_*$, then (u_1,u_2) is said to be *nonoscillatory.*

In this paper, sufficient conditions are obtained for the oscillation of proper solutions of system (1.1) which make the results contained in [1, 2] more complete.

Throughout the paper we will assume that the inequalities

$$\begin{aligned} &f_1(t,x_1,\dots,x_m,y_1,\dots,y_m)\operatorname{sgn} y_1\geq\sum_{i=1}^m p_i(t)|y_i|,\\ &f_2(t,x_1,\dots,x_m,y_1,\dots,y_m)\operatorname{sgn} x_1\leq-\sum_{i=1}^m q_i(t)|x_i| \end{aligned} \tag{1.2}$$

hold for $t\in\mathbb{R}_+$, $x_1x_i>0$, $y_1y_i>0$ $(i=1,\dots,m)$, where $p_i,\,q_i\in L_{loc}(\mathbb{R}_+;\mathbb{R}_+)$ $(i=1,\dots,m)$, and we will use the notation

$$p(t)=\sum_{i=1}^m p_i(t),\quad q(t)=\sum_{i=1}^m q_i(t),\quad h(t)=\int_0^t p(s)\,ds.$$

Theorem 1.1. *Let*

$$h(+\infty)=+\infty, \tag{1.3}$$

$$\int^{+\infty} h_0(t)q(t)\,dt=+\infty, \tag{1.4}$$

where $h_0(t)=\min\{h(t),\,h(\tau_i(t))\ :\ \ i=1,\dots,m\}$, *and there exist a nondecreasing function* $\sigma\in C(\mathbb{R}_+;\mathbb{R}_+)$ *such that* $\sigma_i(t)\leq\sigma(t)\leq t$ *for* $t\in\mathbb{R}_+$ $(i=1,\dots,m)$,

$$\limsup_{t\to+\infty}\frac{h(\tau(\sigma(t)))}{h(t)}<+\infty. \tag{1.5}$$

If, moreover, there exists $\varepsilon_0>0$ *such that for any* $\lambda\in\,]0,1]$,

$$\liminf_{t\to+\infty}\ h^{\varepsilon_0}(t)h^{1-\lambda}(\tau(\sigma(t)))\int_{\tau(\sigma(t))}^{+\infty}p(s)h^{-2-\varepsilon_0}(s)g(s,\lambda)\,ds>1, \tag{1.6}$$

where

$$\tau(t) = \max\Big[\max\big\{\tau_i(s),\, \eta(s):\ \ i=1,\dots,m\big\}:\ \ 0 \le s \le t\Big],$$
$$\eta(t) = \sup\big\{s:\ \ \sigma(s) < t\big\},$$
$$g(t,\lambda) = \int\limits_0^{\sigma(t)} h(s) \sum_{i=1}^m q_i(s) h^\lambda(\tau_i(s))\, ds, \tag{1.7}$$

then every proper solution of system (1.1) *is oscillatory.*

Theorem 1.2. *Let conditions* (1.3)–(1.5) *be fulfilled, where the function* $\sigma \in C(\mathbb{R}_+;\mathbb{R}_+)$ *is nondecreasing,* $\sigma_i(t) \le \sigma(t) \le t$ *for* $t \in \mathbb{R}_+$ $(i=1,\dots,m)$. *If, moreover, there exists* $\varepsilon > 0$ *such that for any* $\lambda \in]0,1]$,

$$\liminf_{t\to+\infty}\ h^{-\lambda}(t) \int\limits_0^{\sigma(t)} h(s) \sum_{i=1}^m q_i(s) h^\lambda(\tau_i(s))\, ds \ge 1 - \lambda + \varepsilon, \tag{1.8}$$

then every proper solution of system (1.1) *is oscillatory.*

Theorem 1.3. *Let conditions* (1.3), (1.4) *be fulfilled,*

$$\limsup_{t\to+\infty} \frac{h(\tau_i(t))}{h(t)} < +\infty \quad (i=1,\dots,m), \tag{1.9}$$

and there exist $\varepsilon > 0$ *such that for any* $\lambda \in]0,1]$,

$$\liminf_{t\to+\infty}\ h^{-1}(t) \int\limits_0^{t} h^2(s) \sum_{i=1}^m q_i(s) \Big[\frac{h(\tau_i(s))}{h(s)}\Big]^\lambda ds \ge \lambda(1-\lambda) + \varepsilon. \tag{1.10}$$

