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A CONTACT PROBLEM OF THE INTERACTION OF A SEMI-FINITE INCLUSION WITH A PLATE

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ABSTRACT. A piecewise-homogeneous plane made up of twodifferent materials and reinforced by an elastic unclusion is considered on a semi-finite section where the different materials join. Vertical and horizontal forces are applied to the inclusion which haz a variable thichness and a variable elasticity modulus.

Under certain conditions the problem is reduced to integrodifferential equations of third order. The solution is constructed effectively by applying the methods of theory of analytic functions to a boundary value problem of the Carleman type for a strip. Asymptotic estimates of normal contact stress are obtained.

We shall consider an elastic composite plate by which we understand an unbounded elastic medium composed of two half-planes $y > 0$ and $y < 0$ having different elastic constants (E_+, μ_+ and E_-, μ_-). It is assumed that the plate is subjected to plane deformation and, on the semi-axis $(0, \infty)$, is strengthened by an inclusion of variable thickness $h_0(x)$, with elasticity modulus $E_0(x)$ and Poisson's ratio ν_0.

Contact problems of the interaction of an elastic body with thin elastic elements in the form of stringers and inclusions as well as relevant bibliographic references are given in the monographs [1–4].

The inclusion is assumed to be a thin plate subjected to the action of vertical and horizontal forces of intensities $p_0(x)$ and $\tau_0(x)$, respectively, while the plate is assumed to be free from action ($p_0(x)$, $\tau_0(x)$ are the continuous functions on the semi-axis). The stress field undergoes discontinuity when passing across the semi-axis, while the stress and displacement fields do not become discontinuous when passing across the remaining part of the Ox-axis.

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The principal equilibrium equations $(0,x)$-part of the inclusion are

$$\begin{gathered}E_1(x)\,\frac{du_1}{dx}=\int\limits_0^x[\tau(t)-\tau_0(t)]dt,\\ \frac{d^2}{dx^2}\,D_1(x)\,\frac{d^2v_1}{dx^2}=p(x)-p_0(x),\quad x>0,\end{gathered}\tag{1}$$

where

$$\begin{gathered}\tau(x)\equiv\tau^-(x)-\tau^+x,\quad E_1(x)=\frac{E_0(x)h_0(x)}{1-\nu_0^2},\\ p(x)\equiv p^-(x)-p^+(x),\quad D_1(x)=\frac{E_0(x)h_0^3(x)}{12(1-\nu_0^2)},\end{gathered}$$

$p^{\pm}(x)$ and $\tau^{\pm}(x)$ are respectively the unknown normal and tangential contact stresses on the upper and the lower inclusion banks; u_1 and v_1 are respectively the horizontal and vertical displacements of inclusion points ($\tau(x)\equiv 0$, $p(x)\equiv 0$, for $x<0$).

The inclusion equilibrium conditions

$$\begin{gathered}\int\limits_0^\infty[\tau(t)-\tau_0(t)]dt=0,\quad \int\limits_0^\infty[p(t)-p_0(t)]dt=0,\\ \int\limits_0^\infty t[p(t)-p_0(t)]dt=0\end{gathered}\tag{2}$$

are obtained assuming that the end cross-sections of the inclusion are free from of external forces.

The deformation of the inclusion is assumed to be compatible with that of the elastic composite plate with a defect on the semi-axis $x>0$. By virtue of the results of [3], [5], to construct discontinuous solutions of the biharmonic equation we write horizontal and vertical deformations of the $0x$-axis as

$$\left.\begin{aligned}u'(x)&=-Ap(x)+\frac{B}{\pi}\int\limits_0^\infty\frac{\tau(t)dt}{t-x},\\ v'(x)&=A\tau(x)+\frac{B}{\pi}\int\limits_0^\infty\frac{p(t)dt}{t-x},\end{aligned}\right.\qquad x>0,\tag{3}$$

where

$$A = \frac{a_+b_-(b_+ + a_-) - a_-b_+(b_- + a_+)}{2c},$$
$$B = \frac{a_+b_-(b_+ + a_-) + a_-b_+(b_- + a_+)}{2c},$$
$$c = 4(c_+ + c_-)^2 - [(c_- - c_+) - (d_- - d_+)]^2, \quad a_\pm = 3c_\pm - d_\pm,$$
$$b_\pm = c_\pm + d_\pm, \quad c_\pm = \frac{1-\mu_\pm^2}{E_\pm}, \quad d_\pm = \frac{\mu_\pm(1+\mu_\pm)}{E_\pm}.$$

