Georgian Mathematical Journal
Volume 8 (2001), Number 1, 135-164

ON PERIODIC SOLUTIONS OF AUTONOMOUS DIFFERENCE EQUATIONS

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Abstract. We establish conditions necessary for the existence of non-constant periodic solutions of non-linear autonomous difference equations with Lipschitzian right-hand sides.

2000 Mathematics Subject Classification: Primary 34K13, 34C25; Secondary 47H07.
Key words and phrases: Periodic solution, autonomous system, difference equation, cone, partially ordered space, fixed point.

In this paper, we obtain lower bounds for the periods of periodic solutions of autonomous difference equations with non-linear terms satisfying a kind of the Lipschitz condition in an abstract Banach space. Surprisingly enough, the main Theorems 2 and 3 from §3 are proved easily by using the appropriate version of the method of periodic successive approximations [1] adopted for the study of periodic difference equations in Banach spaces (see §2), which, besides the results mentioned, has allowed us to improve some statements of [3, 2, 4, 5] concerning the convergence of approximate solutions. For this purpose, some results from [6, 7, 8, 9] have been used.

Note that, for ordinary differential equations in the space of bounded real sequences, a similar method was first developed in [10].

The estimates obtained here are close to some results of the works [11, 12, 13, 14, 15, 16, 17] motivated mainly by the papers [11, 12], where it was proved that the autonomous system $x' = f(x)$, in which, for some $l \in (0, +\infty)$, the mapping $f : X \to X$ (X is a Hilbert space and $\|\cdot\|_X = \sqrt{(\cdot,\cdot)}$) satisfies the Lipschitz condition with respect to the norm,

$$\|f(x_1) - f(x_2)\|_X \leq l_{f;X} \|x_1 - x_2\|_X \qquad (\forall \{x_1, x_2\} \subset X), \tag{1}$$

has no non-constant periodic solutions with period less than $2\pi/l$.

Here, we establish similar estimates for difference equations assuming an abstract two-sided Lipschitz condition formulated in terms of an *abstract modulus* (see §2). The latter serving as a natural generalisation of the notion of norm, the use of a more general conception for the Lipschitz condition allows us to obtain statements that, in many cases, improve the earlier results, which are optimal within the framework of the traditional condition (1)—the circumstance best illustrated by Corollary 3 from §3, which has no analogue in the scalar case.

ISSN 1072-947X / \$8.00 / © Heldermann Verlag *www.heldermann.de*

1. Notations

Let X, X_1, X_2 be Banach spaces. Then:

(1) $\|\cdot\|_X$ denotes the norm in X.
(2) $\mathcal{L}(X)$ is the algebra of linear, continuous mappings $X \to X$.
(3) I_X is the identity mapping of X.
(4) $\|X\|_{X_1 \to X_2}$ is the norm of the linear, continuous operator $X_1 \to X_2$.
(5) $\ker L$ and $\operatorname{im} L$, as usual, denote the image space and the kernel of an element $L \in \mathcal{L}(X)$.
(6) $\sigma(L)$ is the spectrum of $L \in \mathcal{L}(X)$.
(7) $r(L)$ is the spectral radius of $L \in \mathcal{L}(X)$.
(8) $r_\sigma(L)$ is the maximal, in modulus, eigen-value of an operator $L \in \mathcal{L}(X)$ (if such a value exists).
(9) The symbol ℓ^p, where $1 \le p < +\infty$ (resp., $p = +\infty$) stands for the usual space of summable in the power p (resp., bounded) real sequences.
(10) For $N \in \mathbb{N}$, the symbol ℓ_N^∞ denotes the linear space $\mathbb{R}^N$ with the norm induced from ℓ^∞: $\|x\|_{\ell_N^\infty} := \max_{1 \le \nu \le N} |x_\nu|$ for $x = (x_\nu)_{\nu=1}^N \in \ell_N^\infty$.
(11) If T is an interval, $C(T, X)$ denotes the Banach space of all continuous X-valued functions on T with the uniform norm $\|x\| := \max_{t \in T} \|x(t)\|_X$, $x \in C(T, X)$. By definition, $C(T) := C(T, \mathbb{R}^1)$.

When referring to different parts of the paper, we use the symbol '§' as an abbreviation of the word 'Section.'

2. Method of Periodic Successive Approximations for Difference Equations

The results of this paper lean upon the scheme of investigating boundary value problems, the original version of which gained the name 'method of periodic successive approximations' [1]. In this Section, we establish a few general statements concerning the construction and application of a similar scheme for studying periodic solutions of difference equations in a Banach space. The notations and definitions related to the theory of partially ordered Banach spaces and explicitly not specified here can be found, e.g., in [20, 19, 18].

Let $\langle X, \preccurlyeq_X, \|\cdot\|_X \rangle$ be a partially ordered Banach space (POBS for short) with the reproducing positive cone X_+ (i.e., $X_+ - X_+ = X$), and $\langle E, \preccurlyeq_E, \|\cdot\|_E \rangle$ be another POBS, in which the positive cone E_+ is normal. (The latter means that order bounded subsets of E are also bounded with respect to the norm $\|\cdot\|_E$.) Suppose that $\mathfrak{m} : X \to E_+$ is an *abstract modulus* [18], i.e., a mapping satisfying the following conditions:

(m_1) $\mathfrak{m}(x) = 0 \Rightarrow x = 0$;
(m_2) $\mathfrak{m}(\lambda x) = |\lambda| \, \mathfrak{m}(x) \quad (\forall \lambda \in \mathbb{R},\ x \in X)$;
(m_3) $\mathfrak{m}(x_1 + x_2) \preccurlyeq_E \mathfrak{m}(x_1) + \mathfrak{m}(x_2) \quad (\forall \{x_1, x_2\} \subset X)$.
(m_4) $\exists \alpha > 0 :\ \mathfrak{m}(x_1) \preccurlyeq_E \alpha \mathfrak{m}(x_2)$ for $\{x_1, x_2\} \subset X$ such that $0 \preccurlyeq_X x_1 \preccurlyeq_X x_2$;
(m_5) $\exists \beta > 0 :\ \|x\|_X \le \beta \, \|\mathfrak{m}(x)\|_E \quad (\forall x \in X)$.

These conditions are assumed in order to guarantee the validity of Lemma 1 below (and, hence, that of other statements depending on it, among which is the main Theorem 2). Note that, in [18], analogous questions are dealt with under very similar assumptions. The essence of the considerations to follow being rather similar, we do not dwell on some technical implications in more or less detail, providing, where appropriate, references to the paper mentioned.

Let us give a few simple examples illustrating the conception of modulus introduced above.

Example 1. If X is a conditionally complete Banach lattice [21], then one can put $E = X$, $E_+ = X_+$, and $\mathfrak{m}(x) = \sup\{x, 0\} + \inf\{x, 0\}$ $(x \in X)$.

This is the simplest, and most used, example of the mapping satisfying conditions (m_1)–(m_5), the fulfillment of which is easily verified in this case.

Example 2. If $X = \ell^\infty$ carries the natural, component-wise partial ordering, one can put, e.g., $\mathfrak{m}(x) = \big(\rho_1|x_1|, \rho_2|x_2|, \dots\big)$ for every $x = (x_1, x_2, \dots)$ from ℓ^∞, where $\rho = (\rho_1, \rho_2, \dots) \in \ell^1$ is a fixed non-negative sequence.

Example 3. Let $X = \mathbb{R}^n$ and $X_+ = K_\sigma$, where K_σ is the cone defined by

$$K_\sigma := \{x = (x_\nu)_{\nu=1}^n : \sigma_\nu x_\nu \geq 0 \text{ for all } \nu\}$$

with $\sigma = (\sigma_\nu)_{\nu=1}^n$ such that $\sigma_1\sigma_2\cdots\sigma_n \neq 0$.

In this case, one can set $E = X$, $E_+ = K_\delta$, where $\delta = (\delta_\nu)_{\nu=1}^n$ satisfies $\delta_1\delta_2\cdots\delta_n \neq 0$, and define an abstract modulus $\mathfrak{m} : \mathbb{R}^n \to K_\delta$ by the formula $\mathfrak{m}(x) = D|Ax|$, in which $D = (d_{\nu\mu})_{\nu,\mu=1}^n$ and $A = (a_{\nu\mu})_{\nu,\mu=1}^n$ are non-singular matrices such that 1° $\delta_\nu d_{\nu\mu} \geq 0$ and 2° $\sigma_\mu a_{\nu\mu} \geq 0$ for all ν and μ from 1 to n.

