GEORGIAN MATHEMATICAL JOURNAL: Vol. 6, No. 5, 1999, 415-420

TRIPLE POSITIVE SOLUTIONS FOR MULTIPOINT CONJUGATE BOUNDARY VALUE PROBLEMS

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ABSTRACT. For the nth order nonlinear differential equation $y^{(n)}(t) = f(y(t))$, $t \in [0,1]$, satisfying the multipoint conjugate boundary conditions, $y^{(j)}(a_i) = 0$, $1 \le i \le k$, $0 \le j \le n_i - 1$, $0 = a_1 < a_2 < \cdots < a_k = 1$, and $\sum_{i=1}^{k} n_i = n$, where $f : \mathbb{R} \to [0,\infty)$ is continuous, growth condtions are imposed on f which yield the existence of at least three solutions that belong to a cone.

1. INTRODUCTION

Let $n \ge 2$ be an integer and $k \in \{2, 3, \ldots, n\}$. Let $0 = a_1 < a_2 < \cdots < a_k = 1$ be k points, $n_i \in \{1, 2, \ldots, n-1\}$, $1 \le i \le k$, be such that $\sum_{i=1}^{k} n_i = n$ and define $\alpha_i = \sum_{j=i+1}^{k-1} n_j$ where $1 \le i \le k$. We are concerned with the existence of multiple solutions for the nth order multipoint conjugate boundary value problem

$$y^{(n)}(t) = f(y(t)), \qquad 0 \le t \le 1, \tag{1.1}$$

$$y^{(j)}(a_i) = 0, \qquad 1 \le i \le k,\ 0 \le j \le n_i - 1, \tag{1.2}$$

where $f : \mathbb{R} \to [0, \infty)$ is continuous. We will impose growth conditions on f which ensure the existence of at least three solutions of (1.1), (1.2) that belong to a cone.

Recent attention has been directed toward obtaining multiple solutions for boundary value problems (BVPs) for ordinary differential equations (ODEs). While his methods are not the same as those applied here, Brykalov [1–3] has established the existence of multiple solutions for certain nonlinear BVPs for ODEs. Closer to this work, we refer the reader to the papers of Avery [4], Chyan and Davis [5], Chyan, Davis, and Yin [6], Davis and

1991 *Mathematics Subject Classification.* Primary: 34B10; Secondary: 34B15.

Key words and phrases. Multiple positive solutions, multipoint conjugate boundary value problem, fixed points.

1072-947X/99/0900-0415\$16.00/0 © 1999 Plenum Publishing Corporation

Henderson [7,8], Henderson and Thompson [9], and Guo and Lakshmikantham [10]. This paper can be considered a complete generalization of [7] which deals with triple positive solutions for two point conjugate BVPs (i.e., when $k = 2$). Multipoint problems for higher order ODEs (specifically, conjugate problems) and the existence of multiple positive solutions have been studied by Eloe and Henderson in [11].

For the most part, each of the papers on the existence of triple positive solutions makes an application of the fixed point theorem by Leggett and Williams [12] which was developed using the fixed point index in ordered Banach spaces. Leggett and Williams [12] applied their fixed point theorem to prove the existence of three positive solutions for Hammerstein integral equations of the form $y(x) = \int_\Omega G(x,s)g(s,y(s))\,ds$, $\Omega \subset \mathbb{R}^N$, by making use of suitable inequalities imposed on G and g.

In Section 2, we provide some definitions and background results, and we state the Leggett–Williams Fixed Point Theorem. Then in Section 3, we impose growth conditions on f which allow us to apply the Leggett–Williams Fixed Point Theorem in obtaining three solutions of (1.1), (1.2) that lie in a cone.

2. Background and Definitions

Our main results will hinge on an application of the Legget–Williams Fixed Point Theorem which deals with fixed points of a cone preserving operator. For the convenience of the reader, we include here the necessary definitions from cone theory in Banach spaces.