Using the conditions of contact between the inclusion and the plate

$$u'(x) = u_1'(x),$$
$$v'(x) = v_1'(x)$$

and substituting formulas (3) into (1), we obtain a system of integrodifferential equations

$$\begin{aligned} -A\psi''(x) + \frac{B}{\pi}\int_0^\infty \frac{\varphi'(t)dt}{t-x} &= \frac{\varphi(x)}{E_1(x)} - \frac{f_1(x)}{E_1(x)}, \\ A\varphi'(x) + \frac{B}{\pi}\int_0^\infty \frac{\psi'''(t)dt}{t-x} &= \frac{\psi(x)}{D_1(x)} - \frac{f_2(x)}{D_1(x)}, \end{aligned} \qquad x > 0, \qquad (4)$$

where

$$\varphi(x) = \int_0^x \tau(t)dt, \quad \psi(x) = \int_0^x dt\int_0^t p(\tau)d\tau$$
$$f_1(x) = \int_0^x \tau_0(t)dt, \quad f_2(x) = \int_0^x dt\int_0^t p_0(\tau)d\tau.$$

The unknown functions are to satisfy the conditions

$$\varphi(0) = 0, \quad \varphi(\infty) = T_0,$$
$$\psi(0) = 0, \quad \psi(\infty) = M_0,$$
$$\psi'(0) = 0, \quad \psi'(\infty) = P_0,$$

where $T_0 = \int_0^\infty \tau_0(t)dt$, $P_0 = \int_0^\infty p_0(t)dt$, $M_0 = \int_0^\infty tp_0(t)dt$.

If the plate is homogeneous $A = 0$, system (4) splits into two independent equations

$$\frac{B}{\pi}\int_0^\infty \frac{\varphi'(t)dt}{t-x} = \frac{\varphi(x)}{E_1(x)} - \frac{f_1(x)}{E_1(x)}, \tag{5}$$

$$x > 0.$$

$$\frac{B}{\pi}\int_0^\infty \frac{\psi'''(t)dt}{t-x} = \frac{\psi(t)}{D_1(x)} - \frac{f_2(x)}{D_1(x)}, \tag{6}$$

In the case $p_0(x) = 0$, i.e., when the inclusion undergoes only tension, we obtain equation (5) considered in [6], while for $\tau_0(x) = 0$, i.e., when the inclusion is bent under the action of vertical forces $p_0(x)$, we have one integrodifferential equation (6).

Let us consider equation (6) under the boundary conditions

$$\psi(0) = 0, \ \ \psi(\infty) = M_0,$$
$$\psi'(0) = 0, \ \ \psi'(\infty) = P_0.$$

After introducing the notation $g(x) = \psi(x) - f_2(x)$, equation (6) takes the form

$$g(x) - \frac{B}{\pi}D_1(x)\int_0^\infty \frac{g'''(t)dt}{t-x} = \frac{BD_1(x)}{\pi}\int_0^\infty \frac{p_0'(t)dt}{t-x}, \quad x > 0, \quad B > 0, \tag{7}$$

provided that $g(0) = g(\infty) = 0$, $g'(0) = g'(\infty) = 0$.

Let the bending rigidity of the inclusion change according to the law $D_1(x) = h_0 x^n$ ($h_0 = const > 0$, $n \geq 0$ is any real number). A solution of equation (7) will be sought for in the class of functions whose second derivative may have nonintegrable singularities at the integration interval ends (i.e., in the class of functions of the type $g''(x) = x^{-3/2}\widetilde{g}_0(x)$, where $\widetilde{g}_0(x)$ is a function satisfying the Hölder condition on the semi-axis $x > 0$), while the corresponding integrals will be understood in the regularized sense [7]. The latter circumstance is important for the problem posed to be correct, since for this class the energy integral of the bent plate converges like the nonproper one, which enables one to investigate the solution uniqueness of the problem posed. Let us assume that the principal vector and the principal moment of external forces acting on the inclusion be equal to zero and that $p_0(0) = 0$, $|p_0(x)| < \frac{c}{x^{2+\delta}}$, $x \to \infty$, $\delta > 0$.

