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THE THREE-DIMENSIONAL PROBLEM OF STATICS OF THE ELASTIC MIXTURE THEORY WITH DISPLACEMENTS GIVEN ON THE BOUNDARY

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ABSTRACT. The first three-dimensional boundary value problem is considered for the basic equations of statics of the elastic mixture theory in the finite and infinite domains bounded by the closed surfaces. It is proved that this problem splits into two problems whose investigation is reduced to the first boundary value problem for an elliptic equation which structurally coincides with an equation of statics of an isotropic elastic body. Using the potential method and the theory of Fredholm integral equations of second kind, the existence and uniqueness of the solution of the first boundary value problem is proved for the split equation.

Basic homogeneous equations of statics of the elastic mixture theory have the form [1]

$$\begin{aligned} &a_1\Delta u' + b_1 \operatorname{grad}\operatorname{div} u' + c\Delta u'' + d \operatorname{grad}\operatorname{div} u'' = 0,\\ &c\Delta u' + d \operatorname{grad}\operatorname{div} u' + a_2\Delta u'' + b_2 \operatorname{grad}\operatorname{div} u'' = 0, \end{aligned} \tag{1}$$

where a_1, b_1, c, d, a_2, b_2 are the coefficients kharacterizing the physical properties of an elastic mixture, u' and u'' are partial displacements.

The problem to be considered in this paper is formulated as follows: given a continuous displacement vector on the boundary S, in the domain D^+ (or D^-) find a solution $u(u',u'') \in C(\overline{D}^{\pm}) \cap C^2(D^{\pm})$ of equation (1). This problem is investigated in the space $C^{1,\alpha}(\overline{D}^{\pm}) \cap C^2(D^{\pm})$ in [1] by the method of potentials and the theory of singular integral equations.

Here we give a different technique of solving the above problem. Our investigation is carried out using Fredholm integral equations of second kind.

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Instead of the vectors u' and u'' we introduce the vectors

$$v' = u' + X_1 u'', \quad v'' = u' + X_2 u'', \tag{2}$$

where X_1 and X_2 are the roots of the quadratic equation

$$\varepsilon_2 X^2 - (\varepsilon_4 - \varepsilon_1)X - \varepsilon_3 = 0. \tag{3}$$

Here the coefficients ε_1, ε_2, ε_3, ε_4, are defined as follows [2]:

$$\begin{aligned}
&\delta_0\varepsilon_1 = 2(a_2b_1 - cd) + b_1b_2 - d^2, \ \ \delta_0\varepsilon_2 = 2(da_1 - cb_1),\\
&\delta_0\varepsilon_3 = 2(da_2 - cb_2), \ \ \delta_0\varepsilon_4 = 2(a_1b_2 - cd) + b_1b_2 - d^2,\\
&\quad \delta_0 = (2a_1+b_1)(2a_2+b_2)-(2c+d)^2 \equiv 4\Delta_0 d_1 d_2, \ \ \Delta_0 = m_1m_3 - m_2^2 > 0,\\
&\quad d_1 = (a_1 + b_1)(a_2 + b_2) - (c + d)^2 > 0, \ \ d_2 = a_1a_2 - c^2 > 0,\\
&\quad m_1 = l_1 + \frac{l_4}{2}, \ \ m_2 = l_2 + \frac{l_5}{2}, \ \ m_3 = l_3 + \frac{l_6}{2}, \ \ l_1 = \frac{a_2}{d_2}, \ \ l_2 = -\frac{c}{d_2},\\
&\quad l_3 = \frac{a_1}{d_2}, \ \ l_1 + l_4 = \frac{a_2 + b_2}{d_1}, \ \ l_2 + l_5 = -\frac{c + d}{d_1}, \ \ l_3 + l_6 = \frac{a_1 + b_1}{d_1}.
\end{aligned} \tag{4}$$