Then every proper solution of system (1.1) *is oscillatory.*

Corollary 1.1. *Let conditions* (1.3), (1.4), (1.9) *be fulfilled and* $\alpha_i \in]0,+\infty[$ $(i=1,\dots,m)$, *where*

$$\alpha_i = \liminf_{t\to+\infty} \frac{h(\tau_i(t))}{h(t)} \quad (i=1,\dots,m). \tag{1.11}$$

If, moreover, there exists $\varepsilon > 0$ *such that for any* $\lambda \in]0,1]$,

$$\liminf_{t\to+\infty}\ h^{-1}(t) \int\limits_0^{t} h^2(s) \sum_{i=1}^m \alpha_i^\lambda q_i(s)\, ds \ge \lambda(1-\lambda) + \varepsilon,$$

then every proper solution of system (1.1) *is oscillatory.*

Corollary 1.2. *Let conditions* (1.3), (1.4), (1.9) *be fulfilled,* $\alpha_i \in]0,+\infty[$ $(i=1,\dots,m)$, $q_i(t) \ge c_i q_0(t)$ *for* $t \in \mathbb{R}_+$ $(i=1,\dots,m)$, *where* α_i $(i=1,\dots,m)$ *are defined by* (1.11), $c_i > 0$ $(i=1,\dots,m)$, *and* $q_0 \in L_{loc}(\mathbb{R}_+;\mathbb{R}_+)$. *Then the condition*

$$\liminf_{t\to+\infty} h^{-1}(t)\int_0^t h^2(s)q_0(s)\,ds > \max\Big\{\lambda(1-\lambda)\Big(\sum_{i=1}^m \alpha_i^\lambda c_i\Big)^{-1} : \ \lambda\in[0,1]\Big\}$$

is sufficient for the oscillation of every proper solution of system (1.1).

Corollary 1.3. *Let* $q_0 \in L_{loc}(\mathbb{R}_+;\mathbb{R}_+)$, $\alpha \in]0,1[$, *and*

$$\liminf_{t\to+\infty} t^{-1}\int_0^t s^{1+\alpha}q_0(s)\,ds > 0.$$

Then every proper solution of the equation

$$u''(t) + q_0(t)u(t^\alpha) = 0$$

is oscillatory.

§ 2. Some Auxiliary Statements

Lemma 2.1. *Let condition* (1.3) *be fulfilled and* (u_1,u_2) *be a nonoscillatory solution of system* (1.1). *Then there exists* $t_0 \in \mathbb{R}_+$ *such that*

$$u_1(t)u_2(t) > 0 \quad \text{for} \quad t \ge t_0. \tag{2.1}$$

If, moreover,

$$\int^{+\infty} h(t)q(t)\,dt = +\infty,$$

then

$$\lim_{t\to+\infty} |u_1(t)| = +\infty.$$

Lemma 2.2. *Let*

$$\int_t^{+\infty} p(s)\,ds > 0, \quad \int_t^{+\infty} q(s)\,ds > 0 \quad \text{for} \quad t\in\mathbb{R}_+. \tag{2.2}$$

Then every weakly oscillatory solution of system (1.1) *is oscillatory.*[1]

[1] For the proofs of Lemma 2.1 and Lemma 2.2 see [2].

