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ON BOUNDED SOLUTIONS OF SYSTEMS OF LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. Sufficient conditions of the existence and uniqueness of bounded on real axis solutions of systems of linear functional differential equations are established.

1. Formulations of the Main Results

Let R be the set of real numbers, $C_{loc}(R,R)$ be the space of continuous functions $u : R \to R$ with the topology of uniform convergence on every compact interval and $L_{loc}(R,R)$ be the space of locally summable functions $u : R \to R$ with the topology of convergence in the mean on every compact interval. Consider the system of functional differential equations

$$x_i'(t) = \sum_{k=1}^{n} l_{ik}(x_k)(t) + q_i(t) \quad (i = 1,\dots,n), \tag{1.1}$$

where $l_{ik} : C_{loc}(R,R) \to L_{loc}(R,R)$ $(i,k = 1,\dots,n)$ are linear continuous operators and $q_i \in L_{loc}(R,R)$ $(i = 1,\dots,n)$. Moreover, there exist linear positive operators $\bar{l}_{ik} : C_{loc}(R,R) \to L_{loc}(R,R)$ $(i,k = 1,\dots,n)$ such that for any $u \in C_{loc}(R,R)$ the inequalities

$$|l_{ik}(u)(t)| \le \bar{l}_{ik}(|u|)(t) \quad (i,k = 1,\dots,n) \tag{1.2}$$

are fulfilled almost everywhere on R.

A simple but important particular case of (1.1) is the linear differential system with deviated arguments

$$x_i'(t) = \sum_{k=1}^{n} p_{ik}(t) x_k(\tau_{ik}(t)) + q_i(t) \quad (i = 1,\dots,n), \tag{1.3}$$

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where $p_{ik} \in L_{loc}(R,R)$, $q_i \in L_{loc}(R,R)$ $(i,k=1,\dots,n)$ and $\tau_{ik} : R \to R$ $(i,k=1,\dots,n)$ are locally measurable functions.

A locally absolutely continuous vector function $(x_i)_{i=1}^n : R \to R$ is called a bounded solution of system (1.1) if it satisfies this system almost everywhere on R and

$$\sup\left\{\sum_{i=1}^n |x_i(t)| : t \in R\right\} < +\infty.$$

I. Kiguradze [1, 2] has established optimal in some sense sufficient conditions of the existence and uniqueness of a bounded solution of the differential system

$$\frac{dx_i(t)}{dt} = \sum_{k=1}^n p_{ik}(t)x_k(t) + q_i(t) \quad (i=1,\dots,n).$$

In the present paper these results are generalized for systems (1.1) and (1.3).

Before formulating the main results we want to introduce some notation.

δ_{ik} is Kronecker's symbol, i.e., $\delta_{ii} = 1$ and $\delta_{ik} = 0$ for $i \neq k$.

$A = (a_{ik})_{i,k=1}^n$ is a $n \times n$ matrix with components a_{ik} $(i,k=1,\dots,n)$.

$r(A)$ is the spectral radius of the matrix A.

If $t_i \in R \cup \{-\infty,+\infty\}$ $(i=1,\dots,n)$, then

$$\mathcal{N}_0(t_1,\dots,t_n) = \{i : t_i \in R\}.$$

If $u \in L_{loc}(R,R)$, then

$$\eta(u)(s,t) = \int_t^s u(\xi)\,d\xi \quad \text{for } t \text{ and } s \in R. \tag{1.4}$$