If the equality $\varepsilon_2 = \varepsilon_3 = 0$ holds, then by (4) we obtain

$$\frac{b_1}{a_1} = \frac{d}{c} = \frac{b_2}{a_2} = \lambda. \tag{5}$$

The substitution of these values into (1) gives

$$\begin{aligned}
&a_1(\Delta u' + \lambda \operatorname{grad}\operatorname{div} u') + c(\Delta u'' + \lambda \operatorname{grad}\operatorname{div} u'') = 0,\\
&c(\Delta u' + \lambda \operatorname{grad}\operatorname{div} u') + a_2(\Delta u'' + \lambda \operatorname{grad}\operatorname{div} u'') = 0.
\end{aligned}$$

Since $a_1a_2 - c^2 > 0$, we now have $\Delta u' + \lambda \operatorname{grad}\operatorname{div} u' = 0$, $\Delta u'' + \lambda \operatorname{grad}\operatorname{div} u'' = 0$, i.e., the basic equations and the first boundary value problem split so that they can be investigated as the first three-dimensional boundary value problem of statics of an isotropic elastic body [3].

In what follows it will be assumed without loss of generality that $\varepsilon_2 \neq 0$. Then the roots of equation (3) can be expressed as

$$\begin{aligned}
2\varepsilon_2 X_1 &= \varepsilon_4 - \varepsilon_1 + \sqrt{(\varepsilon_1 - \varepsilon_4)^2 + 4\varepsilon_2\varepsilon_3},\\
2\varepsilon_2 X_2 &= \varepsilon_4 - \varepsilon_1 - \sqrt{(\varepsilon_1 - \varepsilon_4)^2 + 4\varepsilon_2\varepsilon_3}.
\end{aligned} \tag{6}$$

Since

$$(\varepsilon_1 - \varepsilon_4)^2 + 4\varepsilon_2\varepsilon_3 = \frac{4}{\delta_0^2 a_1 a_2}\Big\{ \big[a_2(da_1 - cb_1) + a_1(da_2 - cb_2)\big]^2 + \\ + d_2(a_1b_2 - a_2b_1)^2 \Big\} > 0,$$

the roots X_1 and X_2 are different real values. Note that conditions (5) are fulfilled if the discriminant of equation (3) is equal to zero. In addition to X_1 and X_2, we also need the values

$$2k_1=\varepsilon_1+\varepsilon_4+\sqrt{(\varepsilon_1-\varepsilon_4)^2+4\varepsilon_2\varepsilon_3},\quad 2k_2=\varepsilon_1+\varepsilon_4-\sqrt{(\varepsilon_1-\varepsilon_4)^2+4\varepsilon_2\varepsilon_3}.$$

We prove that

$$-1<k_j<1,\quad j=1,2, \tag{7}$$

and

$$(1-k_1)(1-k_2)=4d_2/\delta_0. \tag{8}$$

Now from (2) we have

$$u'=\frac{-X_2v'+X_1v''}{X_1-X_2},\quad u''=\frac{v'-v''}{X_1-X_2}. \tag{9}$$

After substituting these experssions into (1) and performing some simple transformations, we obtain

$$\begin{aligned}(c-a_1X_2)(\Delta v'+M_1\operatorname{grad}\operatorname{div}v')+(a_1X_1-c)(\Delta v''+M_2\operatorname{grad}\operatorname{div}v'')=0,\\(a_2-cX_2)(\Delta v'+M_1\operatorname{grad}\operatorname{div}v')+(cX_1-a_2)(\Delta v''+M_2\operatorname{grad}\operatorname{div}v'')=0,\end{aligned} \tag{10}$$

where

$$M_1=\frac{d-b_1X_2}{c-a_1X_2}=\frac{b_2-dX_2}{a_1-cX_2},\quad M_2=\frac{b_1X_1-d}{a_1X_1-c}=\frac{dX_1-b_2}{cX_1-a_2}. \tag{11}$$

Let us consider equations (10) as a system with respect to $\Delta v'+M_1\operatorname{grad}\operatorname{div}v'$ and $\Delta v''+M_2\operatorname{grad}\operatorname{div}v''$. Since

$$(c-a_1X_2)(cX_1-a_2)-(a_2-cX_2)(a_1X_1-c)=d_2(X_2-X_1)\neq 0,$$

from (10) we have

$$\Delta v'+M_1\operatorname{grad}\operatorname{div}v'=0, \tag{12}$$

$$\Delta v''+M_2\operatorname{grad}\operatorname{div}v''=0. \tag{13}$$

Thus we have shown that the three-dimensional boundary value problem of statics of the theory of elastic mixtures with given displacements on the boundary splits in the general case.