Definition 2.1. Let $\mathcal{B}$ be a Banach space over $\mathbb{R}$. A nonempty, closed set $\mathcal{P} \subset \mathcal{B}$ is said to be a*cone* provided

(a) $\alpha\mathbf{u} + \beta\mathbf{v} \in \mathcal{P}$ for all $\mathbf{u}, \mathbf{v} \in \mathcal{P}$ and all $\alpha, \beta \geq 0$, and
(b) $\mathbf{u}, -\mathbf{u} \in \mathcal{P}$ implies $\mathbf{u} = \mathbf{0}$.

Definition 2.2. A Banach space $\mathcal{B}$ is called a*partially ordered Banach space* if there exists a partial ordering $\preceq$ on $\mathcal{B}$ satisfying

(a) $\mathbf{u} \preceq \mathbf{v}$, for $boldu, \mathbf{v} \in \mathcal{B}$ implies $t\mathbf{u} \preceq t\mathbf{v}$, for all $t \geq 0$, and
(b) $\mathbf{u}_1 \preceq \mathbf{v}_1$ and $\mathbf{u}_2 \preceq \mathbf{v}_2$, for $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2 \in \mathcal{B}$ imply $\mathbf{u}_1 + \mathbf{u}_2 \preceq \mathbf{v}_1 + \mathbf{v}_2$.

Let $\mathcal{P} \subset \mathcal{B}$ be a cone and define $\mathbf{u} \preceq \mathbf{v}$ if and only if $\mathbf{v} - \mathbf{u} \in \mathcal{P}$. Then $\preceq$ is a partial ordering on $\mathcal{B}$ and we will say that $\preceq$ is the partial ordering induced by $\mathcal{P}$. Moreover, $\mathcal{B}$ is a partially ordered Banach space with respect to $\preceq$.

We also state the following definitions for future reference.

Definition 2.3. The map α is a*nonnegative continuous concave functional on* $\mathcal{P}$ provided $\alpha : \mathcal{P} \to [0, \infty)$ is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in \mathcal{P}$ and $0 \leq t \leq 1$.

Definition 2.4. Let $0 < a < b$ be given and α be a nonnegative continuous concave functional on the cone $\mathcal{P}$. Define the convex sets $\mathcal{P}_r$ and $\mathcal{P}(\alpha, a, b)$ by

$$\mathcal{P}_r = \{y \in \mathcal{P} \mid \ \|y\| < r\} \ \text{ and } \ \mathcal{P}(\alpha, a, b) = \{y \in \mathcal{P} \mid a \leq \alpha(y), \ \|y\| \leq b\}.$$

Next we state the Leggett–Williams Fixed Point Theorem. The proof can be found in Deimling's text [13] and utilizes the fixed point index in ordered Banach spaces.

Theorem 2.1 (Leggett–Williams Fixed Point Theorem). *Let $\mathcal{A} : \overline{\mathcal{P}}_c \to \overline{\mathcal{P}}_c$ be a completely continuous operator and let α be a nonnegative continuous concave functional on $\mathcal{P}$ such that $\alpha(y) \leq \|y\|$ for all $y \in \overline{\mathcal{P}}_c$. Suppose there exist $0 < a < b < d \leq c$ such that*

(C1) $\{y \in \mathcal{P}(\alpha, b, d) \mid \alpha(y) > b\} \neq \varnothing$ *and* $\alpha(\mathcal{A}y) > b$ *for* $y \in \mathcal{P}(\alpha, b, d)$,
(C2) $\|\mathcal{A}y\| < a$ *for* $\|y\| \leq a$, *and*
(C3) $\alpha(\mathcal{A}y) > b$ *for* $y \in \mathcal{P}(\alpha, b, c)$ *with* $\|\mathcal{A}y\| > d$.

Then $\mathcal{A}$ has at least three fixed points y_1, y_2, and y_3 such that $\|y_1\| < a$, $b < \alpha(y_2)$, and $\|y_3\| > a$ with $\alpha(y_3) < b$.

3. Triple Positive Solutions

In this section, we will impose growth conditions on f which allow us to apply Theorem 2.1 in regard to obtaining three solutions of (1.1), (1.2). We will apply Theorem 2.1 to a completely continuous operator whose kernel is the Green's function, $G(t, s)$, for

$$y^{(n)}(t) = 0, \tag{3.1}$$

satisfying the boundary conditions (1.2). It is fairly well known [14] that

$$(-1)^{\alpha_i} G(t, s) > 0, \quad \text{for } (t, s) \in (a_i, a_{i+1}) \times (0, 1), \ 1 \leq i \leq k-1. \tag{3.2}$$

For $s \in (0, 1)$, define

$$\|G(\cdot, s)\| = \max_{t \in [0,1]} |G(t, s)|. \tag{3.3}$$

Eloe and Henderson [15] proved the following theorem which is a key estimate for our main result.