By change of the variables $x = e^{\xi}$, $t = e^{\zeta}$ in equation (7) we have

$$g_0(\xi) - \frac{h_0 B}{\pi} e^{n\xi} \int_{-\infty}^{\infty} \frac{[g_0'''(\zeta) - 3g_0''(\zeta) + 2g_0'(\zeta)]e^{-3\zeta} d\zeta}{1 - e^{\xi-\zeta}} =$$

$$= \frac{Bh_0}{\pi} e^{n\xi} \int_{-\infty}^{\infty} \frac{\widetilde{p}_0'(\zeta) d\zeta}{e^{\zeta} - e^{\xi}}, \quad -\infty < \xi < \infty,$$

where $g_0(\xi) = g_0(e^{\xi})$, $\widetilde{p_0}(\xi) = p_0(e^{\xi})$.

Rewrite this equation as

$$g_0(\xi)e^{-k\xi} - \frac{h_0 B}{\pi} \int_{\infty}^{\infty} \frac{[g_0'''(\zeta) - 3g_0''(\zeta) + 2g_0'(\zeta)]e^{-3(\xi-\zeta)} d\zeta}{1 - e^{\xi-\zeta}} =$$

$$= e^{3\xi} \frac{Bh_0}{\pi} \int_{-\infty}^{\infty} \frac{\widetilde{p}_0'(\zeta) d\zeta}{e^{\zeta} - e^{\xi}}, \quad -\infty < \xi < \infty, \tag{8}$$

where $k = n - 3$, $g_0(\pm\infty) = 0$, $g_0'(\pm\infty) = 0$.

If we consider the case with k as a positive integer, i.e., $n > 3$, and perform the Fourier transform of both sides of equation (8), then we obtain

$$\Psi(s + ik) + \lambda s \coth \pi s \ (is + 1)(is + 2)\Psi(s) = F(s), \quad -\infty < s < \infty, \tag{9}$$

where $\lambda = \frac{Bh_0}{\sqrt{2\pi}}$, $\Psi(s)$ is the Fourier transform of the function $g_0(\xi)$ we are seeking for, while $F(s)$ is the Fourier transform of the right-hand side of equation (8) whose representation implies that $F(s)$ is analytically extendable in a strip $-1 < \operatorname{Im} x \leq 2$ and, for sufficiently large $|s|$, has the form $F(s) = O(1/|s|^{3+\varepsilon})$, where ε is an arbitrarily small positive integer.

In equation (8), for $\xi = \zeta$ the integral is understood in a sense of the Cauchy principal value, while the Fourier transform means a generalized transform.

The problem is posed as follows: find a function $\Psi(z)$ which is analytic in a strip, continuously extendable on the strip boundary, vanishes at infinity and satisfies condition (9).

The problem coefficient can be written as

$$s \coth \pi s \ (s - i)(s - 2i) =$$

$$= -is \coth \pi s \tanh \frac{\pi s}{2k} \frac{\sinh \frac{\pi}{2k}(s + ik)}{\sinh \frac{\pi}{2k} s} \frac{s - i}{s + 2i} \frac{2s + ik}{2s - ik} (s^2 + 4) \frac{2s - ik}{2s + ik}.$$

The function $G_0(s) \equiv \coth \pi s \tanh \frac{\pi s}{2k} \frac{s-i}{s+2i} \frac{2s+ik}{2s-ik}$ is continuous on the entire exis and $G_0(\infty) = G_0(-\infty) = 1$. It is easy to verify that $\operatorname{Ind} G_0(s) = 0$

and the branch of the function $\ln G_0(s)$ that vanishes at infinity is integrable on the entire axis.