Equations (12) and (13) can be combined as one equation

$$\Delta v+M\operatorname{grad}\operatorname{div}v=0, \tag{14}$$

where $v=v'$ for $M=M_1$ and $v=v''$ for $M=M_2$. It is obvious that equation (14) is an elliptic system if $1+M>0$, i.e., $1+M_1>0$ and $1+M_2>0$.

Let us show that these conditions hold for M_1 and M_2. To this end, we have to write the expressions of M_1 and M_2 in a different form. From (11) it obviously follows that $M_1 = \frac{(b_2-dX_2)(cX_1-a_2)}{(a_2-cX_2)(cX_1-a_2)}$. Taking into account

$$X_1 + X_2 = (\varepsilon_4 - \varepsilon_1)/\varepsilon_2, \quad X_1X_2 = -\varepsilon_3/\varepsilon_2,$$

and performing some obvious calculations, from equation (3) we obtain

$$M_1 = (a_2b_1 - cd + (da_1 - cb_1)X_1)/d_2. \tag{15}$$

In a similar manner we have

$$M_2 = (a_2b_1 - cd + (da_1 - cb_1)X_2)/d_2. \tag{16}$$

The substitution of the values X_1 and X_2 from (6) into (15) and (16) gives

$$\begin{aligned} M_1 &= \frac{a_1b_2 + a_2b_1 - 2cd}{2d_2} + \frac{\delta_0}{4d_2}\sqrt{(\varepsilon_1-\varepsilon_4)^2 + 4\varepsilon_2\varepsilon_3}, \\ M_2 &= \frac{a_1b_2 + a_2b_1 - 2cd}{2d_2} - \frac{\delta_0}{4d_2}\sqrt{(\varepsilon_1-\varepsilon_4)^2 + 4\varepsilon_2\varepsilon_3}. \end{aligned} \tag{17}$$

Now using (8) and carrying out some elementary transformations we have

$$M_1 = 2k_1/(1-k_1), \quad M_2 = 2k_2/(1-k_2). \tag{18}$$

Hence by virtue of (7) we obtain

$$1 + M_1 = \frac{1+k_1}{1-k_1} > 0, \quad 1 + M_2 = \frac{1+k_2}{1-k_2} > 0. \tag{19}$$

It has thus been shown that (14) is an elliptic system. Now setting

$$v = \Delta\Psi - M(M+1)^{-1}\operatorname{grad}\operatorname{div}\Psi$$

in (14), we obtain the equation

$$\Delta\Delta\Psi = 0. \tag{20}$$

that defines the unknown vector Ψ.

For equation (14) we introduce the generalized stress vector

$$\overset{\varkappa}{T}v = (1+\varkappa)\frac{\partial v}{\partial n} + (M-\varkappa)n\operatorname{div}v + \varkappa[n\operatorname{rot}v],$$

where $\varkappa$ is an arbitrary real constant. Now in (20) let

$$\Psi = Er/2, \tag{21}$$

where E is the three-dimensional unit matrix,

$$r = \sqrt{\sum_{k=1}^{3}(x_k - y_k)^2}, \tag{22}$$

x_1, x_2, x_3 and y_1, y_2, y_3 are the coordinates of the points x and y, respectively. By virtue of (21) and (22) we can rewrite the basic fundamental matrix for equation (14) as $\Gamma(x-y) = \|\Gamma_{kj}\|_{3\times 3}$, where

$$\Gamma_{kj} = \frac{\delta_{kj}}{r} - \frac{M}{2(1+M)}\frac{\partial^2 r}{\partial x_k \partial x_j} \equiv \frac{2+M}{2(1+M)}\frac{\delta_{kj}}{r} + \frac{M}{2(1+M)}\frac{1}{r}\frac{\partial r}{\partial x_k}\frac{\partial r}{\partial x_j}.$$