Lemma 2.3. *Let condition* (1.3) *be fulfilled and* (u_1, u_2) *be a nonoscillatory solution of system* (1.1). *Then either the inequality*

$$|u_1(t)| < h(t)|u_2(t)| \tag{2.3}$$

is fulfilled for sufficiently large t *or there exists* $t_0 \in \mathbb{R}_+$ *such that*

$$|u_1(t)| \geq h(t) \int\limits_t^{+\infty} \frac{p(s)}{h^2(s)} \int\limits_{t_0}^{\sigma(s)} h(\xi) \sum_{i=1}^m q_i(\xi) \big|u_1(\tau_i(\xi))\big|\, d\xi\, ds \quad \textit{for} \ \ t \geq t_0, \tag{2.4}$$

where $\sigma \in C(\mathbb{R}_+; \mathbb{R}_+)$ *is a nondecreasing function and* $\sigma_i(t) \leq \sigma(t) \leq t$ *for* $t \in \mathbb{R}_+$ $(i = 1, \dots, m)$.

Proof. Since (u_1, u_2) is a nonoscillatory solution of system (1.1) and condition (1.3) is fulfilled, by Lemma 2.1 there exists $t_* \in \mathbb{R}_+$ such that condition (2.1) holds for $t \geq t_*$. By (1.2) and (2.1), from (1.1) we have

$$\left.\begin{aligned} |u_1(t)|' &\geq \sum_{i=1}^m p_i(t)\big|u_2(\sigma_i(t))\big| \\ |u_2(t)|' &\leq -\sum_{i=1}^m q_i(t)\big|u_1(\tau_i(t))\big| \end{aligned}\right. \quad \text{for} \quad t \geq t_*. \tag{2.5}$$

Consider the function $\rho(t) = |u_1(t)| - h(t)|u_2(t)|$. Taking into account (2.5) and the fact that $|u_2(t)|$ is a nonincreasing function, we get

$$\begin{aligned} \rho'(t) &= |u_1(t)|' - h(t)|u_2(t)|' - p(t)|u_2(t)| \geq \\ &\geq \sum_{i=1}^m p_i(t)\big|u_2(\sigma_i(t))\big| - p(t)|u_2(t)| - h(t)|u_2(t)|' \geq \\ &\geq -h(t)|u_2(t)|' \geq 0 \quad \text{for} \quad t \geq t_*. \end{aligned}$$

Thus there exists $t_1 > t_*$ such that either

$$|u_1(t)| - h(t)|u_2(t)| < 0 \quad \text{for} \quad t \geq t_1 \tag{2.6}$$

or

$$|u_1(t)| - h(t)|u_2(t)| \geq 0 \quad \text{for} \quad t \geq t_1. \tag{2.7}$$

If condition (2.6) holds, then the validity of the lemma is obvious. Thus assume that (2.7) is fulfilled and show that in that case estimate (2.4) is valid.

Multiplying the second inequality of system (2.5) by $h(t)$ and integrating from t_1 to $\sigma(t)$, we obtain

$$\int\limits_{t_1}^{\sigma(t)} h(s)\sum_{i=1}^{m} q_i(s)\big|u_1(\tau_i(s))\big|\,ds \le -\int\limits_{t_1}^{\sigma(t)} h(s)|u_2(s)|'\,ds =$$

$$= -h(\sigma(t))\big|u_2(\sigma(t))\big| + \int\limits_{t_1}^{\sigma(t)} p(s)|u_2(s)|\,ds + h(t_1)|u_2(t_1)| \quad \text{for} \quad t \ge t_1.$$