Theorem 1.1. *Let there exist $t_i \in R \cup \{-\infty,+\infty\}$ $(i=1,\dots,n)$, a nonnegative constant matrix $A = (a_{ik})_{i,k=1}^n$ and a nonnegative number a such that*

$$r(A) < 1, \tag{1.5}$$

$$\left|\int_{t_i}^t \exp\left(\int_s^t l_{ii}(1)(\xi)d\xi\right)\left[\bar{l}_{ii}(|\eta(\bar{l}_{ik}(1))(\cdot,s)|)(s) + (1-\delta_{ik})|\bar{l}_{ik}(1)(s)|\right]ds\right| \leq$$
$$\leq a_{ik} \quad \textit{for } t \in R \ (i,k=1,\dots,n), \tag{1.6}$$

$$\sum_{i=1}^n \left|\int_{t_i}^t \exp\left(\int_s^t l_{ii}(1)(\xi)d\xi\right)\left[\bar{l}_{ii}(|\eta(|q_i|)(\cdot,s)|)(s) + |q_i(s)|\right]ds\right| \leq$$
$$\leq a \quad \textit{for } t \in R \tag{1.7}$$

and

$$\sup\left\{\int\limits_{t_i}^{t} l_{ii}(1)(s)ds : t\in R\right\} < +\infty \quad \textit{for } i\in\mathcal{N}_0(t_1,\dots,t_n). \tag{1.8}$$

Then for any $c_i \in R$ $(i\in\mathcal{N}_0(t_1,\dots,t_n))$ *system* (1.1) *has at least one bounded solution satisfying the conditions*

$$x_i(t_i)=c_i \ \textit{ for } \ i\in\mathcal{N}_0(t_1,\dots,t_n). \tag{1.9}$$

Theorem 1.2. *Let all the conditions of Theorem* 1.1 *be fulfilled and*

$$\liminf_{t\to t_i}\int\limits_{t}^{0} l_{ii}(1)(s)ds=-\infty \ \textit{ for } \ i\in\{1,\dots,n\}\setminus\mathcal{N}_0(t_1,\dots,t_n). \tag{1.10}$$

Then for any $c_i\in R$ $(i\in\mathcal{N}_0(t_1,\dots,t_n))$ *system* (1.1) *has one and only one bounded solution satisfying conditions* (1.9).

If $t_i\in\{-\infty,+\infty\}$ $(i=1,\dots,n)$, then $\mathcal{N}_0(t_1,\dots,t_n)=\varnothing$. In that case in Theorems 1.1 and 1.2 conditions (1.8) and (1.9) become unnecessary so that these theorems are formulated as follows.

Theorem 1.1′. *Let there exist* $t_i\in\{-\infty,+\infty\}$ $(i=1,\dots,n)$, *a nonnegative constant matrix* $A=(a_{ik})_{i,k=1}^n$ *and a nonnegative number* a *such that conditions* (1.5)–(1.7) *are fulfilled. Then system* (1.1) *has at least one bounded solution.*

Theorem 1.2′. *Let all the conditions of Theorem* 1.1′ *be fulfilled and*

$$\liminf_{t\to t_i}\int\limits_{t}^{0} l_{ii}(1)(s)\,ds=-\infty \quad (i=1,\dots,n).$$

Then system (1.1) *has one and only one bounded solution.*

The above theorems yield the following statements for system (1.3).

Corollary 1.1. *Let there exist* $t_i\in R\cup\{-\infty,+\infty\}$ $(i=1,\dots,n)$, *a nonnegative constant matrix* $A=(a_{ik})_{i,k=1}^n$ *and a nonnegative number* a

such that $r(A) < 1$,

$$\left|\int_{t_i}^{t} \exp\left(\int_{s}^{t} p_{ii}(\xi)d\xi\right)\left[\left|p_{ii}(s)\int_{s}^{\tau_{ii}(s)} |p_{ik}(\xi)|d\xi\right| + (1-\delta_{ik})|p_{ik}(s)|\right]ds\right| \leq$$
$$\leq a_{ik} \quad \text{for } t \in R \ \ (i,k = 1,\dots,n), \tag{1.11}$$

$$\sum_{i=1}^{n}\left|\int_{t_i}^{t} \exp\left(\int_{s}^{t} p_{ii}(\xi)d\xi\right)\left[\left|p_{ii}(s)\int_{s}^{\tau_{ii}(s)} |q_i(\xi)|d\xi\right| + |q_i(s)|\right]ds \leq$$
$$\leq a \quad \text{for } t \in R \tag{1.12}$$

and

$$\sup\left\{\int_{t_i}^{t} p_{ii}(s)ds : t \in R\right\} < +\infty \quad \text{for } i \in \mathcal{N}_0(t_1,\dots,t_n). \tag{1.13}$$

Then for any $c_i \in R$ $(i \in \mathcal{N}_0(t_1,\dots,t_n))$ *system* (1.3) *has at least one bounded solution satisfying conditions* (1.9).