Explanation. Obviously, it suffices to consider the case when $\{\sigma_\nu, \delta_\nu : \nu = 1, 2, \dots, n\} \subset \{-1, 1\}$, which is assumed below. Assumption 1° guarantees that $Dx \in K_\delta$ whenever x has non-negative components. Let us verify condition (m_4). Indeed, if x and y satisfy the inequality

$$0 \leq \sigma_\nu x_\nu \leq \sigma_\nu y_\nu \qquad \text{for every } \nu, \tag{2}$$

we have to show that, again for all ν, the relation $\sum_{k=1}^n \delta_\nu d_{\nu k}\Big|\sum_{\mu=1}^n a_{\nu\mu}x_\mu\Big| \leq \sum_{k=1}^n \delta_\nu d_{\nu k}\Big|\sum_{\mu=1}^n a_{\nu\mu}y_\mu\Big|$ holds. By condition 1°, the coefficients outside the absolute value sign are non-negative, and, in view of 2°, $a_{\nu\mu}x_\mu \leq a_{\nu\mu}y_\mu$ for all ν and μ whenever x and y are related by (2). Since, by 2° and (2), $a_{\nu\mu}x_\mu = \sigma_\mu a_{\nu\mu} \cdot \sigma_\mu x_\mu \geq 0$, we have the required relation, and (m_4) is thus satisfied. The fulfillment of condition (m_5) follows analogously to Lemma 3 in [18], because, in a finite-dimensional space, all norms are equivalent. □

Remark 1. In Example 3, assumptions 1° and 2° guarantee that $Dx \in K_\delta$ and $A^*x \in K_\sigma$ whenever $x \in K_{(1,1,\dots,1)}$, i.e., x has non-negative components.

This list of examples can be continued infinitely. We note only that, although there is no general recipe for constructing mappings with properties (m_1)–(m_5), in concrete problems, a suitable definition often 'arises by itself'.

Remark 2. Although the corresponding assumption in [18] is (m_4) with $\alpha = 1$, it can be readily shown that all the statements relying upon this condition by minor modifications in the proofs can be established also for the case when $\alpha \neq 1$.

With the above conventions adopted, consider the problem on finding an N-periodic solution of the difference equation

$$x_{n+1} - x_n = f_n(x_n) \qquad (n \in \mathbb{Z}), \tag{3}$$

in which the mapping $\mathbb{Z} \ni n \mapsto f_n(x) \in X$ is periodic with period N for every $x \in X$,

$$f_{n+N}(x) = f_n(x) \qquad (\forall n \in \mathbb{Z},\ x \in X),$$

and $X \ni x \mapsto f_n(x)$ satisfies the (generalised) Lipschitz condition with respect to the modulus $\mathfrak{m}$, i.e.,

$$\mathfrak{m}(f_n(x) - f_n(y)) \preccurlyeq_E L_n\, \mathfrak{m}\,(x - y) \qquad (\forall \{x, y\} \subset E,\ n \in \mathbb{Z}), \tag{4}$$

where the sequence $\{L_n\}_{n\in\mathbb{Z}} \subset \mathcal{L}(X)$ is such that $L_n E_+ \subset E_+$ ($\forall n \in \mathbb{Z}$).

Similarly to the 'continuous' version of the method of periodic successive approximations, (see, e.g., [1, 18]), we seek for an N-periodic solution of (3) among the sequences $\{x_n(a) : n \in \mathbb{N}\}$ satisfying the difference equation

$$x_n(a) = \begin{cases} a + \sum_{\nu=0}^{n-1} f_\nu(x_\nu(a)) - nN^{-1} \sum_{\nu=0}^{N-1} f_\nu(x_\nu(a)) & \text{if } 1 \le n < N, \\ a & \text{if } n = 0, \end{cases} \tag{5}$$

where a is a parameter from X. The condition setting off the N-periodic solutions of (3) from the rest of solutions of (5) then has the form

$$\Delta_N(a) := \frac{1}{N} \sum_{\nu=0}^{N-1} f_\nu(x_\nu(a)) = 0. \tag{6}$$

Equation (6) is natural to be called the 'determining equation' with respect to the unknown parameter $a \in X$.

For an arbitrary $a \in X$, every single solution of (5) is necessarily N-periodic. It is essential for our consideration that this (actually, almost obvious) fact is motivated by the coincidence of the solution set of (5) with the set of fixed points of a certain non-linear operator, whose range consists of N-periodic sequences. Rewriting these reasonings in a formal manner and complementing them by a statement on the solvability of equation (5), we arrive at the scheme of the method of periodic successive approximations. Let us describe this scheme in the form convenient for further reference.

In the sequel, we shall identify an N-periodic sequence $\{x_n\}_{n\in\mathbb{Z}} \subset X$ with the vector $\vec{x} = (x_0, x_1, \ldots, x_{N-1})$ and use the notation $\vec{x}(n) := x_n$. When possible, we shall also write $x(n)$ and $f\,(n, \vec{x}(n))$ along with x_n and $f_n(\vec{x}(n))$. The right-hand side of (3) will be written shortly by using the superposition operator $\mathfrak{f}$ generated by f according to the formula

$$(\mathfrak{f}x)\,(n) := f_n\,(n, x(n)) \qquad (n \in \mathbb{Z}).$$

The linear operator $\mathcal{P}_N$ on the space of bounded sequences $\{x_n\}_{n\in\mathbb{Z}} \subset X$, defined as

$$(\mathcal{P}_N x)(n) := x(n) - \frac{n}{N}[x(N) - x(0)] \qquad (n \in \mathbb{Z}), \tag{7}$$

is idempotent and such that $\operatorname{im} \mathcal{P}_N = \{x : x(0) = x(N)\} = \ker \mathcal{Q}_N$, where $(\mathcal{Q}_N x)(n) := nN^{-1}(x(N) - x(0))$ $(n \in \mathbb{Z})$, so that $\mathcal{P}_N x + \mathcal{Q}_N x = x$. For difference equations, the mapping $\mathcal{P}_N$ serves as a 'discrete' analogue of the canonical projection onto the space of continuous periodic functions on an interval used in the 'continuous' case [18].

Replacing an integral in a definition from [18] by a finite sum according to the lower rectangle formula,

$$(\mathcal{J}_N \vec{x})(n) := \begin{cases} \frac{1}{N}\sum_{\nu=0}^{n-1} \vec{x}(\nu) & \text{if } 1 \le n < N, \\ 0 & \text{if } n = 0, \end{cases}$$

we introduce the 'integration' operator $\mathcal{J}_N$. For the sake of convenience, we also define the forward difference operator, $(\nabla x)(n) := x(n+1) - x(n)$ $(n \in \mathbb{Z})$.

In these notations, we can state

Theorem 1. *In order that a sequence* $\{x_n\}_{n\in\mathbb{Z}} \subset X$ *be a solution of the* N*-periodic boundary value problem*

$$\nabla x = \mathfrak{f}x, \tag{8}$$

$$x(0) = x(N), \tag{9}$$

it is sufficient that there exist an $a \in X$*, for which*

$$x = a + \mathcal{P}_N \mathcal{J}_N \mathfrak{f}x, \tag{10}$$

$$\mathcal{Q}_N \mathcal{J}_N \mathfrak{f}x = 0. \tag{11}$$

Conversely, if a sequence $\{x_n\}_{n\in\mathbb{Z}}$ *is a solution of problem* (8), (9)*, then it also satisfies system* (10), (11) *with* $a = x(0)$*.*

This lemma can be proved by a reasoning very similar to that in the proof of Theorem 8 from [18]. We omit the proof here.

Remark 3. An analogous statement holds for the higher order equation of the form $\nabla^k x = \mathfrak{f}x$ with an arbitrary $k \in \mathbb{N}$: *The periodic boundary value problem*

$$\nabla^k x = \mathfrak{f}x, \qquad \nabla^\nu x(0) = \nabla^\nu x(N) \quad (\nu = 0, 1, \ldots, k-1)$$

is equivalent, in the above sense, to the system of equations

$$x = a + (\mathcal{P}_N \mathcal{J}_N)^k \mathfrak{f}x, \qquad \mathcal{Q}_N \mathcal{J}_N \mathfrak{f}x = 0. \tag{12}$$

The corresponding reasoning is also omitted, because, firstly, it is also similar to that given in [18] and, secondly, because the most attention is paid in the sequel to equation (3).

Remark 4. It is not difficult to see from (7) that (11) is nothing but another form of equation (6).