As shown in [8, 9], the function $G_0(s)$ can be represented as

$$G_0(s) = \frac{\chi_0(s+ik)}{\chi_0(s)}, \quad -\infty < s < \infty, \tag{10}$$

where

$$\chi_0(z) = \exp\Big\{\frac{1}{2ki}\int\limits_{-\infty}^{\infty} \coth\frac{\pi}{k}(t-z)\ln G_0(t)dt\Big\}, \quad 0 < \operatorname{Im} z < k.$$

The function s^2+4 can be written as

$$s^2+4 = \frac{\chi_1(s+ik)}{\chi_1(s)}, \tag{11}$$

where $\chi_1(z) = K^{\frac{-2iz-k}{k}}\frac{\Gamma\left(\frac{2-iz}{k}\right)}{\Gamma\left(\frac{k+2+iz}{k}\right)}$, and the number λ as

$$\lambda = \frac{\chi_2(s+ik)}{\chi_2(s)} \tag{12}$$

where $\chi_2(z) = \exp(-iz\ln\sqrt[k]{\lambda})$, $0 < \operatorname{Im} z < k$.

If we substitute (10), (11) and (12) into condition (9) and introduce the notation

$$\chi_3(z) = \frac{\chi_0(z)\chi_1(z)\chi_2(z)\sinh\frac{\pi z}{2k}}{z(z-ik/2)},$$

we obtain

$$\frac{\Psi(s+ik)}{\chi_3(s+ik)} - \frac{\Psi(s)(k-is)}{\chi_3(s)} = \frac{F(s)}{\chi_3(s+ik)}, \quad -\infty < s < \infty. \tag{13}$$

The function $k - is$ can be represented as

$$k - is = \frac{\chi_4(s+ik)}{\chi_4(s)},$$

where $\chi(z) = K^{-iz/k}\Gamma\left(\frac{k-iz}{k}\right)$, $0 < \operatorname{Im} z < k$.

If we introduce one more notation

$$\chi(z) = \chi_3(z)\chi_4(z),$$

then condition (13) takes the form

$$\frac{\Psi(s+ik)}{\chi(s+ik)} - \frac{\Psi(s)}{\chi(s)} = \frac{F(s)}{\chi(s+ik)}, \quad -\infty < s < \infty. \tag{14}$$

The function $\chi(z)$ is holomorphic in a strip $0 < \operatorname{Im} z < k$ except for the point $z = ik/2$ at which it has a pole of first order. Let us investigate its behavir for large $|z|$.

The functions $\chi_0(z)$ and $\chi_2(z)$ are bounded throughout the entire strip, while for sufficiently large $|z|$ the functions $\chi_1(z)$ and $\chi_4(z)$ admit estimates $\chi_1(z) = O\big(|t|^{2\tau/k-1}\big)$, $\chi_4(z) = O\big(|t|^{1/2+\tau/k}\big)e^{-\frac{\pi}{2k}|t|}$, $z = t + i\tau$, $0 \le \tau \le k$. Hence it follows that for sufficiently large $|z|$ the function $\chi(z)$ admits an estimate

$$|\chi(z)| = O\big(|t|^{3\tau/k-5/2}\big), \quad 0 \le \tau \le k. \tag{15}$$

Thus the function $\frac{\Psi(z)}{\chi(z)}$ is holomorphic in the strip and the solution of problem (15) can be represented as

$$\Psi(z) = \frac{\chi(z)\cosh\frac{\pi}{k}z}{2ik}\int_{-\infty}^{\infty}\frac{F(t)dt}{\chi(t+ik)\cosh\frac{\pi}{k}t\sinh\frac{\pi}{k}(t-z)}, \quad 0 < \operatorname{Im} z < k. \tag{16}$$

By virtue of formulas analogous to Sokhotskii–Plemelj ones, representation (16) yields

$$\Psi(t_0) = \frac{\chi(t_0)F(t_0)}{2\chi(t_0+ik)} + \frac{\chi(t_0)\cosh\frac{\pi}{k}t_0}{2ik}\int_{-\infty}^{\infty}\frac{F(t)dt}{\chi(t+ik)\cosh\frac{\pi}{k}t\sinh\frac{\pi}{k}(t-t_0)},$$

$$\Psi(t_0+ik) = -\frac{F(t_0)}{2} + \frac{\chi(t_0+ik)\cosh\frac{\pi}{k}t_0}{2ik}\int_{-\infty}^{\infty}\frac{F(t)dt}{\chi(t+ik)\cosh\frac{\pi}{k}t\sinh\frac{\pi}{k}(t-t_0)}.$$