Theorem 3.1. *Suppose $y \in C^{(n)}[0, 1]$ is such that $y^{(n)}(t) \geq 0$, $t \in [0, 1]$, and that y satisfies the multipoint conjugate boundary conditions* (1.2). *Then, for each $1 \leq i \leq k-1$,*

$$(-1)^{\alpha_i} G(t, s) \geq (a/4)^m \|G(\cdot, s)\|, \qquad t \in S_i, \ s \in (0, 1), \tag{3.4}$$

where $S_i = [\frac{3a_i+a_{i+1}}{4}, \frac{a_i+3a_{i+1}}{4}]$, $a = \min_{1\le i\le k-1}\{a_{i+1}-a_i\}$, *and* $m = \max\{n-n_1, n-n_k\}$.

Next, we define

$$K = \left(\max_{t\in[0,1]} \int_0^1 G(t,s)\,ds\right)^{-1}, \tag{3.5}$$

$$L = \left(\min_{1\le i\le k-1} \min_{t\in S_i} \int_{S_i} G(t,s)\,ds\right)^{-1}. \tag{3.6}$$

Let $\mathcal{B}$ denote the Banach space $C[0,1]$ with the maximum norm $\|y\| = \max_{t\in[0,1]} |y(t)|$ and define the cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} := \{y \in \mathcal{B} \mid (-1)^{\alpha_i} y(t) \ge 0,\ t \in [a_i, a_{i+1}] \text{ for } 1 \le i \le k-1\}.$$

Let $\alpha : \mathcal{P} \to [0,\infty)$ be the nonnegative continuous concave functional

$$\alpha(y) = \min_{1\le i\le k-1} \min_{t\in S_i} |y(t)|, \qquad \text{for } y \in \mathcal{P},$$

and let $\mathcal{A} : \mathcal{B} \to \mathcal{B}$ be the operator

$$\mathcal{A}y(t) = \int_0^1 G(t,s) f(y(s))\,ds.$$

We now present the main result of the paper.

Theorem 3.2. *Let* $0 < a < b < \left(\frac{4}{a}\right)^m b \le c$ *be such that* f *satisfies*

(i) $f(w) < Ka$, *for* $0 \le |w| \le a$,
(ii) $f(w) \ge Lb$, *for* $b \le |w| \le \left(\frac{4}{a}\right)^m b$, *and*
(iii) $f(w) \le Kc$, *for* $0 \le |w| \le c$.

Then, the boundary value problem (1.1), (1.2) *has three positive solutions* y_1, y_2, *and* y_3 *satisfying*

$$\|y_1\| < a,\ b < \min_{1\le i\le k-1} \min_{t\in S_i} |y_2(t)|,\ \|y_3\| > a \text{ with } \min_{1\le i\le k-1} \min_{t\in S_i} |y_3(t)| < b.$$

Proof. We seek fixed points of $\mathcal{A}$ which satisfy the conclusion of the theorem. We observe first from the positivity of f and (3.2) that, for $y \in \mathcal{P}$, $(-1)^{\alpha_i}\mathcal{A}y(t) \ge 0$ for $t \in [a_i, a_{i+1}]$. Thus, $\mathcal{A} : \mathcal{P} \to \mathcal{P}$.