Corollary 1.2. *Let all the conditions of Corollary* 1.1 *be fulfilled and*

$$\liminf_{t \to t_i} \int_{t}^{0} p_{ii}(s)ds = -\infty \ \ \text{for } i \in \{1,\dots,n\} \setminus \mathcal{N}_0(t_1,\dots,t_n). \tag{1.14}$$

Then for any $c_i \in R$ $(i \in \mathcal{N}_0(t_1,\dots,t_n))$ *system* (1.3) *has one and only one bounded solution satisfying conditions* (1.9).

Corollary 1.3. *Let there exist* $t_i \in R \cup \{-\infty,+\infty\}$, $b_i \in [0,+\infty[$, $b_{ik} \in [0,+\infty[$ $(i,k = 1,\dots,n)$ *such that the real part of every eigenvalue of the matrix* $(-\delta_{ik}b_i + b_{ik})_{i,k=1}^{n}$ *is negative and the inequalities*

$$\sigma(t,t_i)p_{ii}(t) \leq -b_i, \quad \left|p_{ii}(t)\int_{t}^{\tau_{ii}(t)} |p_{ik}(s)|ds\right| + (1-\delta_{ik})|p_{ik}(t)| \leq$$
$$\leq b_{ik} \quad (i,k = 1,\dots,n)$$

hold almost everywhere on R, *where* $\sigma(t,t_i) \equiv \operatorname{sgn}(t-t_i)$ *if* $t_i \in R$, $\sigma(t,t_i) \equiv 1$ *if* $t_i = -\infty$ *and* $\sigma(t,t_i) \equiv -1$ *if* $t_i = +\infty$. *Moreover, let*

$$\sup\left\{\int_{t}^{t+1}\left[\left|p_{ii}(s)\int_{s}^{\tau_{ii}(s)} |q_i(\xi)|d\xi\right| + |q_i(s)|\right]ds : t \in R\right\} <$$
$$< +\infty \quad (i = 1,\dots,n). \tag{1.15}$$

Then for any $c_i \in R$ $(i \in \mathcal{N}_0(t_1,\dots,t_n))$ *system* (1.3) *has one and only one bounded solution satisfying conditions* (1.9).

Corollary 1.1′. *Let there exist* $t_i \in \{-\infty,+\infty\}$ $(i=1,\dots,n)$, *a nonnegative constant matrix* $A=(a_{ik})_{i,k=1}^n$ *and a nonnegative number* a *such that* $r(A)<1$ *and conditions* (1.11) *and* (1.12) *are fulfilled. Then system* (1.3) *has at least one bounded solution.*

Corollary 1.2′. *Let all the conditions of Corollary* 1.1′ *be fulfilled and*

$$\liminf_{t\to t_i} \int_t^0 p_{ii}(s)\,ds = -\infty \quad (i=1,\dots,n).$$

Then system (1.3) *has one and only one bounded solution.*

Corollary 1.3′. *Let there exist* $\sigma_i \in \{-1,1\}$, $b_i \in [0,+\infty[$, $b_{ik} \in [0,+\infty[$ $(i,k=1,\dots,n)$ *such that the real part of every eigenvalue of the matrix* $(-\delta_{ik}b_i+b_{ik})_{i,k=1}^n$ *is negative and the inequalities*

$$\sigma_i p_{ii}(t) \le -b_i, \quad \left| p_{ii}(t) \int_t^{\tau_{ii}(t)} |p_{ik}(s)|ds \right| + (1-\delta_{ik})|p_{ik}(t)| \le \\ \le b_{ik} \quad (i,k=1,\dots,n)$$

hold almost everywhere on R. *Moreover, if conditions* (1.15) *are fulfilled, then system* (1.3) *has one and only one bounded solution.*