Multiplying the latter inequality by $\frac{p(t)}{h^2(t)}$, integrating from t to $+\infty$, and taking into account (2.5), (2.7) and the fact that $|u_2(t)|$ is a nonincreasing function, we get

$$\int\limits_{t}^{+\infty} \frac{p(s)}{h^2(s)} \int\limits_{t_1}^{\sigma(s)} h(\xi)\sum_{i=1}^{m} q_i(\xi)\big|u_1(\tau_i(\xi))\big|\,d\xi\,ds \le$$

$$\le \int\limits_{t}^{+\infty} \frac{p(s)}{h^2(s)} \int\limits_{t_1}^{\sigma(s)} p(\xi)|u_2(\xi)|\,d\xi\,ds - \int\limits_{t}^{+\infty} \frac{p(s)}{h^2(s)}\, h(\sigma(s))\big|u_2(\sigma(s))\big|\,ds +$$

$$+h(t_1)|u_2(t_1)| \int\limits_{t}^{+\infty} \frac{p(s)}{h^2(s)}\,ds = \int\limits_{t}^{+\infty} \frac{p(s)}{h^2(s)} \int\limits_{t_1}^{\sigma(s)} p(\xi)|u_2(\xi)|\,d\xi\,ds -$$

$$-\int\limits_{t}^{+\infty} \frac{p(s)}{h^2(s)}\bigg[\int\limits_{0}^{s} p(\xi)\,d\xi - \int\limits_{\sigma(s)}^{s} p(\xi)\,d\xi\bigg]\big|u_2(\sigma(s))\big|\,ds + \frac{h(t_1)|u_2(t_1)|}{h(t)} \le$$

$$\le \int\limits_{t}^{+\infty} \frac{p(s)}{h^2(s)} \int\limits_{t_1}^{\sigma(s)} p(\xi)\big|u_2(\sigma(\xi))\big|\,d\xi\,ds - \int\limits_{t}^{+\infty} \frac{p(s)}{h(s)}\,\big|u_2(\sigma(s))\big|\,ds +$$

$$+\int\limits_{t}^{+\infty} \frac{p(s)}{h^2(s)} \int\limits_{\sigma(s)}^{s} p(\xi)\big|u_2(\sigma(\xi))\big|\,d\xi\,ds + \frac{h(t_1)|u_2(t_1)|}{h(t)} =$$

$$= \int\limits_{t}^{+\infty} \frac{p(s)}{h^2(s)} \int\limits_{t_1}^{s} p(\xi)\big|u_2(\sigma(\xi))\big|\,d\xi\,ds -$$

$$-\int\limits_{t}^{+\infty} \frac{1}{h(s)}\,d\int\limits_{t_1}^{s} p(\xi)\big|u_2(\sigma(\xi))\big|\,d\xi + \frac{h(t_1)|u_2(t_1)|}{h(t)} =$$

$$= \frac{1}{h(t)} \int_{t_1}^{t} p(s)\big|u_2(\sigma(s))\big|\,ds + \frac{h(t_1)|u_2(t_1)|}{h(t)} \le$$

$$\le \frac{1}{h(t)}\left[|u_1(t_1)| + \int_{t_1}^{t} \sum_{i=1}^{m} p_i(s)\big|u_2(\sigma_i(s))\big|\,ds\right] \le \frac{|u_1(t)|}{h(t)} \quad \text{for} \quad t \ge t_0,$$

where $t_0 > t_1$ is a sufficiently large number. Therefore (2.4) is fulfilled. Thus the lemma is proved. □

Lemma 2.4. *Let $t_0 \in \mathbb{R}_+$, $\varphi,\ \psi \in C([t_0,+\infty[\,;\,]0,+\infty[\,)$,*

$$\liminf_{t\to+\infty} \varphi(t) = 0, \quad \psi(t) \uparrow +\infty \quad \textit{for} \quad t \uparrow +\infty, \tag{2.8}$$

and

$$\lim_{t\to+\infty} \widetilde{\varphi}(\sigma(t))\psi(t) = +\infty, \tag{2.9}$$

where

$$\widetilde{\varphi}(t) = \min\big\{\varphi(s): \quad t_0 \le s \le t\big\} \tag{2.10}$$

and $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing continuous function such that $\sigma(t) \le t$ for $t \in \mathbb{R}_+$, $\lim\limits_{t\to+\infty} \sigma(t) = +\infty$. Then there exists a sequence of points $\{t_k\}_{k=1}^{+\infty}$ such that $t_k \uparrow +\infty$ for $k \uparrow +\infty$ and

$$\widetilde{\varphi}(\sigma(t_k))\psi(t_k) \le \widetilde{\varphi}(\sigma(s))\psi(s) \quad \textit{for} \quad s \ge t_k,$$
$$\widetilde{\varphi}(\sigma(t_k)) = \varphi(\sigma(t_k)) \quad (k = 1, 2, \dots).$$