Let us now calculate, with respect to the coordinates of x, the generalized stress operator of the basic fundamental matrix. After some obvious calculations we obtain $\overset{\varkappa}{T}_x \Gamma(x-y) = \|(\overset{\varkappa}{T}_x \Gamma^{(j)})_k\|_{3\times 3}$, where

$$(\overset{\varkappa}{T}_x \Gamma^{(j)})_k = \Big[(1+\varkappa)\frac{2+M}{2(1+M)} - \varkappa\Big]\delta_{kj}\frac{\partial}{\partial n(x)}\frac{1}{r} + (1+\varkappa)\frac{3M}{2(1+M)} \times$$
$$\times \frac{\partial r}{\partial x_k}\frac{\partial r}{\partial x_j}\frac{\partial}{\partial n(x)}\frac{1}{r} + \frac{1}{2(1+M)}[\varkappa(2+M) - M]\Big(n_j\frac{\partial}{\partial x_k} - n_k\frac{\partial}{\partial x_j}\Big)\frac{1}{r}.$$

For the matrix $\overset{\varkappa}{T}_x \Gamma(x-y)$ to contain only a weak singulary at $x=y$ it is necessary and sufficient that $\varkappa(2+M) - M = 0$, i.e., that

$$\varkappa = M/(M+2). \tag{23}$$

When $\varkappa$ is defined by (23), the generalized stress operator will be denoted by N. We have

$$N_x\Gamma(x-y) = \frac{1}{2+M}\Big\|2\delta_{kj} + 3M\frac{\partial r}{\partial x_k}\frac{\partial r}{\partial x_j}\Big\|_{3\times 3}\cdot\frac{\partial}{\partial n(x)}\frac{1}{r}, \quad k,j=1,2,3.$$

It is obvious that

$$N_y\Gamma(y-x) = \frac{1}{2+M}\Big\|2\delta_{kj} + 3M\frac{\partial r}{\partial x_k}\frac{\partial r}{\partial x_j}\Big\|_{3\times 3}\cdot\frac{\partial}{\partial n(y)}\frac{1}{r}. \tag{24}$$

By direct calculations it is proved that each column of matrix (24) is a solution of equation (14) with respect to x when $x \neq y$. Taking into account formulas (18) and (19), we obtain

$$\varkappa_1 = k_1 = M_1/(2+M_1), \quad \varkappa_2 = k_2 = M_2/(2+M_2).$$

Green's formula for the operator N is obtained in a usual manner [2] and for the regular vector v has the form

$$\int_{D^+} N(v,v)dy_1\, dy_2 = \int_S vNv\, ds, \tag{25}$$

where v is a solution of equation (14) and

$$N(v,v) = \Big(\frac{1-2k}{3} + \frac{2k}{1-k}\Big)(\operatorname{div} v)^2 + \frac{1+k}{3}\Big[\Big(\frac{\partial v_1}{\partial y_1} - \frac{\partial v_2}{\partial y_2}\Big)^2 + \Big(\frac{\partial v_1}{\partial y_1} - \frac{\partial v_3}{\partial y_3}\Big)^2 +$$

$$+\Big(\frac{\partial v_2}{\partial y_2}-\frac{\partial v_3}{\partial y_3}\Big)^2\Big]+\frac{1+k}{2}\Big[\Big(\frac{\partial v_2}{\partial y_1}+\frac{\partial v_1}{\partial y_2}\Big)^2+\Big(\frac{\partial v_3}{\partial y_1}+\frac{\partial v_1}{\partial y_3}\Big)^2+\Big(\frac{\partial v_3}{\partial y_2}+$$
$$+\frac{\partial v_2}{\partial y_3}\Big)^2\Big]+\frac{1-k}{2}\Big[\Big(\frac{\partial v_2}{\partial y_1}-\frac{\partial v_1}{\partial y_2}\Big)^2+\Big(\frac{\partial v_3}{\partial y_2}-\frac{\partial v_2}{\partial y_3}\Big)^2+\Big(\frac{\partial v_1}{\partial y_3}-\frac{\partial v_3}{\partial y_1}\Big)^2\Big]. \quad (26)$$