2. Lemma of the Existence of a Bounded Solution of System (1.1)

Let $t_i \in R \cup \{-\infty,+\infty\}$ $(i=1,\dots,n)$ and $(t_{0m})_{m=1}^{+\infty}$ and $(t_m^0)_{m=1}^{+\infty}$ be arbitrary sequences of real numbers such that

$$\begin{gathered} t_{0m} < t_m^0, \quad t_{0m} \le t_i \le t_m^0 \quad \big(i \in \mathcal{N}_0(t_1,\dots,t_n);\ m=1,2,\dots\big), \\ \lim_{m\to+\infty} t_{0m} = -\infty, \quad \lim_{m\to+\infty} t_m^0 = +\infty. \end{gathered} \tag{2.1}$$

For any natural number m and arbitrary functions $u \in C_{loc}(R,R)$ and $h \in L_{loc}(R,R)$ set

$$t_{im} = \begin{cases} t_i & \text{for } t_i \in R \\ t_{0m} & \text{for } t_i = -\infty \\ t_m^0 & \text{for } t_i = +\infty \end{cases}, \tag{2.2}$$

$$e_m(u)(t) = \begin{cases} u(t) & \text{for } t_{0m} \le t \le t_m^0 \\ u(t_{0m}) & \text{for } t < t_{0m} \\ u(t_m^0) & \text{for } t > t_m^0 \end{cases}, \tag{2.3}$$

$$l_{ikm}(u)(t) = l_{ik}(e_m(u))(t) \quad (i,k = 1,\dots,n) \tag{2.4}$$

and

$$\nu_{im}(h) = \max\bigg\{ \Big| \int_{t_{im}}^{t} \exp\Big(\int_s^t l_{ii}(1)(\xi)\,d\xi \Big) \big[\bar{l}_{ii}\big(|\eta(|h|)(\cdot,s)|\big)(s) + +|h(s)| \big]\,ds \Big| : \ t_{0m} \le t \le t_m^0 \bigg\}. \tag{2.5}$$

On the interval $[t_{0m}, t_m^0]$ consider the boundary value problem

$$y_i'(t) = \sum_{k=1}^n l_{ikm}(y_k)(t) + h_i(t) \quad (i = 1,\dots,n), \tag{2.6$_m$}$$

$$\begin{aligned} y_i(t_{im}) &= c_i \quad \text{for} \quad i \in \mathcal{N}_0(t_1,\dots,t_n), \\ y_i(t_{im}) &= 0 \quad \text{for} \quad i \in \{1,\dots,n\} \setminus \mathcal{N}_0(t_1,\dots,t_n). \end{aligned} \tag{2.7$_m$}$$

Lemma 2.1. *Let there exist a positive number ρ such that for any $h_i \in L_{loc}(R,R)$ $(i = 1,\dots,n)$, $c_i \in R$ $(i \in \mathcal{N}_0(t_1,\dots,t_n))$ and natural m every solution $(y_i)_{i=1}^n$ of problem* (2.6$_m$), (2.7$_m$) *admits the estimate*

$$\sum_{i=1}^n |y_i(t)| \le \rho \sum_{i=1}^n \big(|c_i| + \nu_{im}(h_i) \big) \quad \text{for} \quad t_{0m} \le t \le t_m^0, \tag{2.8}$$

where $c_i = 0$ as $i \in \{1,\dots,n\} \setminus \mathcal{N}_0(t_1,\dots,t_n)$. Moreover, let conditions (1.7) *hold. Then for any $c_i \in R$ $(i \in \mathcal{N}_0(t_1,\dots,t_n))$ system* (1.1) *has at least one bounded solution satisfying conditions* (1.9).