Proof. Define the sets E_i $(i = 1, 2)$ in the following manner:

$$t \in E_1 \Longleftrightarrow \widetilde{\varphi}(\sigma(t))\psi(t) \le \widetilde{\varphi}(\sigma(s))\psi(s) \quad \text{for} \quad s \ge t,$$
$$t \in E_2 \Longleftrightarrow \widetilde{\varphi}(\sigma(t)) = \varphi(\sigma(t)).$$

In view of (2.8)–(2.10) it is clear that

$$\sup E_i = +\infty \quad (i = 1, 2). \tag{2.11}$$

Show that $E_1 \cap E_2$ is a nonempty set. Let $m \in \mathbb{N}$. According to (2.11) there exist $t_m^{(i)} \in E_i$ $(i = 1, 2)$ such that $m \le t_m^{(2)} \le t_m^{(1)}$. Suppose that $t_m^{(1)} \notin E_2$. Then we can find $t_m^* \in [t_m^{(2)}, t_m^{(1)}[$ such that

$$\widetilde{\varphi}(\sigma(t)) = \widetilde{\varphi}(\sigma(t_m^{(1)})) \quad \text{for} \quad t \in [t_m^*, t_m^{(1)}] \tag{2.12}$$

and

$$\widetilde{\varphi}(\sigma(t_m^*)) = \varphi(\sigma(t_m^*)). \tag{2.13}$$

On the other hand, since $t_m^{(1)} \in E_1$, on account of (2.8), (2.12), we have

$$\widetilde{\varphi}(\sigma(t_m^*))\psi(t_m^*) \le \widetilde{\varphi}(\sigma(s))\psi(s) \quad \text{for} \quad s \ge t_m^*. \tag{2.14}$$

By virtue of (2.13) and (2.14), $t_m^* \in E_1 \cap E_2$. Taking into account the arbitrariness of m, by the above reasoning we can easily conclude that $\sup E_1 \cap E_2 = +\infty$. This implies that the lemma is valid. $\square$

§ 3. Proof of the Main Results

Proof of Theorem 1.1. Let (u_1, u_2) be a proper solution of system (1.1). Suppose that this solution is not oscillatory. From (1.3), (1.4) follow inequalities (2.2). Thus by Lemma 2.2 (u_1, u_2) is nonoscillatory. Therefore, due to Lemma 2.1, one can find $t_0 \in \mathbb{R}_+$ so that condition (2.1) will be fulfilled for $t \ge t_0$ and

$$\lim_{t\to+\infty} |u_1(t)| = +\infty. \tag{3.1}$$

By (1.2) and (2.1), from (1.1) we have

$$\left.\begin{aligned} |u_1(t)|' &\ge \sum_{i=1}^m p_i(t)\big|u_2(\sigma_i(t))\big| \\ |u_2(t)|' &\le -\sum_{i=1}^m q_i(t)\big|u_1(\tau_i(t))\big| \end{aligned}\right. \quad \text{for} \quad t \ge t_0. \tag{3.2}$$

Since (u_1, u_2) is a nonoscillatory solution of system (1.1) and condition (1.3) holds, by Lemma 2.3 either (2.3) or (2.4) is fulfilled.