For the infinite domain D^- we have

$$\int_{D^-} N(v,v)dy_1\, dy_2 = -\int_S vNv\, ds. \quad (27)$$

In this case the vector v satisfies at infinity the conditions

$$v = O(\rho^{-1}), \quad \frac{\partial v}{\partial x_k} = O(\rho^{-2}), \quad k = 1,2,3,$$

where $\rho = \sqrt{x_1^2 + x_2^2 + x_3^2}$. To use formulas (25) and (27) in the proof of the uniqueness theorems it is necessary that expression (26) have a positively defined form both for k_1 and k_2. Since (7) is fulfilled, all the terms in (26) except for the first one are positive. From (17) we obtain

$$M_1 + \frac{1}{2} = \frac{a_1(b_2-\lambda_5)+a_2(b_1-\lambda_5)-2c(d+\lambda_5)+\Delta_1}{2d_2} +$$
$$+ \frac{\delta_0}{4d_2}\sqrt{(\varepsilon_1-\varepsilon_4)^2+4\varepsilon_2\varepsilon_3} > 0,$$

where $\Delta_1 = \mu_1\mu_2 - \mu_3^2 > 0$ and $a_1(b_2-\lambda_5)+a_2(b_1-\lambda_5)-2c(d+\lambda_5)>0$ [2]. By the first formula of (18) we have $k_1 > -\frac{1}{3}$. Therefore $-\frac{1}{3} < k_1 < 1$. Then

$$\frac{1-2k_1}{3} + \frac{2k_1}{1-k_1} = \frac{(2k_1+1)(k_1+1)}{3(1-k_1)} > 0.$$

For the expression $\frac{1-2k_2}{3} + \frac{2k_2}{1-k_2} = \frac{(2k_2+1)(k_2+1)}{3(1-k_2)}$ to be positive it is necessary and sufficient that $(2k_2+1)(k_2+1) > 0$. After obvious transformations the expanded form of this inequality is

$$\frac{1}{\delta_0}\big[9(b_1b_2-d^2)+6(a_1b_2+a_2b_1-2cd)+4d_2\big] \equiv$$
$$\equiv \frac{1}{\delta_0}\big[(2a_1+3b_1)(2a_2+3b_2)-(2c+3d)^2\big] > 0, \quad (28)$$

where δ_0 is given by (4). When (28) is fulfilled, the first term of formula (26) will be positive, too, for $k = k_2$. In what follows it will be assumed that (28) is valid.

Solutions of the first boundary value problem for equations (12) and (13) are sought for in the form

$$v'(x) = \frac{1}{2\pi}\int_S N_y^{(1)}\Gamma^{(1)}(y-x)g'(y)\, ds, \quad (29)$$

$$v''(x) = \frac{1}{2\pi}\int_S N_y^{(2)}\Gamma^{(2)}(y-x)g''(y)\,ds, \tag{30}$$

where $g'(y)$ and $g''(y)$ are the unknown vector functions and

$$N_y^{(i)}\Gamma^{(i)}(y-x) = \left\|(1-k_i)\delta_{kj} + 3k_i\frac{\partial r}{\partial x_k}\frac{\partial r}{\partial x_j}\right\|_{3\times 3}\cdot\frac{\partial}{\partial n(y)}\frac{1}{r}, \quad i=1,2.$$