Proof. If $c_i = 0$ and $h_i(t) \equiv 0$ $(i = 1,\dots,n)$, then (2.8) implies that $y_i(t) \equiv 0$ $(i = 1,\dots,n)$, i.e., the homogeneous problem

$$y_i'(t) = \sum_{k=1}^n l_{ikm}(y_k)(t) \quad (i = 1,\dots,n),$$
$$y_i(t_{im}) = 0 \quad (i = 1,\dots,n)$$

has only the trivial solution. On the other hand, by (1.2), (2.3) and (2.4) for any $u \in C([t_{0m}, t_m^0], R)$ the inequalities

$$\big| l_{ikm}(u)(t) \big| \le \bar{l}_{ik}(1)(t) \|u\| \quad (i,k = 1,\dots,n) \tag{2.9}$$

hold almost everywhere on $[t_{0m}, t_m^0]$, where $\|u\| = \max\{|u(t)| : \ t_{0m} \le t \le t_m^0\}$. These facts imply that for any $h_i \in L_{loc}(R,R)$, $c_i \in R$ $(i \in$

$\mathcal{N}_0(t_1,\dots,t_n)$) and natural m the boundary value problem has one and only one solution (see [3], Theorem 1.1).

For arbitrarily fixed $c_i \in R$ ($i \in \mathcal{N}_0(t_1,\dots,t_n)$) and natural m denote by $(x_{im})_{i=1}^{m}$ the solution of the problem

$$x'_{im}(t) = \sum_{k=1}^{n} l_{ikm}(x_{km})(t) + q_i(t) \quad (i = 1,\dots,n), \tag{2.10}$$

$$x_{im}(t_{im}) = c_i \quad (i = 1,\dots,n), \tag{2.11}$$

where

$$c_i = 0 \quad \text{as} \quad i \in \{1,\dots,n\} \setminus \mathcal{N}_0(t_1,\dots,t_n),$$

and extend x_{im} ($i = 1,\dots,n$) on R by the equalities

$$x_{im}(t) = e_m(x_{im})(t) \quad \text{for} \quad t \in R \ \ (i = 1,\dots,n). \tag{2.12}$$

Then according to (1.7), (2.5) and (2.8) we have

$$\sum_{i=1}^{n} |x_{im}(t)| \le \rho \sum_{i=1}^{n} \big(|c_i| + \nu_{im}(q_i)\big) \le \rho^* \quad \text{for} \quad t \in R \ \ (m = 1,2,\dots), \tag{2.13}$$

where $\rho^* = \rho(\sum\limits_{i=1}^{n} |c_i| + a)$ is a nonnegative number independent of m.

By virtue of (2.9) and (2.13) from (2.10) we obtain

$$\sum_{i=1}^{n} |x'_{im}(t)| \le q(t) \quad \text{for almost all} \quad t \in R \ \ (m = 1,2,\dots),$$

where

$$q(t) = \sum_{i=1}^{n} \Big[\rho^* \sum_{k=1}^{n} \bar{l}_{ik}(1)(t) + |q_i(t)|\Big]$$

and $q \in L_{loc}(R,R)$. Consequently, the sequences $(x_{im})_{m=1}^{+\infty}$ ($i=1,\dots,n$) are uniformly bounded and equicontinuous on every compact interval. Without loss of generality, by Arzela–Ascoli's lemma we can assume that $(x_{im})_{m=1}^{+\infty}$ ($i = 1,\dots,n$) are uniformly convergent on every compact interval. Put

$$\lim_{m\to+\infty} x_{im}(t) = x_i(t) \quad \text{for} \quad t \in R \ \ (i = 1,\dots,n). \tag{2.14}$$

Then by (2.1), (2.3), and (2.12)

$$\lim_{m\to+\infty} e_m(x_{im})(t) = x_i(t)$$
$$\text{uniformly on every compact interval} \quad (i = 1,\dots,n). \tag{2.15}$$

Let m_0 be a natural number such that

$$t_{0m} < 0 < t_m^0 \quad (m = m_0, m_0+1,\dots).$$

Then by (2.10) we have

$$x_{im}(t) = x_{im}(0) + \int_0^t \Big[\sum_{k=1}^n l_{ikm}(x_{km})(s) + q_i(s)\Big]\,ds$$
$$\text{for}\quad t_{0m} \le t \le t_m^0 \quad (i = 1,\dots,n).$$