By virtue of the properties of the operator $\mathcal{P}_N$, the expression in the right-hand side of (10) determines an N-periodic sequence, whence it follows that equation (10) can be considered as a system of N non-linear equations in N unknowns from the space X. Furthermore, it is clear that, under the N-periodicity assumption of f with respect to the discrete variable, it suffices to specify the Lipschitz operator L_ν in condition (4) for $\nu = 0, 1, \ldots, N-1$ only. This simple consideration complemented by the usual estimations similar to those carried out in [8, 18] in the case of an analogous problem for an ordinary differential equation allows us to establish the following

Lemma 1. *If the spectral radius of the linear, continuous operator $\mathcal{K}_N : X^N \to X^N$ defined by the matrix*

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \left(1-\frac{1}{N}\right)L_0 & \frac{1}{N}L_1 & \ldots & \frac{1}{N}L_{N-2} & \frac{1}{N}L_{N-1} \\ \left(1-\frac{2}{N}\right)L_0 & \left(1-\frac{2}{N}\right)L_1 & \ldots & \frac{2}{N}L_{N-2} & \frac{2}{N}L_{N-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \left(1-\frac{N-2}{N}\right)L_0 & \left(1-\frac{N-2}{N}\right)L_1 & \ldots & \frac{N-1}{N}L_{N-2} & \frac{N-2}{N}L_{N-1} \\ \left(1-\frac{N-1}{N}\right)L_0 & \left(1-\frac{N-1}{N}\right)L_1 & \ldots & \left(1-\frac{N-1}{N}\right)L_{N-2} & \frac{N-1}{N}L_{N-1} \end{bmatrix}$$

satisfies the relation

$$r\left(\mathcal{K}_N\right) < 1, \tag{13}$$

then equation (10) *has a unique solution for an arbitrary $a \in X$, and this solution can be approximately found by iteration.*

We recall that the mapping f generating the superposition operator $\mathfrak{f}$ is assumed to satisfy the abstract Lipschitz condition (4) with some sequence of linear, positive operators $L_n : E \to E$ $(n \in \mathbb{Z})$.

Remark 5. The assertion of Lemma 1 implies that, under condition (13), the function $\Delta_N : X \to X$ in (11) is well-defined and single-valued.

Lemma 1 is an immediate consequence of the following

Lemma 2. *For an arbitrary N-periodic sequence $\vec{x} = (x_0, x_1, \ldots, x_{N-1})$ of elements from X and an arbitrary $k \in \mathbb{N}$, the component-wise estimate*

$$[\mathcal{P}_N \mathcal{J}_N]^k \vec{x} \preccurlyeq_X A_N^k \vec{x}$$

holds, where $A_N : X^N \to X^N$ is the linear operator defined with the formula

$$\left(A_N \vec{x}\right)(n) := \begin{cases} \frac{1}{N}\left(1 - nN^{-1}\right)\sum_{i=0}^{n-1} x_i + \frac{n}{N^2}\sum_{i=n}^{N-1} x_i & \text{if } 1 \le n < N, \\ 0 & \text{if } n = 0. \end{cases} \tag{14}$$

Remark 6. It is not difficult to verify that operator (14) leaves invariant the cone $X_+^N = X_+ \times X_+ \times \cdots \times X_+$.

Proof of Lemma 2. This statement is obtained readily by taking into account that (10) can be rewritten as (5). □

Proof of Lemma 1. In view of the restrictions imposed on the POBS X, E and the mapping $\mathfrak{m}$, the assertion of Lemma 1 is derived from Lemma 2 and Remark 6 by using of the generalised version of the Banach fixed point theorem (see Theorem 6.2 in [22]) similarly to the proof of Theorem 7 from [18]. □

For our applications, the autonomous case is of major interest, when the right-hand side of (3) is defined by a mapping $f : X \to X$. As is shown below, the assumption of Lemma 1 is greatly simplified therewith.

Corollary 1. *Suppose that, in the Lipschitz condition* (4)*,* $L_0 = L_1 = \cdots = L_{N-1} = L$ *and, furthermore,* $r(L) < N/r_\sigma(Q_N)$*, where*

$$Q_N := \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ N-2 & 2 & \dots & 2 & 2 \\ \dots & \dots & \dots & \dots & \dots \\ 2 & 2 & \dots & N-2 & N-2 \\ 1 & 1 & \dots & 1 & N-1 \end{bmatrix}. \tag{15}$$

Then the conclusion of Lemma 1 *holds.*

Proof. Obviously, the operator $\mathcal{K}_N$ can be decomposed as

$$\mathcal{K}_N = \operatorname{diag}(L, \dots, L) \circ \Lambda_N,$$

where

$$\Lambda_N := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \left(1-\frac{1}{N}\right) I_E & \frac{1}{N} I_E & \dots & \frac{1}{N} I_E & \frac{1}{N} I_E \\ \left(1-\frac{2}{N}\right) I_E & \left(1-\frac{2}{N}\right) I_E & \dots & \frac{2}{N} I_E & \frac{2}{N} I_E \\ \dots & \dots & \dots & \dots & \dots \\ \left(1-\frac{N-2}{N}\right) I_E & \left(1-\frac{N-2}{N}\right) I_E & \dots & \frac{N-2}{N} I_E & \frac{N-2}{N} I_E \\ \left(1-\frac{N-1}{N}\right) I_E & \left(1-\frac{N-1}{N}\right) I_E & \dots & \left(1-\frac{N-1}{N}\right) I_E & \frac{N-1}{N} I_E \end{bmatrix}.$$

It is not difficult to show that $r(\Lambda_N) = r(\Pi_N)$, where Π_N is the linear operator in the Cartesian product of $N-1$ copies of the space E defined by the formula

$$\Pi_N := \begin{bmatrix} \frac{1}{N} & \frac{1}{N} & \dots & \frac{1}{N} & \frac{1}{N} \\ \left(1-\frac{2}{N}\right) & \frac{2}{N} & \dots & \frac{2}{N} & \frac{2}{N} \\ \dots & \dots & \dots & \dots & \dots \\ \left(1-\frac{N-2}{N}\right) & \left(1-\frac{N-2}{N}\right) & \dots & \frac{N-2}{N} & \frac{N-2}{N} \\ \left(1-\frac{N-1}{N}\right) & \left(1-\frac{N-1}{N}\right) & \dots & \left(1-\frac{N-1}{N}\right) & \frac{N-1}{N} \end{bmatrix} \otimes I_E.$$

Taking into account the obvious identity $r(\operatorname{diag}(L, \dots, L)) = r(L)$, the fact that the operators Λ_N and $\operatorname{diag}(L, \dots, L)$ commute, and the relation $r(\Pi_N) = N^{-1} r_\sigma(Q_N)$ derived easily from definition (15), by virtue of the well-known result of functional analysis (see §149 in [23]), we obtain $r(\mathcal{K}_N) \leq r(L) \cdot r(\Lambda_N) = r(L) \cdot r(\Pi_N) = r(L) \cdot \frac{r_\sigma(Q_N)}{N}$, and it remains to apply Lemma 1. □

Identifying every stationary sequence $x, x, x, \ldots$ with the element $x \in X$, we can establish the following result, which can be regarded as a 'discrete' analogue of Lemma 6 from [18].

Lemma 3. $\ker(\mathcal{P}_N \mathcal{J}_N)^k = X$ *for an arbitrary* $k \in \mathbb{N}$.

The *proof* of Lemma 3 is rather similar to that of the above-mentioned Lemma 6 from [18] and, therefore, is omitted.

3. Necessary Conditions for the Existence of Periodic Solutions

Let us now turn to the autonomous difference equation (3),

$$x(n+1) - x(n) = f(x(n)) \qquad (n \in \mathbb{Z}), \tag{16}$$

where $f : X \to X$ satisfies the generalised two-sided Lipschitz condition of the form

$$\mathfrak{m}(f(z_1) - f(z_2)) \preccurlyeq_E L\mathfrak{m}(z_1 - z_2) \qquad (\forall \{z_1, z_2\} \subset X) \tag{17}$$

with a linear and continuous operator $L : E \to E$ positive with respect to the cone E_+. From POBS X, E and the abstract modulus $\mathfrak{m} : X \to E_+$, we require the fulfillment of the assumptions formulated in §2.

Theorem 2. *Assume that equation* (16) *is known to have a non-constant periodic solution with the minimal period* N. *Then necessarily* $r(L) \geq N/r_\sigma(Q_N)$, *or, which is the same,*

$$N \geq \frac{c_N}{r(L)}, \tag{18}$$

where

$$c_N := N^2/r_\sigma(Q_N) \qquad (N = 2, 3, \ldots). \tag{19}$$

Remark 7. As will be shown in §4, the equality $r(L) = 0$ is impossible under the conditions of Theorem 2, because, in the contrary case, (16) cannot have any non-constant periodic solutions.

It should be noted that estimate (18) essentially depends upon the choice of the partial orderings $\preccurlyeq_X$, $\preccurlyeq_E$, abstract modulus $\mathfrak{m} : X \to E_+$, and the Lipschitz operator L in condition (17). Due to this circumstance, the assertion of Theorem 2 does not follow from Theorem 4.1 of [13] (see also [15]), which is claimed to provide the best possible estimate. The cause of the seeming contradiction lies in the use of different formalisations of the *two-sided Lipschitz condition*—under this term, close in the spirit but essentially different notions are implied in [16, 15] and in this paper.

It seems to us that the latter, more general, notion is preferable in this context, for it is natural to expect it to comprise much more information about the non-linear mapping f than the traditional condition (1) with respect to the norm (see also Remark 9). On the other hand, *a priori* knowledge of such kind is no more difficult to obtain on practice, a good evidence being provided by the example of condition (17) in a finite-dimensional space understood according to the coordinate-wise partial ordering and absolute value.