Suppose that (2.3) is fulfilled. Then taking into account (3.2) and the fact that $|u_2(t)|$ is a nonincreasing function, we obtain

$$\begin{aligned} \Big(\frac{|u_1(t)|}{h(t)}\Big)' &= \frac{|u_1(t)|'h(t) - p(t)|u_1(t)|}{h^2(t)} \ge \\ &\ge \frac{h(t)\sum\limits_{i=1}^m p_i(t)|u_2(\sigma_i(t))| - p(t)|u_1(t)|}{h^2(t)} \ge \\ &\ge \frac{p(t)[h(t)|u_2(t)| - |u_1(t)|]}{h^2(t)} \ge 0 \quad \text{for} \quad t \ge t_1, \end{aligned}$$

where $t_1 > t_0$ is a sufficiently large number. Thus there exist $c > 0$ and $t^* \ge t_1$ such that

$$\big|u_1(\tau_i(t))\big| \ge ch(\tau_i(t)) \quad \text{for} \quad t \ge t^* \quad (i = 1, \dots, m).$$

In view of the latter inequalities, from the second inequality of system (3.2) we get

$$|u_2(t^*)| \geq c \int\limits_{t^*}^{+\infty} \sum_{i=1}^{m} q_i(s)h(\tau_i(s))\,ds \geq c \int\limits_{t^*}^{+\infty} h_0(s)q(s)\,ds.$$

But the latter inequality contradicts (1.4). Therefore below (2.4) will be assumed to be fulfilled.

Denote by Δ the set of all $\lambda \in]0,1]$ satisfying

$$\liminf_{t\to+\infty} \frac{|u_1(t)|}{h^{\lambda}(t)} = 0.$$

By (3.1) it is obvious that $0 \notin \Delta$, and by using (1.4) we can easily show that $1 \in \Delta$.

Let $\lambda_0 = \inf\Delta$. Then by (1.6) there exist $\lambda^* \in]0,1] \cap [\lambda_0,1]$ and $\varepsilon_1 \in]0,\varepsilon_0]$ such that $\lambda^* - \varepsilon_1 \in [0,1]$,

$$\liminf_{t\to+\infty} \frac{|u_1(t)|}{h^{\lambda^*}(t)} = 0, \quad \lim_{t\to+\infty} \frac{|u_1(t)|}{h^{\lambda^*-\varepsilon_1}(t)} = +\infty, \tag{3.3}$$

and

$$\liminf_{t\to+\infty} h^{\varepsilon_1}(t)h^{1-\lambda^*}\big(\tau(\sigma(t))\big) \int\limits_{\tau(\sigma(t))}^{+\infty} p(s)h^{-2-\varepsilon_1}(s)g(s,\lambda^*)\,ds > 1, \tag{3.4}$$

where $g(t,\lambda)$ is defined by (1.7).

Introduce the notation

$$\widetilde{\varphi}(t) = \min\left\{ \frac{|u_1(\tau(s))|}{h^{\lambda^*}(\tau(s))} \;:\; t_0 \leq s \leq t \right\}.$$

It is obvious that $\widetilde{\varphi}(t) \downarrow 0$ for $t \uparrow +\infty$ and

$$\frac{|u_1(\tau_i(t))|}{h^{\lambda^*}(\tau_i(t))} \geq \widetilde{\varphi}(t) \quad \text{for} \quad t \geq t_0 \quad (i = 1,\dots,m). \tag{3.5}$$

By virtue of (1.3), (1.5) and (3.3) all the conditions of Lemma 2.4 are fulfilled. Thus there exists a sequence of points $\{t_k\}_{k=1}^{+\infty}$ such that $t_k \uparrow +\infty$ for $k \uparrow +\infty$,

$$\widetilde{\varphi}(\sigma(t_k))h^{\varepsilon_1}(t_k) \leq \widetilde{\varphi}(\sigma(s))h^{\varepsilon_1}(s) \quad \text{for} \quad s \geq t_k, \tag{3.6}$$

$$\widetilde{\varphi}(\sigma(t_k)) = \frac{|u_1(\tau(\sigma(t_k)))|}{h^{\lambda^*}(\tau(\sigma(t_k)))} \quad (k = 1,2,\dots). \tag{3.7}$$