Taking into account (15) and the behavior of $F(t)$ for large $|t|$, by the latter formulas we conclude that for $0 < \operatorname{Im} z < k$ the function $\Psi(z)$ represented by (16) vanishes at infinity with order greater than three.

Condition (9) implies that for $0 < k \le 2$ the function $\Psi(z)$ is analytically continuable in the strip $0 < \operatorname{Im} z \le 2$.

For $k > 2$, using the formula $g''(x) = \frac{g_0''(\ln x) - g_0'(\ln x)}{x^2}$ and recalling the nature of $\Psi(z)$, by the Cauchy formula and the inverse Fourier transform we obtain

$$g_0'(\ln x) = \frac{-i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} t\Psi(t)e^{-it\ln x}dt = -\frac{-ix^k}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(t+ik)\Psi(t+ik)e^{-it\ln x}dt,$$

$$g_0''(\ln x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} t^2\Psi(t)e^{-it\ln x}dt = \frac{x^k}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(t+ik)^2\Psi(t+ik)e^{-it\ln x}dt,$$

while the contact force we are seeking behaves in the neighborhood of the point $x=0$ in the manner as follows:

$$p(x)-p_0(x)\equiv g''(x)=x^{k-2}\widetilde{g}(x),\quad k>2,$$

where $\widetilde{g}(x)$ is a continuous function on the semi-axis $x>0$.

For $0<k\le 2$, recalling that $\Psi(z)$ is analytically continuable in the strip $0<\operatorname{Im} z\le 2$, by the Cauchy formula we obtain as above $g''(x)=O(1)$ in the neighborhood of $x=0$.

Now let us consider the case with $k<0$, i.e., with $0\le n<3$. Condition (9) takes the form

$$\Psi(s-ip)+\lambda s\coth\pi s\ (is+1)(is+2)\Psi(s)=F(s),\quad -\infty<s<\infty, \tag{9'}$$

where $p=-k$, $p>0$.

The problem is formulated as follows: find a function which is analytic in the strip $-p<\operatorname{Im} z<p$ except for a finite number of points lying in the strip $0<\operatorname{Im} z<p$ at which this function may have poles, is continuously extendable on the strip boundary, vanishes at infinity and satisfies $(9')$.

Obviously, if we can find a function which is holomorphic in the strip $-p<\operatorname{Im} z<0$, continuous on the strip boundary and satisfies condition $(9')$, then the function

$$\Psi_1(z)=\begin{cases}\Psi(z), & -p<\operatorname{Im} z<0,\\[2mm] \dfrac{F(z)-\Psi(z-ip)}{\lambda z\coth\pi z\ (iz+1)(iz+2)}, & 0<\operatorname{Im} z<p,\end{cases}$$

with poles at the points $z=im/2$, $m=1,3,\dots$, will be a solution of the problem under consideration.

After writing the coefficient of problem $(9')$ in the form

$$s\coth\pi s(s-i)(s-2i)=$$
$$=is\coth\pi s\tanh\frac{\pi s}{2p}\,\frac{\sinh\frac{\pi(s-ip)}{2p}}{\sinh\frac{\pi s}{2p}}\,\frac{s-i}{s+2i}\,\frac{2s+ip}{2s-ip}\,(s^2+4)\,\frac{2s-ip}{2s+ip},$$

the function $\widetilde{G}_0(s)\equiv\coth\pi s\tanh\frac{\pi s}{2p}\frac{s-i}{s+2i}\frac{2s+ip}{2s-ip}$ can be represented as

$$\widetilde{G}_0(s)=\frac{\widetilde{\chi}_0(s-ip)}{\widetilde{\chi}_0(s)},\ \ -\infty<s<\infty, \tag{10'}$$

where

$$\widetilde{\chi}_0(z)=\exp\bigg\{-\frac{1}{2pi}\int\limits_{-\infty}^{\infty}\ln\widetilde{G}_0(t)\coth\frac{\pi(t-z)}{p}dt\bigg\},\quad 0<\operatorname{Im} z<p.$$