Remark 8. If, in addition to the conditions assumed, the cone E_+ is reproducing in E, then, by virtue of Theorem 1 from [24], the Lipschitz operator L in relation (17) is necessarily continuous, and the corresponding continuity assumption in (17) turns out to be superfluous.

Remark 9. The use of the generalised Lipschitz condition often guarantees better estimates. For example, if we carried out estimations with respect to the ℓ^1-norm in $\mathbb{R}^{N-1}$, i.e., $\|x\|_{\ell^1} := |x_1|+|x_2|+\cdots+|x_{N-1}|$ for $x = (x_i)_{i=1}^{N-1} \in \mathbb{R}^{N-1}$, then, as is well-known, $\|B\|_{\ell^1\to\ell^1} = \max_{j=1,2,\ldots,N-1} \sum_{i=1}^{N-1} |B_{ij}|$ for an arbitrary matrix $B = (B_{ij})_{i,j=1}^{N-1}$. It is easy to compute the ℓ^1-norm of the matrix Q_N : $\|Q_N\|_{\ell^1\to\ell^1} = \frac{1}{2}N(N-1)$. Thus, $r_\sigma(Q_N) \leq N(N-1)/2$, which yields

$$c_N = \frac{N^2}{r_\sigma(Q_N)} \geq \frac{2N}{N-1}. \tag{20}$$

Inequality (20) means that, in this case, estimate (18) is more accurate that its 'scalar' analogue corresponding to the choice $E = \mathbb{R}$, $E_+ = \mathbb{R}_+$, and $\mathfrak{m} = |\cdot|$.

The following statement is an immediate consequence of Theorem 2.

Corollary 2. *For equation* (16) *to have a non-constant solution periodic with the minimal period* 2*, it is necessary that the Lipschitz operator L in condition* (17) *satisfy the relation* $r(L) \geq 2$.

Remark 10. The assertion of Corollary 2 is strict in the sense that the inequality $r(L) > 2$ may not hold under the conditions specified. Indeed, consider the system of two scalar difference equations

$$u_{n+1} = u_n + hv_n, \qquad v_{n+1} = -v_n, \tag{21}$$

in which $n \in \mathbb{Z}$ and h is an arbitrary number. This linear system can be rewritten as (16) with $E = X = \mathbb{R}^2$, $E_+ = X_+ = \mathbb{R}^2_+$, and f given by $f(u,v) = \left(\begin{smallmatrix} hv \\ -2v \end{smallmatrix}\right)$ for $\{u,v\} \subset \mathbb{R}^1$. It is clear that the linear mapping f satisfies the Lipschitz condition (17) with $L = \left[\begin{smallmatrix} 0 & |h| \\ 0 & 2 \end{smallmatrix}\right]$ in terms of the natural, component-wise, modulus $\mathfrak{m} : \mathbb{R}^2 \to \mathbb{R}^2_+$ defined by $\mathfrak{m}(u,v) = \left(\begin{smallmatrix} |u| \\ |v| \end{smallmatrix}\right)$ for all $\{u,v\} \subset \mathbb{R}^1$.

Obviously, L leaves invariant the (normal and reproducing) cone $\mathbb{R}^2_+$ and, furthermore, $r(L) = 2$. However, every solution of system (21) satisfying the condition $u_0 v_0 \neq 0$ is periodic with the minimal period 2.

Another interesting particular case of Theorem 2 is that when the Lipschitz operator for f does not have non-zero points of spectrum. For this kind of equations, we have

Corollary 3. *The autonomous difference equation* (16)*, in which f satisfies condition* (17) *with a quasi-nilpotent operator $L \in \mathcal{L}(E)$ leaving invariant the cone E_+, cannot have periodic solutions other than constant ones.*

Let us now make a few comments. Theorem 2 is close to some statements from [16, 14, 15, 13], where it is proved that the inequality

$$l \geq N/r_\sigma(M_N) \tag{22}$$

is necessary for the existence of a non-constant periodic solution of (16) with f satisfying condition (1). Herein, $\{M_N\}_{N=2}^{\infty}$ are certain matrices,

$$M_2 = [1]\,, \quad M_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}, \quad \dots,$$

the analytic formulae for the components of which are rather complicated.

Bringing (22) to the form more similar to our estimate (18), we obtain

$$N \geq \mu_N/l, \tag{23}$$

where $\mu_N := N^2/r_\sigma(M_N)$. We see that relation (18) indeed resembles (23). One may say that, in Theorem 2, the role of $\{M_N\}_{N\geq 2}$ from [15, 16] is played by the matrices

$$Q_2 = [1]\,, \quad Q_3 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad Q_4 = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 3 \end{bmatrix}, \quad \dots,$$

defined with formula (15). The meaning of the eigen-values of these matrices, is, of course, different.

A comparison of Theorem 4.1 from [16] with our Theorem 2 shows that the latter contains the result of [16] for $N = 2$ (cf. Corollary 2) and complements it for $N \geq 3$.

Besides being of use as a means for quick detecting ‘non-resonant’ equations (16), the main result of this paper allows one to improve some earlier statements similar to Lemma 1. More precisely, Corollary 1, from which Theorem 2 will be derived in §4, strengthens the results of [3, 2], where the standard scheme of the method of periodic successive approximations is established on the assumption that $Nr(L) < 3$. In fact, it suffices to require the inequality $Nr(L) < c_N$, where c_N satisfies $c_N \geq 3.38113277396367\ldots$ for $N \geq 5$ (see §5).

As an application of Theorem 2, knowing a lower bound for the period of all non-constant periodic solutions of an autonomous difference system, we can prove the solvability of some non-linear equations. Being similar to a result of [14], the corresponding statement has some relation to the conjecture of Schauder discussed in [25].

Corollary 4. *Let a continuous mapping $\mathcal{T}$ of the Banach space X into itself be such that $I_X + \mathcal{T}$ leaves invariant some closed and convex set $\mathcal{O} \subset X$. Assume* *that the mapping* $(I_X+\mathcal{T})^N$ *is compact for some* $N\in\mathbb{N}\setminus\{1\}$ *and, furthermore,* $\mathcal{T}$ *satisfies the Lipschitz condition*

$$\mathfrak{m}(\mathcal{T}x_1-\mathcal{T}x_2)\preccurlyeq_E L\mathfrak{m}(x_1-x_2)\qquad(\forall\{x_1,x_2\}\subset\mathcal{O})$$

with some positive operator $L\in\mathcal{L}(X)$ *such that*

$$r(L)<c_N/N. \tag{24}$$

Then the equation $\mathcal{T}x=0$ *has a solution in* $\mathcal{O}$.

Proof. Let us argue similarly to [14]. In view of the Schauder fixed point theorem, there exists a $y\in\mathcal{O}$ such that $y=(I_X+\mathcal{T})^N y$. By virtue of Theorem 2, inequality (24) guarantees the absence of non-constant N-periodic solutions of the difference equation $y_{n+1}=y_n+\mathcal{T}(y_n)$ $(n\in\mathbb{Z})$. This implies that y is a fixed point of $I_X+\mathcal{T}$ and, therefore, $\mathcal{T}y=0$. □

A result analogous to Theorem 2 holds also for higher order equations. Thus, for the difference equation of the form

$$[\nabla^k x](n)=f\left(x(n)\right)\qquad(n=0,\pm1,\pm2,\dots), \tag{25}$$

where $k\in\mathbb{N}$ is fixed and f satisfies (17), we have the following

Theorem 3. *The minimal period,* N*, of every non-constant periodic solution of* (25) *is not less than* $c_N/r(L)^{\frac{1}{k}}$.

In this context, the phrase *"The solution,* x*, of equation* (25) *is periodic with period* N*"* should be understood in the sense that the sequence $\{x(n):n\in\mathbb{Z}\}$ satisfies (25) and the equalities

$$\nabla^\nu x(0)=\nabla^\nu x(N)\qquad(\nu=0,1,\dots,N-1).$$

In particular, for the difference equation

$$x(n+2)-2x(n+1)+x(n)=f(x(n))\qquad(n\in\mathbb{Z})$$

with f satisfying (17), the assertion of Theorem 3 consists in the estimate $N\ge c_N/\sqrt{r(L)}$, which holds whenever some non-constant solution, x, of this equation is known to satisfy the relations $x(0)=x(N)$ and $x(1)-x(0)=x(N+1)-x(N)$.

Remark 11. It is not difficult to verify that the expression in the left-hand side of (25) can be rewritten as $[\nabla^k x](n)=-\sum_{\nu=0}^{k}(-1)^\nu\binom{k}{\nu}x(n+\nu)$, where $n\in\mathbb{Z}$, and $\binom{k}{\nu}$, $\nu=0,1,\dots,k$, are the corresponding binomial coefficients.