According to conditions (2.1), (2.4), (2.15) and the continuity of the operators $l_{ik} : C_{loc}(R,R) \to L_{loc}(R,R)$ $(i,k = 1,\dots,n)$ these equalities imply that

$$x_i(t) = x_i(0) + \int_0^t \Big[\sum_{k=1}^n l_{ik}(x_k)(s) + q_i(s)\Big]\,ds \quad \text{for} \quad t \in R \quad (i = 1,\dots,n),$$

i.e., $(x_i)_{i=1}^n$ is a solution of system (1.1). On the other hand, by virtue of (2.1), (2.2), and (2.14) from (2.11) and (2.13) we conclude that the vector function $(x_i)_{i=1}^n$ is bounded and satisfies conditions (1.9). □

3. Proof of the main results

Along with the notation introduced in Section 1, we shall also use some additional notation.

Z^{-1} is the matrix, inverse to the nonsingular $n \times n$ matrix Z.

E is the $n \times n$ unit matrix.

If $Z = (z_{ik})_{i,k=1}^n$, then $\|Z\| = \sum_{i,k=1}^n |z_{ik}|$.

The inequalities between the real column vectors $z = (z_i)_{i=1}^n$ and $\overline{z} = (\overline{z}_i)_{i=1}^n$ are understood componentwise, i.e.,

$$z \le \overline{z} \Leftrightarrow z_i \le \overline{z}_i \quad (i = 1,\dots,n).$$

Proof of Theorem 1.1. By (1.8) there exists a constant $\rho_0 > 1$ such that

$$\exp\left(\int_{t_i}^t p_{ii}(s)ds\right) < \rho_0 \quad \text{for} \quad t \in R \quad (i \in \mathcal{N}_0(t_i,\dots,t_n)). \tag{3.1}$$

On the other hand, by (1.5) the matrix $E - A$ is nonsingular and its inverse matrix $(E - A)^{-1}$ is nonnegative. Put

$$\rho = \rho_0 \|(E - A)^{-1}\|. \tag{3.2}$$

Let $(t_{0m})_{m=1}^\infty$ and $(t_m^0)_{m=1}^\infty$ be arbitrary sequences of real numbers satisfying conditions (2.1) and t_{im}, e_m, l_{ikm} and ν_{im} $(i,k = 1,2;\ m = 1,2,\dots)$ are the numbers and operators given by equalities (2.2)–(2.5). By Lemma 2.1, to prove Theorem 1.1 it is sufficient to show that for any $h_i \in L_{loc}(R,R)$

$(i=1,\dots,n)$, $c_i \in R$ $(i \in \mathcal{N}_0(t_1,\dots,t_n))$ and natural m an arbitrary solution $(y_i)_{i=1}^n$ of problem (2.6_m), (2.7_m) admits estimate (2.8), where $c_i = 0$ as $i \in \{1,\dots,n\} \setminus \mathcal{N}_0(t_1,\dots,t_n)$.

By (1.4) and (2.4), equation (2.6_m) implies

$$\begin{aligned}
l_{iim}(y_i)(t) &= l_{iim}(1)(t)y_i(t) + l_{iim}\big(y_i(\cdot) - y_i(t)\big)(t) = \\
&= l_{ii}(1)(t)y_i(t) + l_{iim}\big(\eta(y_i')(\cdot,t)\big)(t) = \\
&= l_{ii}(1)(t)y_i(t) + \sum_{k=1}^m l_{iim}\big(\eta(l_{ikm}(y_k))(\cdot,t)\big)(t) + \\
&\quad + l_{iim}\big(\eta(h_i)(\cdot,t)\big)(t), \\
y_i'(t) &= l_{ii}(1)(t)y_i(t) + \widetilde{h}_i(t) \quad (i=1,\dots,n),
\end{aligned}$$