The function s^2+4 has a representation

$$s^2+4=\frac{\widetilde{\chi}_1(s)}{\widetilde{\chi}_1(s-ip)}, \tag{11'}$$

where $\widetilde{\chi}_1(z)=P^{\frac{-2iz+p}{p}}\frac{\Gamma\left(\frac{2+p-iz}{p}\right)}{\Gamma\left(\frac{2+iz}{p}\right)}$, and the number λ is represented as

$$\lambda=\frac{\widetilde{\chi}_2(s)}{\widetilde{\chi}_2(s-ip)}, \quad -\infty<s<\infty, \tag{12'}$$

where $\widetilde{\chi}_2(z)=\exp(-iz\ln\sqrt[p]{\lambda})$, $-p<\operatorname{Im}z<0$.

On substituting (10′), (11′) and (12′) into condition (9′) and introducing the notation

$$\widetilde{\chi}_3(z)=\frac{\widetilde{\chi}_0(z)\sinh\frac{\pi z}{2p}}{z\widetilde{\chi}_1(z)\widetilde{\chi}_2(z)}(z+ip/2),$$

we obtain

$$\frac{\Psi(s-ip)}{\widetilde{\chi}_3(s-ip)}-\frac{\Psi(s)(p+is)}{\widetilde{\chi}_3(s)}=\frac{F(s)}{\widetilde{\chi}_3(s-ip)}, \quad -\infty<s<\infty. \tag{13'}$$

The function $p+is$ is represented as

$$p+is=\frac{\widetilde{\chi}_4(s-ip)}{\widetilde{\chi}_4(s)},$$

where $\widetilde{\chi}_4(z)=P^{iz/p}\Gamma\left(\frac{p+iz}{p}\right)$, $-p<\operatorname{Im}z<0$. If we introduce the notation

$$\widetilde{\chi}(z)=\widetilde{\chi}_3(z)\widetilde{\chi}_4(z),$$

then condition (13′) can be rewritten as

$$\frac{\Psi(s-iz)}{\widetilde{\chi}(s-ip)}-\frac{\Psi(s)}{\widetilde{\chi}(s)}=\frac{F(s)}{\widetilde{\chi}(s-ip)}, \quad -\infty<s<\infty. \tag{14'}$$

The function $\widetilde{\chi}(z)$ is holomorphic in the strip $-p<\operatorname{Im}z<0$, the functions $\widetilde{\chi}_0(z)$ and $\widetilde{\chi}_2(z)$ are bounded throughout the strip, for sufficiently large $|z|$ the functions $\widetilde{\chi}_1(z)$ and $\widetilde{\chi}_4(z)$ admit the estimates $\widetilde{\chi}_1(z)=O\left(|t|^{2\tau/p+1}\right)$, $\widetilde{\chi}_4(z)=O\left(|t|^{1/2-\tau/p}\right)e^{-\frac{\pi}{2p}|t|}$, $z=t+i\tau$, and for large $|z|$ the function $\widetilde{\chi}(z)$ admits an estimate

$$|\widetilde{\chi}(z)|=O(|t|^{-3\tau/p-1/2}), \quad -p\le\tau\le 0. \tag{15'}$$

The function $\frac{\Psi(z)}{\widetilde{\chi}(z)}$ is holomorphic in the strip $-p<\operatorname{Im}z<0$ except for the point $z=-ip/2$ at which it may have a pole of first order, and the solution of the boundary value problem $(14')$ is given by the formula

$$\Psi(z)=\frac{\widetilde{\chi}(z)\cosh\frac{\pi}{p}z}{2ip}\int\limits_{-\infty}^{\infty}\frac{F(t)dt}{\widetilde{\chi}(t-ip)\cosh\frac{\pi}{p}t\sinh\frac{\pi}{p}(t-z)}+$$
$$+C_0\tanh\frac{\pi}{p}z\widetilde{\chi}(z)+C_1\widetilde{\chi}(z),\quad -p<\operatorname{Im}z<0, \tag{16'}$$

where C_0 and C_1 are any constants.