For the sake of completeness, we establish also the following

Corollary 5. *Under the above assumptions on the spaces X and E, let $L \in \mathcal{L}(X)$ be such that*

$$L^{\sharp}\mathfrak{m}(x) - \mathfrak{m}(Lx) \in E_{+} \qquad (\forall x \in X) \tag{26}$$

with some $L^{\sharp} \in \mathcal{L}(E)$ leaving invariant the cone E_+ and $\mathfrak{m} : X \to E_+$ satisfying conditions (m_1)–(m_5) from §2. Then the difference equation

$$\sum_{\nu=0}^{k} (-1)^{\nu} \binom{k}{\nu} x(n+\nu) + Lx(n) = 0 (n \in \mathbb{Z})$$

cannot have periodic solutions with period, N, less than $c_N / r\left(L^{\sharp}\right)^{\frac{1}{k}}$.

Proof. It suffices to take into account Remark 11 and apply Theorem 3. □

In the most frequently encountered situation when $X = E = \mathbb{R}^n$ and $X_+ = E_+ = \mathbb{R}^n_+$, the linear mapping $L : \mathbb{R}^n \to \mathbb{R}^n$ is determined by a square matrix $L = (l_{\nu\mu})^n_{\nu,\mu=1}$, and it is customary to put in (26) $\mathfrak{m}(x_1, x_2, \ldots, x_n) := \begin{pmatrix} |x_1| \\ \cdots \\ |x_n| \end{pmatrix}$ for every $(x_\nu)^n_{\nu=1} \in \mathbb{R}^n$ and define the Lipschitz operator, $L^{\sharp}$, as the multiplication by the matrix $L^{\sharp} := (|l_{\nu\mu}|)^n_{\nu,\mu=1}$.

Remark 12. One can extend the above reasonings to a more general situation. The exact formulations are omitted, because this paper makes no use of them.

To conclude the section, we note that, for $N \geq 3$, the estimates provided by Theorems 2 and 3, unfortunately, are not optimal. As we shall see in §4, further improvement of these theorems is closely allied to the problem of refinement of the method of periodic successive approximations (or, more precisely, Lemma 2), the applicability of which is proved at present under conditions of Lemma 1. The exact convergence estimates for this method, however, have not yet been obtained.

4. Proof of Theorems 2 and 3

Taking into account Remark 3 from §2, we turn directly to the proof of the more general Theorem 3. It suffices to consider the case when $\sigma(L) \neq \{0\}$ only.

Thus, let x be a non-constant solution of (25) periodic with the minimal period N. Assume that the theorem is wrong. Then

$$N < c_N / r\,(L)^{\frac{1}{k}}. \tag{27}$$

According to Remark 3 from §2 concerning the generalisation of Theorem 1 to equations of the form (25), it follows that the pair x and $a := x(0)$ should satisfy system (12).

By taking into account Lemma 2 and arguing similarly to the proof of Corollary 1, we can show that, under condition (27), the first equation in system (12) has a unique solution, say x_a. Furthermore, the successive approximations

$$x_{m;a} = a + \mathcal{P}_N \mathcal{J}_N \mathfrak{f} x_{m-1;a} \qquad (m \in \mathbb{N}) \tag{28}$$

converge to x_a independently of the choice of $x_{0;a}$. However, putting $x_{0;a} := (a, a, a, \dots)$ and sequentially applying Lemma 3, we find that, in fact, all the members of sequence (28) coincide: $x_{m;a} = x_a$ ($\forall m \in \mathbb{N}$), which yields immediately $x_a = (a, a, a, \dots)$.

Since $N^{-1} \sum_{\nu=0}^{N-1} f(x_a(\nu)) = f(a)$, we see that condition (11) determining whether the sequence x_a belongs to the set of N-periodic solutions of (17) has the form

$$f(a) = 0.$$

This means that the periodic solution x of (25) is constant, which contradicts our basic assumption. Indeed, on the contrary, we have two N-periodic solutions of the latter equation—a constant one, $x_a = (a, a, \dots)$, and a non-constant one, x. Furthermore, x has the initial data $x_0 = a$. Since both sequences satisfy system (12), it follows from the Lipschitz condition (17) and Lemma 2 that, component-wise,

$$\begin{aligned} \mathfrak{m}(x - a) = \mathfrak{m}\Big[(\mathcal{P}_N \mathcal{J}_N)^k (\mathfrak{f}x - \mathfrak{f}x_a) \Big] &\preccurlyeq_E A_N^k \mathfrak{m}(x - x_a) \\ = A_N^k \mathfrak{m}(x - a). & \end{aligned} \tag{29}$$

In view of Lemma 4 from §5 below, relations (29) and (27) yield $r(A_N^k) < 1$ and, therefore, $\mathfrak{m}(x - x_a) = 0$. Axiom (m_1) then implies $x - x_a = 0$, i.e., $x = (a, a, \dots)$, a contradiction.

In such a manner we have shown that, under condition (27), equation (25) cannot have non-constant periodic solutions with period N. Theorem 3 is thus proved, and Theorem 2 follows as a corollary.

5. On the Sequence $\{c_N\}_{N\geq 2}$

Here, we establish some properties of the sequence $\{c_N\}_{N\geq 2}$ introduced in §2. First we introduce a definition: let c_∞ denote the minimal (obiously, positive) root of the equation

$$1/c_\infty = \int_0^{\frac{1}{2}} e^{\tau(\tau-1)c_\infty} \, \mathrm{d}\tau. \tag{30}$$

Let us prove the following lemma, which, in particular, explains the strange, at first glance, notation adopted for the minimal root of equation (30).

Lemma 4. *For every $N \geq 2$, $c_N = 1/r(A_N)$, where A_N is the operator defined by* (14)*. Furthermore, $\lim_{N\to\infty} c_N = c_\infty$.*

Proof. Since $c_\infty = 1/r(A)$ (see, e.g., [8, 18]), where A is the linear integral operator in $C([0,1], \mathbb{R}^1)$ defined by the formula

$$(Ax)(t) := (1 - t) \int_0^t x(s) \, \mathrm{d}s + t \int_t^1 x(s) \, \mathrm{d}s \qquad (t \in [0, 1]), \tag{31}$$

and the difference equation (16) can be considered as a 'discrete' analogue of system (41) studied in [9, 18], it is natural to expect that the numbers c_N are 'responsible' for the spectra of some linear difference operators playing the rôle of the comparison operator A (cf. §6.1 and [8]) on the appropriate spaces of sequences. More precisely, we shall show that such operators can be constructed as 'approximations' to A; in fact, these will be the operators $\{A_N\}_{N\geq 2}$ defined by equality (14). Thus, the first claim of the lemma will follow as a consequence of the definition (14) of the operator A_N (cf. matrix (35)).

Let us establish the convergence of the sequence $\{c_N\}_{N=2}^{\infty}$. For this purpose, we shall use the general theory of approximate methods of developed by Kantorovich (see, e.g., [26]). Consider the integral equation

$$x - \mu A x = y, \tag{32}$$

where x and y belong to the space $C(S^1)$ of continuous functions on the circle $S^1 := \mathbb{R}/\mathbb{Z}$, the integral operator A in $C(S^1)$ is defined by (31), and μ is a non-negative scalar parameter.

Let us apply the following formulae of approximate integration:

$$\int_0^t x(s)\,\mathrm{d}s \approx \frac{1}{N}\sum_{\nu=0}^{n-1} x\left(\nu N^{-1}\right), \qquad \int_t^1 x(s)\,\mathrm{d}s \approx \frac{1}{N}\sum_{\nu=n}^{N-1} x\left(\nu N^{-1}\right),$$

Herein, $n = \lfloor Nt+1 \rfloor$ for $t \in [0,1)$ ($\lfloor s \rfloor$ is the integer part of s) and, by definition, $\sum_{i=\nu_1}^{\nu_2} := 0$ whenever $\nu_1 > \nu_2$. As a result, we obtain a 'discrete' analogue of equation (32):

$$\vec{x} - \mu A_N \vec{x} = \varphi_N y, \tag{33}$$

where $\vec{x} = (x_0, x_1, \ldots, x_{N-1})$ is the element of the Banach space ℓ_N^∞ of all N–dimensional vectors with the norm $\|\vec{x}\|_{\ell_N^\infty} := \max\{|x_\nu| : \nu = 0, 1, \ldots, N-1\}$, $A_N : \mathbb{R}^N \to \mathbb{R}^N$ is the mapping defined by equality (14), and $\varphi_N : C(S^1) \to \ell_N^\infty$ is the 'discretisation operator'

$$\varphi_N : C(S^1) \ni x \longmapsto \varphi_N x := \left(x(0),\ x\left(\tfrac{1}{N}\right), \ldots,\ x\left(\tfrac{N-1}{N}\right)\right). \tag{34}$$

It can be readily seen that the finite-dimensional (in fact, $\dim \operatorname{im} A_N = N-1$) operator A_N is completely determined by the matrix

$$\frac{1}{N^2}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ N-1 & 1 & \ldots & 1 & 1 \\ N-2 & N-2 & \ldots & 2 & 2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 2 & 2 & \ldots & N-2 & N-2 \\ 1 & 1 & \ldots & 1 & N-1 \end{bmatrix}, \tag{35}$$

and, therefore, in view of (14), (15), and (19), $r(A_N) = N^{-2} r_\sigma(Q_N) \equiv 1/c_N$.