and

$$\begin{aligned}
y_i(t) =& c_i \exp\Bigg(\int_{t_{im}}^t l_{ii}(1)(\xi)d\xi\Bigg) + \\
&+ \int_{t_{im}}^t \exp\Bigg(\int_s^t l_{ii}(1)(\xi)d\xi\Bigg)\widetilde{h}_i(s)ds \quad (i=1,\dots,n),
\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}
\widetilde{h}_i(t) = \sum_{k=1}^m \Big[l_{iim}\big(\eta(l_{ikm}(y_k))(\cdot,t)\big)(t) + (1-\delta_{ik})l_{ikm}(y_k)(t)\Big] + \\
+ l_{iim}\big(\eta(h_i)(\cdot,t)\big)(t) + h_i(t) \quad (i=1,\dots,n).
\end{aligned}$$

Set

$$\gamma_i = \max\big\{|y_i(t)| : t_{0m} \le t \le t_m^0\big\}, \quad \gamma = (\gamma_i)_{i=1}^n. \tag{3.4}$$

Then according to (1.2), (1.4), (2.3), and (2.4) we obtain

$$\begin{aligned}
|\widetilde{q}_i(t)| \le \sum_{k=1}^m \Big[\bar{l}_{ii}\big(|\eta(\bar{l}_{ik}(1))(\cdot,t)|\big)(t) + (1-\delta_{ik})\bar{l}_{ik}(1)(t)\Big]\gamma_k + \\
+ \bar{l}_{ii}\big(|\eta(|h_i|)(\cdot,t)|\big)(t) + |h_i(t)| \quad (i=1,\dots,n).
\end{aligned}$$

If along with these inequalities we take into account conditions (1.6) and (3.1) and notation (2.5), then from (3.3) we find

$$\gamma_i \le \sum_{k=1}^n a_{ik}\gamma_k + \rho_0|c_i| + \nu_{im}(h_i) \le \sum_{k=1}^n a_{ik}\gamma_k + \rho_0\big(|c_i| + \nu_{im}(h_i)\big),$$

i.e.,

$$(E-A)\gamma \le \rho_0\big(|c_i| + \nu_{im}(h_i)\big)_{i=1}^n.$$

But, as mentioned above, the matrix $E-A$ is nonsingular and $(E-A)^{-1}$ is nonnegative. Therefore the last inequality implies that

$$\gamma \le \rho_0(E-A)^{-1}\big(|c_i| + \nu_{im}(h_i)\big)_{i=1}^n.$$

Hence by (3.2) and (3.4) we obtain estimate (2.8). □

Proof of Theorem 1.2. By Theorem 1.1, system (1.1) has at least one bounded solution satisfying conditions (1.9). Consequently, to prove Theorem 1.2 it is sufficient to show that the homogeneous problem

$$x_i'(t) = \sum_{k=1}^n l_{ik}(x_k) \quad (i=1,\dots,n), \tag{3.5}$$

$$x_i(t_i) = 0 \quad \text{for} \quad i \in \mathcal{N}_0(t_1,\dots,t_n) \tag{3.6}$$

has no nontrivial bounded solution.

Let $(x_i)_{i=1}^n$ be a bounded solution of problem (3.5), (3.6) and

$$\gamma_i = \sup\big\{|x_i(t)| : t \in R\big\}, \quad \gamma = (\gamma_i)_{i=1}^n.$$

Then

$$\begin{aligned} l_{ii}(x_i)(t) &= l_{ii}(1)(t)x_i(t) + l_{ii}(x_i(\cdot) - x_i(t))(t) = \\ &= l_{ii}(1)(t)x_i(t) + l_{ii}\big(\eta(x_i')(\cdot,t)\big)(t) = \\ &= l_{ii}(1)(t)x_i(t) + \sum_{k=1}^n l_{ii}\big(\eta(l_{ik}(x_k))(\cdot,t)\big)(t) \end{aligned}$$

and

$$x_i'(t) = l_{ii}(1)(t)x_i(t) + \Delta_i(t) \quad (i=1,\dots,n), \tag{3.7}$$

where

$$\Delta_i(t) = \sum_{k=1}^n \Big[l_{ii}\big(\eta(l_{ik}(x_k))(\cdot,t)\big)(t) + (1-\delta_{ik})l_{ik}(x_k)(t)\Big] \quad (i=1,\dots,n)$$

and

$$|\Delta_i(t)| \le \sum_{k=1}^n \Big[\bar{l}_{ii}\big(|\eta(l_{ik}(1))(\cdot,t)|\big)(t) + (1-\delta_{ik})\bar{l}_{ik}(1)(t)\Big]\gamma_k \quad (i=1,\dots,n). \tag{3.8}$$