Taking into account conditions (3.5)–(3.7), for sufficiently large k from (2.4) we get

$$
\begin{aligned}
&\big|u_1\big(\tau(\sigma(t_k))\big)\big| \geq \\
&\geq h\big(\tau(\sigma(t_k))\big) \int\limits_{\tau(\sigma(t_k))}^{+\infty} \frac{p(s)}{h^2(s)} \int\limits_{t_0}^{\sigma(s)} h(\xi) \sum_{i=1}^m q_i(\xi)\, \frac{|u_1(\tau_i(\xi))|}{h^{\lambda^*}(\tau_i(\xi))}\, h^{\lambda^*}(\tau_i(\xi))\, d\xi\, ds \geq \\
&\geq h\big(\tau(\sigma(t_k))\big) \int\limits_{\tau(\sigma(t_k))}^{+\infty} \frac{p(s)}{h^2(s)} \int\limits_{t_0}^{\sigma(s)} \widetilde{\varphi}(\xi) h(\xi) \sum_{i=1}^m q_i(\xi) h^{\lambda^*}(\tau_i(\xi))\, d\xi\, ds \geq \\
&\geq h\big(\tau(\sigma(t_k))\big) \int\limits_{\tau(\sigma(t_k))}^{+\infty} p(s) h^{-2}(s) \widetilde{\varphi}(\sigma(s)) \int\limits_{t_0}^{\sigma(s)} h(\xi) \sum_{i=1}^m q_i(\xi) h^{\lambda^*}(\tau_i(\xi))\, d\xi\, ds \geq \\
&\geq \widetilde{\varphi}(\sigma(t_k)) h^{\varepsilon_1}(t_k) h\big(\tau(\sigma(t_k))\big) \int\limits_{\tau(\sigma(t_k))}^{+\infty} p(s) h^{-2-\varepsilon_1}(s) \times \\
&\times \int\limits_{t_0}^{\sigma(s)} h(\xi) \sum_{i=1}^m q_i(\xi) h^{\lambda^*}(\tau_i(\xi))\, d\xi\, ds = \\
&= \big|u_1\big(\tau(\sigma(t_k))\big)\big| h^{\varepsilon_1}(t_k) h^{1-\lambda^*}\big(\tau(\sigma(t_k))\big) \times \\
&\times \int\limits_{\tau(\sigma(t_k))}^{+\infty} p(s) h^{-2-\varepsilon_1}(s) \int\limits_{t_0}^{\sigma(s)} h(\xi) \sum_{i=1}^m q_i(\xi) h^{\lambda^*}(\tau_i(\xi))\, d\xi\, ds.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&h^{\varepsilon_1}(t_k) h^{1-\lambda^*}\big(\tau(\sigma(t_k))\big) \int\limits_{\tau(\sigma(t_k))}^{+\infty} p(s) h^{-2-\varepsilon_1}(s) \times \\
&\times \int\limits_{t_0}^{\sigma(s)} h(\xi) \sum_{i=1}^m q_i(\xi) h^{\lambda^*}(\tau_i(\xi))\, d\xi\, ds \leq 1.
\end{aligned}
$$

But the latter inequality contradicts (3.4). The contradiction obtained proves that the theorem is valid. $\square$