Now we need to specify a suitable subspace of $C(S^1)$ isomorphic to ℓ_N^∞. For such a subspace, we take $C_N(S^1)$, which, by definition, consists of all the continuous real-valued functions on the circle S^1, linear on the intervals

$[iN^{-1}, (i+1)N^{-1})$ $(i = 0, 1, \ldots, N-2, N-1)$. It is easy to verify that the restriction, $\varphi|_{C_N(S^1)} =: \hat{\varphi}_N$, of the discretisation operator (34) to the subspace $C_N(S^1)$ carries out the isomorphism $C_N(S^1) \cong \ell^\infty_N$. It is also clear that $\|\hat{\varphi}_N^{-1}\|_{C_N(S^1)\to\ell^\infty_N} = 1$.

In order to apply the corresponding result of Kantorovich (see [26], Ch. XIV, §1), it remains to make sure that the following two conditions are fulfilled:

$$\exists \eta_1 > 0 \ \forall \tilde{x} \in C_N(S^1) : \|A_N\hat{\varphi}_N\tilde{x} - \varphi_N A\tilde{x}\|_{\ell^\infty} \le \eta_1 \|\tilde{x}\|_{C(S^1)}\,, \tag{36}$$

$$\exists \eta_2 > 0 \ \forall x \in C(S^1) \ \exists \tilde{x} \in C_N\left(S^1\right) : \ \|Ax - \tilde{x}\|_{\ell^\infty} \le \eta_2. \tag{37}$$

We select an arbitrary piece-wise linear function $\tilde{x}$ in $C_N(S^1)$ and put $\vec{x} = (x_0, x_1, \ldots, x_{N-1}) = \varphi_N\tilde{x}$. Then, for all $n = 1, 2, \ldots N-1$,

$$(\varphi_N A\tilde{x})(n) = \frac{1}{N}\left(1 - nN^{-1}\right)\sum_{i=0}^{n-1}\frac{x_i + x_{i+1}}{2} + \frac{n}{N^2}\sum_{i=n}^{N-1}\frac{x_i + x_{i+1}}{2}. \tag{38}$$

In view of (38), we have

$$\begin{aligned}
\|A_N\hat{\varphi}_N\tilde{x} - \varphi_N A\tilde{x}\|_{\ell^\infty} &= \frac{1}{2}\max_{0\le n\le N-1}\left|\frac{1}{N}\left(1 - \frac{n}{N}\right)[x_n - x_0] + \frac{n}{N^2}[x_N - x_n]\right| \\
&= \max_{0\le n\le N-1}|x_N - x_n|\frac{N-2n}{2N^2} \\
&\le \frac{1}{N}\max_{0\le n\le N-1}|x_n| = \frac{1}{N}\|\vec{x}\|_{\ell^\infty}\,.
\end{aligned}$$

Consequently, we can set $\eta_1 = N^{-1}$ in (37). Let us now verify condition (36). For this purpose, we take an arbitrary function $x \in C(S^1)$ and estimate the continuity modulus, $(0, +\infty) \ni \delta \mapsto w(\delta) := \sup_{t_1,t_2:|t_1-t_2|\le\delta}|u(t_1) - u(t_2)|$, of the function $u := Ax$. By virtue of definition (31) of the operator A,

$$u(t_1) - u(t_2) = (1 - t_1)\int_0^{t_1} x(s)\,\mathrm{d}s + t_1\int_{t_1}^{1} x(s)\,\mathrm{d}s - (1 - t_2)\int_0^{t_2} x(s)\,\mathrm{d}s. \tag{39}$$

Inserting the expressions $\pm(1 - t_1)\int_0^{t_2} x(s)\,\mathrm{d}s$ and $\pm t_1\int_{t_2}^1 x(s)\,\mathrm{d}s$ into the right-hand side of (39), we obtain that, for all $\{t_1, t_2\} \subset [0, 1]$,

$$u(t_1) - u(t_2) = (1 - t_1)\int_{t_2}^{t_1} x(s)\,\mathrm{d}s + (t_2 - t_1)\int_0^{t_2} x(s)\,\mathrm{d}s + t_1\int_{t_1}^{t_2} x(s)\,\mathrm{d}s + (t_1 - t_2)\int_{t_2}^{1} x(s)\,\mathrm{d}s. \tag{40}$$

Relation (40) yields $|u(t_1) - u(t_2)| \le 2\,|t_1 - t_2| \cdot \|x\|_{C(S^1)}$ for all $\{t_1, t_2\} \subset [0, 1]$. (Note that the constant '2' in the latter inequality is actually the least possible, because it is realised on the constant elements of $C(S^1)$.) As a result, for an arbitrary δ, the quantity $w(\delta)$ admits the estimate $w(\delta) \le 2\delta$ and, therefore, one can put $\eta_2 = 2N^{-1}$ in condition (36) [26, pp. 540–541]. Thus, all the assumptions of the theorems from §1 in Ch. XIV of [26] hold, and application of Corollary 2 [26, p. 526] completes the proof of the lemma. □

Remark 13. Besides the above-stated assertion, Theorem 1a from [26, p. 529] allows one to claim also that if $\lambda \in \varrho(A)$ and N satisfies the inequality $N > \Big[1+ 2\|I - \hat{\varphi}_N^{-1}\varphi_N\|\Big] \|\, (\lambda I - A)^{-1} \,\|$,then $\lambda \in \varrho(A_N)$. In particular, this is true when $\|\, (\lambda I - A)^{-1} \,\| < N/5$. Consequently, if $N_0 \in \mathbb{N}$ is such that $\|\, (\lambda I - A)^{-1} \,\| < N_0/5$ for all $\lambda \in [1/c_5, 1/2]$, then $c_N \ge c_5$ for $N \ge N_0$. This inequality, however, is crude enough: it can be shown that 656 is the least integer greater than $\|(c_5^{-1}I - A)^{-1}\|$ and, hence, the estimate $c_N \ge c_5$ is guaranteed for $N \ge 3280$ only. The estimate of Lemma 6 below, established in a different way, is more accurate.

Remark 14. The convergence of the sequence $\{c_N\}_{N\ge 2}$ can also be proved by using the theory of compact operator approximation developed by Vainikko [27] (see also Theorem 18.1 in [22]).

Thus, the sequence $c_2, c_3, c_4, \dots$ crucial to estimate (18) for the period of the periodic solutions of the difference system (16), converges (and, moreover, as can be shown, converges monotonically) to a certain limit—the fact which is in accordance with the heuristic conception of the closeness of a discrete model to some continuous whenever the number of nodes is large enough. It is interesting to note that, as a reflection of this closeness, and in view of Lemma 4, one can use Theorem 2 to prove the corresponding estimate for the periods of periodic solutions of ordinary differential equations [18, 9], which resembles the reasonings at the beginning of the paper [13]. More precisely, we have

Theorem 4. *Either the dynamical system*

$$x' = f(x), \tag{41}$$

where the mapping $f : X \to X$ *satisfies* (17)*, has no periodic orbits other than equilibria, or the minimal period* ω *of every non-constant periodic solution of* (41) *satisfies the relation* $\omega \ge c_\infty / r(L)$*,where* $c_\infty = \lim_{N\to+\infty} c_N$*. When* $\sigma(L) = \{0\}$*, system* (41) *cannot have any non-constant periodic solutions.*

Remark 15. The numerical investigation of the transcendent equation (30) from §5 shows (see, e.g., [8, 7, 18]) that $c_\infty \approx 3.4161306$.

The data presented in Table 1 suggest the following[1]

Lemma 5. $\min_{N\ge 2} c_N = c_5 = 3.38113277396367\dots$.

In fact, this statement can be proved rigorously. Let us first establish an estimate a little weaker than that of Lemma 5.

[1] Only the last cell of Table 1 deserves comment. In this relation, see Remark 15.

N	c_N	N	c_N
2	4	50	3.415602628856188
3	3.437694101250946	60	3.415763647669043
4	3.381197846482995	70	3.415860870512664
5	3.381132773963670	80	3.415924025588460
6	3.387542305176701	90	3.415967348911256
7	3.393374797824043	100	3.415998350004724
8	3.397884447992579	200	3.416097540016316
9	3.401284588368706	300	3.416115919915052
10	3.403863389821704	500	3.416125331798814
20	3.412879718581394	700	3.416127925032506
30	3.414671300096892	$\vdots$	$\vdots$
40	3.415306932199579	∞	3.416130626392786

TABLE 1. The numbers c_N for some N.

Lemma 6. $c_N \geq 3.3813473 \qquad (\forall N \geq 810)$.