By (1.10) there exist $t_{im} \in R$ $(i \in \{1,\dots,n\} \setminus \mathcal{N}_0(t_1,\dots,t_n);\ m = 1,2,\dots)$ such that

$$\lim_{m\to+\infty} t_{im} = t_i, \quad \lim_{m\to+\infty} \int_{t_{im}}^{0} l_{ii}(1)(s)ds = -\infty$$
$$\text{for} \quad i \in \{1,\dots,n\} \setminus \mathcal{N}_0(t_1,\dots,t_n). \tag{3.9}$$

Set

$$t_{im} = t_i \quad (i \in \mathcal{N}_0(t_1,\dots,t_n);\ m = 1,2,\dots). \tag{3.10}$$

From (3.7) we have

$$x_i(t) = x_i(t_{im}) \exp\Bigg(\int_{t_{im}}^{t} l_{ii}(1)(\xi)d\xi\Bigg) +$$
$$+ \int_{t_{im}}^{t} \exp\Bigg(\int_{s}^{t} l_{ii}(1)(\xi)d\xi\Bigg)\Delta_i(s)ds \quad (i = 1,\dots,n).$$

Hence by virtue of conditions (3.6) and (3.8)–(3.10) we find

$$x_i(t) = \int_{t_i}^{t} \exp\Bigg(\int_{s}^{t} l_{ii}(1)(\xi)d\xi\Bigg)\Delta_i(s)ds \quad (i = 1,\dots,n).$$

These equalities and conditions (1.6), (3.6) and (3.8)–(3.10) yield

$$\gamma_i \le \sum_{k=1}^{n} a_{ik}\gamma_k \quad (i = 1,\dots,n),$$

i.e.,

$$(E - A)\gamma \le 0.$$

Hence the nonnegativity of the matrix $(E-A)^{-1}$ and vector γ implies that $\gamma = 0$, i.e., $x_i(t) \equiv 0$ $(i = 1,\dots,n)$. $\square$

If

$$l_{ik}(u)(t) \equiv p_{ik}(t)u(\tau_{ik}(t)) \quad (i,k = 1,\dots,n),$$

then system (1.1) admits form (1.3). In that case

$$\overline{l}_{ik}(u)(t) \equiv |p_{ik}(t)|u(\tau_{ik}(t)) \quad (i,k = 1,\dots,n),$$

$$\overline{l}_{ii}\big(|\eta(\overline{l}_{ik}(1))(\cdot,t)|\big)(t) \equiv \Bigg|p_{ii}(t) \int_{t}^{\tau_{ii}(t)} |p_{ik}(\xi)|d\xi\Bigg| \quad (i,k = 1,\dots,n)$$

and

$$\bar{l}_{ii}\big(|\eta(|q_i|)(\cdot,t)|\big)(t) \equiv \left|p_{ii}(t)\int\limits_t^{\tau_{ii}(t)} |q_i(\xi)|d\xi\right| \quad (i=1,\dots,n)$$

and conditions (1.6)–(1.8) and (1.10) take the form of (1.11)–(1.13) and (1.14). Theorems 1.1 and 1.2 (Theorems 1.1′ and 1.2′) give rise to Corollaries 1.1 and 1.2 (Corollaries 1.1′ and 1.2′).

Finally, note that if the conditions of Corollary 1.3 (Corollary 1.3′) hold, then the conditions of Corollary 1.2 (Corollary 1.2′) hold too.*

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*See [2], the proof of Corollary 6.11.