We now show that the conditions of Theorem 2.1 are satisfied. Choose $y \in \overline{\mathcal{P}_c}$. Then, $\|y\| \le c$ and by assumption (iii), $f(y(s)) \le Kc$, $s \in [0,1]$. Thus, from (3.5)

$$\begin{aligned}\|\mathcal{A}y\| &= \max_{t\in[0,1]} \int_0^1 G(t,s) f(y(s))\,ds \le \max_{t\in[0,1]} \int_0^1 |G(t,s)| f(y(s))\,ds \\ &\le \max_{t\in[0,1]} \int_0^1 |G(t,s)| Kc\,ds = c.\end{aligned}$$

Hence, $\mathcal{A} : \overline{\mathcal{P}_c} \to \overline{\mathcal{P}_c}$. In a similar way, if $y \in \overline{\mathcal{P}_a}$, then assumption (i) yields $f(y(s)) < Ka$, $s \in [0,1]$, and it follows as above that $\mathcal{A} : \overline{\mathcal{P}_a} \to \mathcal{P}_a$. Consequently, condition (C2) of Theorem 2.1 is fulfilled.

To verify property (C1) of Theorem 2.1, we note that if we let

$$x(t) = \begin{cases} (-1)^{\alpha_i}(4/a)^m b, & t \in S_i,\ 1 \le i \le k-1, \\ \frac{x(a_{i+1})-x(a_i)}{a_{i+1}-a_i}(t-a_i) + x(a_i), & t \in \overline{[0,1] \setminus S_i},\ 1 \le i \le k-1, \end{cases}$$

then $x(t) \in \mathcal{P}(\alpha, b, \left(\frac{4}{a}\right)^m b)$. Moreover, $\alpha(x) = \left(\frac{4}{a}\right)^m b > b$. Hence

$$\{y \in \mathcal{P}(\alpha, b, (4/a)^m b) \,|\, \alpha(y) > b\} \neq \varnothing.$$

Furthermore, if we choose $y \in \mathcal{P}(\alpha, b, \left(\frac{4}{a}\right)^m b)$, then

$$\alpha(y) = \min_{1 \le i \le k-1} \min_{t \in S_i} |y(t)| \ge b,$$

and so $b \le |y(s)| \le \left(\frac{4}{a}\right)^m b$, $s \in S_i$, $1 \le i \le k-1$. Thus, for any $y \in \mathcal{P}(\alpha, b, \left(\frac{4}{a}\right)^m b)$, assumption (ii) yields $f(y(s)) \ge Lb$, $s \in S_i$, $1 \le i \le k-1$, and by (3.6) we have

$$\begin{aligned} \alpha(\mathcal{A}y) &= \min_{1 \le i \le k-1} \min_{t \in S_i} |\mathcal{A}y| = \\ &= \min_{1 \le i \le k-1} \min_{t \in S_i} \int_0^1 (-1)^{\alpha_i} G(t,s) f(y(s))\, ds > \\ &> \min_{1 \le i \le k-1} \min_{t \in S_i} \int_{S_i} (-1)^{\alpha_i} G(t,s) f(y(s))\, ds \ge \\ &\ge \min_{1 \le i \le k-1} \min_{t \in S_i} \int_{S_i} (-1)^{\alpha_i} G(t,s) Lb\, ds = b. \end{aligned}$$

Hence, condition (C1) of Theorem 2.1 is satisfied.

We finally exhibit that (C3) of Theorem 2.1 is satisfied. (In particular, we show, if $y \in \mathcal{P}(\alpha, b, c)$ and $\|\mathcal{A}y\| > \left(\frac{4}{a}\right)^m b$, then $\alpha(\mathcal{A}y) > b$.) Thus we choose $y \in \mathcal{P}(\alpha, b, c)$ such that $\|\mathcal{A}y\| > \left(\frac{4}{a}\right)^m b$. Then, from (3.4),

$$\begin{aligned} \alpha(\mathcal{A}y) &= \min_{1 \le i \le k-1} \min_{t \in S_i} \int_0^1 (-1)^{\alpha_i} G(t,s) f(y(s))\, ds \ge \\ &\ge \left(\frac{a}{4}\right)^m \int_0^1 \|G(\cdot, s)\| f(y(s))\, ds \ge \\ &\ge \left(\frac{a}{4}\right)^m \max_{t \in [0,1]} \int_0^1 |G(t,s)| f(y(s))\, ds = \\ &= \left(\frac{a}{4}\right)^m \|\mathcal{A}y\| > b, \end{aligned}$$

and (C3) of Theorem 2.1 is satisfied. Hence an application of Theorem 2.1 completes the proof. $\square$

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(Received 14.12.1998)

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