Proof. We shall use the subsidiary results established in §6 below. Consider the function $\alpha_3 : [0,1] \to \mathbb{R}_+$ (see formula (49) in §6.1) and apply to it the discretisation operator (34). Lemmata 12 and 13 then yield

$$A_N \varphi_N \alpha_3 \leq \lambda^* \varphi_N \alpha_3 + \frac{\max_{t\in[0,1]} |\alpha_3'(t)|}{N} \varphi_N \alpha, \tag{42}$$

$$A_N \varphi_N \alpha \leq \frac{5}{\sqrt{2}} \varphi_N \alpha_3 + \frac{1}{2N} \varphi_N \alpha, \tag{43}$$

where $\lambda^* \approx 3.4046$ is defined in assertion 1° of Corollary 9 from §6.1.

Introduce the operator $\tilde{A}_N := \left[\begin{smallmatrix} A_N & 0 \\ 0 & A_N \end{smallmatrix}\right]$ and set $\tilde{\alpha} := \left[\begin{smallmatrix} \alpha_3 \\ \alpha \end{smallmatrix}\right]$. System (42), (43) then rewrites as $\tilde{A}_N \tilde{\alpha} \leq \Upsilon_N \tilde{\alpha}$, where

$$\Upsilon_N := \begin{bmatrix} \lambda^* & \frac{1}{N} \max_{t\in[0,1]} |\alpha_3'(t)| \\ \frac{5}{\sqrt{2}} & \frac{1}{2N} \end{bmatrix}.$$

Consequently, in order to estimate $r(A_N)$ from above, one can apply the appropriate statements from [28, 22] (see, e.g., Theorem 5.17 in [22]), which results in the inequality

$$r(A_N) \leq r(\Upsilon_N) = r_\sigma(\Upsilon_N). \tag{44}$$

Let us compute $r_\sigma(\Upsilon_N)$. We have

$$2r_\sigma(\Upsilon_N) = \lambda^* + \frac{1}{2N} + \sqrt{\left(\lambda^* + \frac{1}{2N}\right)^2 - \frac{2\lambda^*}{N} + \frac{20}{N\sqrt{2}} \max_{t\in[0,1]} |\alpha_3'(t)|}$$

$$= \lambda^* + \sqrt{\lambda^{*2} + \frac{1}{4N^2} + \frac{10\sqrt{2}\max_{t\in[0,1]}|\alpha_3'(t)| - \lambda^*}{N}} + \frac{1}{2N}$$

$$= \lambda^* + \sqrt{\lambda^{*2} + \frac{1}{4N^2} + \frac{10\sqrt{2}\max_{t\in[0,1]}|\alpha_3'(t)| - \lambda^*}{N}} + \frac{1}{2N} \tag{45}$$

$$= \lambda^* + \sqrt{\lambda^{*2} + \frac{1}{4N^2} + \frac{\frac{8}{3\sqrt{3}}\sqrt{1+\frac{1}{\sqrt{3}}} - \lambda^*}{N}} + \frac{1}{2N}. \tag{46}$$

When passing from (45) to (46), we have used assertion 3∘ of Corollary 8, §6.1.

Equality (46) implies, in particular, $r_\sigma(\Upsilon_{810}) \approx 0.2957401$ and, therefore,

$$1/r_\sigma(\Upsilon_{810}) \approx 3.3813473. \tag{47}$$

Thus, by virtue of (44), we have the estimate $c_N \geq 1/r_\sigma(\Upsilon_{810})$ valid for all $N \geq 810$, which, together with (47), yields the conclusion desired. □

Since $c_5 \approx 3.381133$, Lemma 6 implies that $c_N \geq c_5$ for $N \geq 810$.

Remark 16. Table 1 suggests that the value $N_0 = 810$ in Lemma 6, which guarantees the estimate $c_N \geq c_5$ for all $N \geq N_0$, should be unnecessarily large. However, the computation shows that, e.g., $1/r_\sigma(\Upsilon_{800}) \approx 3.3810592 < c_5$ and, therefore, the least N_0 that can be obtained by the reasoning above is thus comprised between 800 and 810.

Remark 17. One can obtain a statement close to Lemma 6 by estimating directly the eigen-functions of operator (31) (which leads to a differential equation of type 2.41 in [29]) and then arguing similarly to [15, p. 173]. The existence of a unique, modulo the norm, eigen-function of operator (31) is guaranteed by Corollary 11 from §6.1.

In such a manner we have shown that c_N is not less than c_5 at least when $N \geq 810$. Lemma 5 stating the same inequality for all $N \geq 2$ can also be considered as proved completely, because c_N can be computed numerically unless N is too large, for which purpose one can use, e.g., Lemma 14 of §6.2.

Thus, we can derive from Lemma 5 the following

Theorem 5. *When $\sigma(L) \neq \{0\}$, the period of every non-constant periodic solution of the autonomous difference equation* (16) *is not less than $c_5/r(L)$.*

Of course, the latter statement is weaker than Theorem 2. However, its intriguing feature is that, knowing only an upper bound for the spectral radius of the Lipschitz operator L corresponding to the non-linear mapping f in the right-hand side of (16), one can specify without any computation a condition sufficient for the absence of periodic solutions with 'small' periods. This resembles the corresponding result for differential equations, Theorem 4, and suggests, in particular, that estimate (18) from Theorem 2, which, unfortunately, is not optimal for $N \geq 3$, should first of all be 'optimised' for N equal to 3, 4, and 5.

6. Subsidiary Statements

Here, we establish several results referred to in the preceding sections. These results concern the study of the sequence of operators $A_2, A_3, A_4, \ldots$ (see formula (14) in §5) and the related objects.

6.1. Properties of operator (31)**.** In this subsection, we prove some statements on the comparison operator (31), which will be used in §6.2.

Consider the function

$$\alpha(t) := 2t(1-t), \qquad t \in [0,1], \tag{48}$$

put $\alpha_1 := \alpha$, and introduce the sequence

$$\alpha_k(t) := (1-t)\int_0^t \alpha_{k-1}(s)\,\mathrm{d}s + t\int_t^1 \alpha_{k-1}(s)\,\mathrm{d}s \qquad (t \in [0,1],\ k \geq 2). \tag{49}$$

In other words, $\alpha_k = A^k \alpha_0$, where $\alpha_0(t) := 1$ for all $t \in [0,1]$ and A is the linear integral operator in $C([0,1], \mathbb{R})$ defined by formula (31). Sequence (49) reflecting a number of important properties of operator (31), the majority of the statements to follow are devoted to the study of the functions $\alpha_1, \alpha_2, \ldots$.

Lemma 7. *Let*

$$\beta(t) := \frac{1}{2}\alpha(t) \equiv t(1-t) \tag{50}$$

for $t \in [0,1]$*. Then*

$$[A\beta^m](t) = \beta(t)\sum_{k=1}^{m+1} a_k \beta^{k-1}(t) \qquad (\forall t \in [0,1],\ m \in \mathbb{N}),$$

where $a_1 = a_2 = \frac{(m!)^2}{(2m+1)!}$, $a_{k+2} = \frac{2^{k+1}(2k+1)!!(m!)^2}{(k+2)!(2m+1)!}$ *for* $0 \leq k \leq m-2$, $a_m = \frac{1}{2(4m^2-1)}$, *and* $a_{m+1} = \frac{2}{2m+1}$.

Proof. Set $\psi := A\beta^m$. It is obvious that $\psi(0) = 0$ and, since

$$\psi'(t) \equiv (1-2t)\,\beta^m(t) - \int_0^t \beta^m(s)\,\mathrm{d}s + \int_t^1 \beta^m(s)\,\mathrm{d}s,$$

the relation $A\beta^m = a_1\beta + a_2\beta^2 + \cdots + a_{m+1}\beta^{m+1}$ is equivalent to

$$\psi(0) = 0, \tag{51}$$

$$\psi'(t) = (1-2t)\left[a_1 + 2a_2\beta(t) + \cdots + (m+1)\,a_{m+1}\beta^m(t)\right]. \tag{52}$$

Similarly, since

$$\psi'(0) = \int_0^1 \beta^m(s)\,\mathrm{d}s = B\,(m+1, m+1) = \frac{\{\Gamma\,(m+1)\}^2}{\Gamma\,(2m+2)} = \frac{(m!)^2}{(2m+1)!} = a_1$$

and $\psi''(t) \equiv m\,(1-2t)^2\,\beta^{m-1}(t) - 4\beta^m(t)$, system (51), (52) is equivalent to

$$\begin{aligned} \psi(0) &= 0, \qquad \psi'(0) = a_1, \\ \psi''(t) &= -2\,(a_1 + 2a_2\beta(t) + \cdots + (m+1)\,a_{m+1}\beta^m(t)) \\ &\quad + (1-2t)^2\left[2a_2 + \cdots + (m-1)\,ma_m\beta^m(t) \,+\, m\,(m+1)\,a_{m+1}\beta^{m-1}(t)\right]. \end{aligned}$$