Proof of Theorem 1.2. By Theorem 1.1 it is sufficient to show that we can find $\varepsilon_0 \in\,]0, \varepsilon]$ such that condition (1.6) will hold. Indeed, choose $\varepsilon_0 \in\,]0, \varepsilon]$

so that

$$\frac{1+\varepsilon}{1+\varepsilon_0}\Big(\frac{1}{2\gamma}\Big)^{\varepsilon_0} > 1, \tag{3.8}$$

where

$$\gamma = \limsup_{t\to+\infty} \frac{h(\tau(\sigma(t)))}{h(t)}\,. \tag{3.9}$$

In view of (1.8), (3.8) and (3.9), we obtain

$$h^{\varepsilon_0}(t)h^{1-\lambda}\big(\tau(\sigma(t))\big)\int\limits_{\tau(\sigma(t))}^{+\infty} p(s)h^{-2-\varepsilon_0}(s)g(s,\lambda)\,ds \geq$$

$$\geq (1-\lambda+\varepsilon)h^{\varepsilon_0}(t)h^{1-\lambda}\big(\tau(\sigma(t))\big)\int\limits_{\tau(\sigma(t))}^{+\infty} h^{\lambda-2-\varepsilon_0}(s)\,dh(s) =$$

$$= \frac{1-\lambda+\varepsilon}{1-\lambda+\varepsilon_0}\,h^{\varepsilon_0}(t)h^{-\varepsilon_0}\big(\tau(\sigma(t))\big) \geq$$

$$\geq \frac{1-\lambda+\varepsilon}{1-\lambda+\varepsilon_0}\Big(\frac{1}{2\gamma}\Big)^{\varepsilon_0} \geq \frac{1+\varepsilon}{1+\varepsilon_0}\Big(\frac{1}{2\gamma}\Big)^{\varepsilon_0} > 1 \quad \text{for} \quad t \geq t_0,$$

where $t_0 \in \mathbb{R}_+$ is a sufficiently large number. Therefore condition (1.6) is fulfilled. Thus the theorem is proved. $\square$

Proof of Theorem 1.3. By virtue of Theorem 1.2 it is sufficient to show that condition (1.8) is fulfilled with $\sigma(t) \equiv t$. Indeed, on account of (1.10) we get[2]

$$h^{-\lambda}(t)\int\limits_0^t h(s)\sum_{i=1}^m q_i(s)h^{\lambda}(\tau_i(s))\,ds =$$

$$= h^{-\lambda}(t)\int\limits_0^t h^{1+\lambda}(s)\sum_{i=1}^m q_i(s)\Big[\frac{h(\tau_i(s))}{h(s)}\Big]^{\lambda}\,ds =$$

$$= h^{-\lambda}(t)\int\limits_0^t h^{\lambda-1}(s)\,d\int\limits_0^s h^2(\xi)\sum_{i=1}^m q_i(\xi)\Big[\frac{h(\tau_i(\xi))}{h(\xi)}\Big]^{\lambda}\,d\xi =$$

$$= h^{-1}(t)\int\limits_0^t h^2(s)\sum_{i=1}^m q_i(s)\Big[\frac{h(\tau_i(s))}{h(s)}\Big]^{\lambda}\,ds +$$

[2] Here we mean that $\lambda < 1$. In the case where $\lambda = 1$ the validity of (1.8) is obvious.

$$+(1-\lambda)h^{-\lambda}(t)\int_0^t p(s)h^{\lambda-2}(s)\int_0^s h^2(\xi)\sum_{i=1}^m q_i(\xi)\Big[\frac{h(\tau_i(\xi))}{h(\xi)}\Big]^\lambda d\xi\,ds \geq$$

$$\geq \big(\lambda(1-\lambda)+\varepsilon\big)+(1-\lambda)\big(\lambda(1-\lambda)+\varepsilon\big)h^{-\lambda}(t)\int_0^t p(s)h^{\lambda-1}(s)\,ds =$$

$$=\big(\lambda(1-\lambda)+\varepsilon\big)\Big(1+\frac{1-\lambda}{\lambda}\Big)=\frac{\lambda(1-\lambda)+\varepsilon}{\lambda}\geq 1-\lambda+\varepsilon \quad \text{for} \quad t\geq t_0,$$

where $t_0 \in \mathbb{R}_+$ is a sufficiently large number. Therefore condition (1.8) holds. Thus the theorem is proved. □

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