Condition (9') implies that the function $\Psi(z)$ is analytically continuable in the strip $-3<\operatorname{Im}z<0$. On choosing constants such that $C_0=C_1=0$, the function $\Psi(z)$ given by $(16')$ will be vanishable at infinity with order greater than three. The unknown function $\Psi_1(z)$ has poles at the points $z=i/2,\,3i/2$ and vanishes at infinity with its order unchanged.

Similarly to the above, using the Cauchy formula and the theorem of residues in the neighborhood of the point $x=0$, we obtain the representation $g''(x)=x^{-3/2}\widetilde{g}_1(x)$, where $\widetilde{g}_1(x)$ is continuous on the semi-axis $x\geq 0$.

The results obtained can be formulated as

Theorem 1. *If the function $p_0(x)$ is integrable and bounded on the semi-axis $x\geq 0$ and, moreover, $p_0(0)=0$, $p_0(x)=O(x^{-2-\delta})$ $(x\to\infty)$, $\int\limits_0^\infty p_0(t)dt=0$, $\int\limits_0^\infty tp_0(t)dt=0$, then in the neighborhood of the point $x=0$ the normal contact stress $p(x)$ admits an estimate*

$$p(x)=p_0(x)+\begin{cases}x^{n-5}\widetilde{p}(x) & for\ \ n>5,\\ O(1) & for\ \ 3<n\leq 5,\\ x^{-3/2}\widetilde{p}_1(x) & for\ \ 0\leq n<3,\end{cases}$$

where $\widetilde{p}(x)$ and $\widetilde{p}_1(x)$ are continuous functions on the semi-axis $x\geq 0$.

Remark. For $n=3$ condition (9) or $(9')$ takes the form

$$\Psi(s)[1+\lambda s\coth\pi s\ \ (is+1)(is+2)]=F(s),\quad -\infty<s<\infty. \tag{9''}$$

By considering the equation

$$1+\lambda s\coth\pi s\ \ (is+1)(is+2)=0 \tag{10''}$$

we can prove that it has no complex root $s_0=\alpha+i\beta$, where $\alpha\geq 0$, $0\leq\beta\leq 1/2$.

Indeed, after isolating the real and the imaginary part from equation $(10'')$ we obtain the system of equations

$$\begin{gathered} 1+\frac{\lambda(P\sin 2\pi\beta - Q\sinh 2\pi\alpha)}{2(\sin^2\pi\beta+\sinh^2\pi\alpha)}=0, \\ \frac{Q\sin 2\pi\beta + P\sinh 2\pi\alpha}{\sin^2\pi\beta+\sinh^2\pi\alpha}=0, \end{gathered} \tag{11''}$$

where

$$\begin{gathered} P=\beta(\beta-1)(\beta-2)+2\alpha^2(1-\beta), \\ Q=6\alpha\beta-3\alpha\beta^2-2\alpha+\alpha^3. \end{gathered}$$

Let system $(11'')$ have a solution (α_0,β_0), where $\alpha_0\ge 0$, $0\le\beta_0\le 1/2$. After finding from the second equation of system $(11'')$ $\sinh 2\pi\alpha_0 = -\frac{Q}{P}\sin 2\pi\beta_0$ and substituting into the first equation, we obtain

$$1+\frac{\lambda(P^2+Q^2)\sin 2\pi\beta_0}{2P(\sin^2\pi\beta_0+\sinh^2\pi\alpha_0)}=0.$$

Since $P>0$ and $\sin 2\pi\beta_0\ge 0$ for $0\le\beta_0\le 1/2$, the latter equality does not hold.

Thus we have proved

Theorem 2. *In the conditions of Theorem* 1 *and for* $n=3$, *the normal contact stress admits, in the neighborhood of the point* $x=0$, *a representation*

$$p(x)=p_0(x)+O(x^{-3/2+\delta_0}),$$

where $\delta_0>0$.

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