In the case of the first internal problem, to define g' and g'', we obtain by virtue of the properties of potentials (29) and (30) [3] the Fredholm integral equation of second kind

$$\begin{aligned} &-g'(z) + \frac{1}{2\pi}\int_S N_y^{(1)}\Gamma^{(1)}(y-z)g'(y)\,ds = f^{(1)}(z),\\ &-g''(z) + \frac{1}{2\pi}\int_S N_y^{(2)}\Gamma^{(2)}(y-z)g''(y)\,ds = f^{(2)}(z), \end{aligned} \tag{31}$$

where the vectors $f'(z)$ and $f''(z)$ given on the boundary S are the boundary values of the vectors v' and v'', respectively. Since it is assumed that condition (28) is fulfilled, the quadratic form (26) is positively defined both for k_1 and k_2. Applying the method developed in [3] to (31), we readily conclude that these equations have unique solutions g' and $g'' \in C^{1,\alpha}(S)$ if $f^{(i)} \in C^{1,\alpha}(S)$ and $S \in C^{2,\beta}$, where $i = 1,2$, $0 < \beta < \alpha \leq 1$.

Similar arguments can be used in considering the first boundary value problem for the infinite domain D^- bounded by the closed surface S. In that case we are to seek for a solution the manner as follows:

$$\begin{aligned} v'(x) &= \frac{1}{2\pi}\int_S N_y^{(1)}\Gamma^{(1)}(y-x)g'(y)ds + \frac{1}{4\pi}\Gamma^{(1)}(x)\int_S N_y^{(1)}\Gamma^{(1)}(y)g'(y)ds,\\ v''(x) &= \frac{1}{2\pi}\int_S N_y^{(2)}\Gamma^{(2)}(y-x)g''(y)ds + \frac{1}{4\pi}\Gamma^{(2)}(x)\int_S N_y^{(2)}\Gamma^{(2)}(y)g''(y)ds. \end{aligned} \tag{32}$$

To define the unknown vectors $g'(y)$ and $g''(y)$ we respectively obtain the Fredholm integral equations of second kind

$$\begin{aligned} &g'(z) + \frac{1}{2\pi}\int_S N_y^{(1)}\Gamma^{(1)}(y-z)g'(y)\,ds +\\ &\qquad + \frac{1}{4\pi}\Gamma^{(1)}(z)\int_S N_y^{(1)}\Gamma^{(1)}(y)g'(y)\,ds = f^{(1)}(z),\\ &g''(z) + \frac{1}{2\pi}\int_S N_y^{(2)}\Gamma^{(2)}(y-z)g''(y)\,ds +\\ &\qquad + \frac{1}{4\pi}\Gamma^{(2)}(z)\int_S N_y^{(2)}\Gamma^{(2)}(y)g''(y)\,ds = f^{(2)}(z), \end{aligned} \tag{33}$$

where $f^{(1)}$, $f^{(2)}$ and S satisfy the above conditions. Again applying the method developed in [3] we find that equations (33) have unique solutions and $g(g', g'') \in C^{1,\alpha}(s)$.

Thus we have proved that the vectors $v'(x)$ and $v''(x)$ are uniquely defined both for the finite domain D^+ and the infinite domain D^-. By virtue of (9) this means that u' and u'', i.e., the vector $u(u',u'')$, are defined uniquely, too.

Using the methods and arguments from [4] and [5, Ch. II, §4], we can prove that the Fredholm equations (31) and (33) are also uniquely solvable in the space of continuous vectors (provided that $f^{(i)} \in C(s)$, $i = 1,2$), while the first three-dimensional problem for equations (1) and (14) (i.e., (12) and (13)) is uniquely solvable in the class $C(\overline{D}^{\pm}) \cap C^2(D^{\pm})$).

Since potentials (29), (30), and (32) with continuous densities are continuous in the respective closed domains, the results obtained above prove the unique solvability of the considered problems in the class $C(\overline{D}^{\pm}) \cap C^2(D^{\pm})$ when the boundary data are continuous.

The above reasoning also enables us to construct effective (explicit) solutions of the first three-dimensional boundary value problem of the elastic mixture theory for those specific domain for which we can effectively construct a solution of the elastostatic problem with given displacements on the boundary.

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