Therefore,

$$\begin{aligned} (1-2t)^2[2a_2 + \cdots + r\,(m-1)\,ma_m\beta^{m-2}(t) + (m\,(m+1)\,a_{m+1} - m)\,\beta^{m-1}(t)] \\ = 2[a_1 + \cdots + ma_m\beta^{m-1}(t) + ((m+1)\,a_{m+1} - 2)\,\beta^m(t)] \end{aligned}$$

and thus

$$\begin{aligned} (1-4\beta)[2a_2 + \cdots + m(m-1)a_m\beta^{m-2} + (m(m+1)a_{m+1} - m)\beta^{m-1}] \\ = 2[a_1 + \cdots + ma_m\beta^{m-1} + ((m+1)a_{m+1} - 2)\beta^m]. \end{aligned} \tag{53}$$

Equating the coefficients at $(\beta^i)_{i=0}^m$ in (53), we obtain

$$\begin{aligned} \text{at } \beta^m&: \quad 2(m+1)a_{m+1} - 4 = 4[m - m(m+1)a_{m+1}]; \\ \text{at } \beta^{m-1}&: \quad -4\,(m-1)\,ma_m + m\,(m+1)\,a_{m+1} - m = 2ma_m; \\ \text{at } (\beta^k)_{k=0}^{m-2}&: \quad -4k\,(k+1)\,a_{k+1} + (k+1)\,(k+2)\,a_{k+2} = 2\,(k+1)\,a_{k+1}, \end{aligned}$$

whence it follows that $a_{m+1} = \frac{2}{2m+1}$, $a_m = \frac{1}{2(4m^2-1)}$, and $a_{k+1} = \frac{k+2}{2(2k+1)}a_{k+2}$ for $k = 0, 1, \ldots, m-2$. Thus,

$$a_{k+2} = \frac{2^{k+1}\,(2k+1)!!}{(k+2)!}a_1 = \frac{2^{k+1}(2k+1)!!(m!)^2}{(k+2)!\,(2m+1)!},$$

which concludes the proof. □

Corollary 6. *For all $t \in [0,1]$ and $m \in \mathbb{N}$, the equality*

$$(A\alpha^m)\,(t) = \sum_{k=1}^{m+1} b_k^m \alpha^k(t),$$

holds, in which $b_1^m = \frac{2^{m-1}(m!)^2}{(2m+1)!}$, $b_2^m = \frac{2^{m-2}(m!)^2}{(2m+1)!}$, $b_m^m = \frac{1}{2(4m^2-1)}$, $b_{m+1}^m = \frac{1}{2m+1}$, and $b_k^m = \frac{(2k-3)!!}{k!}b_1^m = \frac{2^{m-1}(m!)^2(2k-3)!!}{(2m+1)!k!}$ for $2 \le k \le m$.

Proof. By virtue of (50), the assertion is readily obtained from Lemma 7. □

Corollary 6 implies, in particular, that $A\alpha = \alpha/6 + \alpha^2/3$, $A\alpha^2 = \alpha/15 + \alpha^2/30 + \alpha^3/5$, and $A\alpha^3 = \alpha/35 + \alpha^2/70 + \alpha^3/70 + \alpha^4/7$.

Corollary 7. *The identity*

$$\alpha_m(t) \equiv [A^m \alpha](t) = \alpha(t) R_{m-1}(\alpha(t)) \qquad (t \in [0,1]),$$

holds, in which $R_{m-1}(\cdot)$ is a certain polynomial of degree $m-1$ having the property that $R_{m-1}(0) = \frac{1}{2}\int_0^1 \alpha_{m-1}(s)\,\mathrm{d}s$.

Remark 18. For every $m \in \mathbb{N}$, the expression $\alpha_m(t) = [A^m\alpha](t)$ is a polynomial of degree $2m$ in t. Thus, Corollary 7 helps significantly is studying the properties of functions (49), because the degree of R_{m-1} is $m-1$.

Corollary 8. *The following equalities hold:*

1° $\alpha_2 = \alpha(1/6 + \alpha/3)$;
2° $\alpha_3 = \alpha(1/20 + \alpha/15 + \alpha^2/15)$;
3○ $\alpha_4 = \alpha(37/2520 + 5\alpha/252 + \alpha^2/70 + \alpha^3/105)$.

Corollary 9. *The following three relations are true:*

1° $\max\limits_{t\in[0,1]} \frac{\alpha_4(t)}{\alpha_3(t)} = 1/\lambda^*$, *where* $\lambda^* = 3.4046234370897\ldots$;
2° $\alpha_3'(t) = \frac{\alpha'(t)}{15}\left[\frac{3}{4} + 2\alpha(t) + 3\alpha^2(t)\right]$ *and* $\alpha_3''(t) = 4\left(\frac{1}{12} - \alpha^2(t)\right)$ *for an arbitrary* $t \in [0,1]$;
3○ $\max\limits_{t\in[0,1]} |\alpha_3'(t)| = \frac{2\sqrt{2}}{15\sqrt{3}}\sqrt{1 + \frac{1}{\sqrt{3}}} = 0.13672791122852\ldots$;
4° $\max_{t\in[0,1]} \frac{\alpha_2(t)}{\alpha_3(t)} = \frac{5}{\sqrt{2}}$.

Note that the value of λ^* in assertion 1° of Corollary 9 can be expressed in radicals. The explicit formula is rather cumbersome and, therefore, we omit it.

Proof. It involves nothing but computation. Let us prove, e.g., 4°. By virtue of Corollary 8, for $w(t) := \alpha_2(t)/\alpha_3(t)$ we have

$$w(t) = \frac{\frac{1}{6} + \frac{\alpha(t)}{3}}{\frac{1}{20} + \frac{1}{15}\alpha(t) + \frac{1}{15}\alpha^2(t)} = \frac{10 + 20\alpha(t)}{3 + 4\alpha(t) + 4\alpha^2(t)} = 10\frac{1 + 2\alpha(t)}{3 + 4\alpha(t) + 4\alpha^2(t)}.$$

The equality $w'(t) = 0$ is equivalent to $3 + 4\alpha(t) + 4\alpha^2(t) = 2(1 + 2\alpha(t))^2$ or, which is the same, $\alpha^2(t) + \alpha(t) = \frac{1}{4}$. Hence, $\max_{t\in[0,1]} w(t) = 10\frac{\sqrt{2}}{4} = \frac{5}{\sqrt{2}}$. □

Corollary 10. $\max_{t\in[0,1]} \alpha_m(t) = \alpha_m(1/2)$ *for every* $m \in \mathbb{N}$.

Proof. Since $\max_{t\in[0,1]} \alpha(t) = \alpha(1/2)$, the result follows from Corollary 7. □

As another useful property of operator (31), we establish its α-positivity understood in the following sense (see [19], Ch. 2, §1.1).

Definition 1. Let $u \succcurlyeq_X 0$ be an element of the POBS $\langle X, \preccurlyeq_X, \|\cdot\|_X\rangle$. A linear operator $A : X \to X$ is said to be u-positive if, for every $x \in X \setminus \{0\}$, there exist some $k(x) \in \mathbb{N}$ and $\{m(x), M(x)\} \subset \mathbb{R}_+$ such that

$$m(x)\,u \preccurlyeq A^{k(x)}x \preccurlyeq M(x)\,u.$$

It is clear that every u-positive operator A is also positive [19], i.e., it leaves invariant the cone X_+.

Lemma 8. *Operator* (31) *is α-positive with respect to function* (48).

Proof. In Definition 1, we can put $k := 1$ and

$$m(x) := \inf_{t\in(0,1)} \frac{[Ax](t)}{\alpha(t)}, \qquad M(x) := \max_{t\in[0,1]} |x(t)|. \tag{54}$$

We need only to make sure that the first quantity in (54) is different from zero. Indeed, taking into account that A maps $C([0,1],\mathbb{R})$ to $C^1([0,1],\mathbb{R})$ and applying the l'Hospital's rule, we obtain

$$\lim_{t\to 0^+} \frac{(Ax)(t)}{\alpha(t)} = \lim_{t\to 0} \frac{(Ax)'(t)}{2-4t} = \frac{1}{2}\left[x(0) + \int_0^1 x(s)\,\mathrm{d}s\right] > 0.$$

Similarly, when $t \to 1^-$,

$$\lim_{t\to 1^-} \frac{[Ax](t)}{\alpha(t)} = \frac{1}{2}\left[x(1) + \int_0^1 x(s)\,\mathrm{d}s\right] > 0.$$

Since $x \neq 0$, this implies $m(x) > 0$ and, by Definition 1, A is α-positive. $\square$

Corollary 11. *Operator* (31) *has a unique, modulo the norm, non-negative eigen-function g corresponding to the eigen-value $r(A)$. This function is such that $m(g)\alpha(t) \leq (Ag)(t) \leq M(g)\alpha(t)$for all $t \in [0,1]$ with $m(g)$ and $M(g)$ as in* (54).

Proof. This statement is an immediate consequence of Theorem 6.2 from [20] and Theorem 2.11 from [19]. $\square$

Lemma 9. $\|A^n\| = \max_{t\in[0,1]} \alpha_n(t)$ *for all $n \in \mathbb{N}$.*

Proof. It suffices to consider definition (31) and take into account the equality $A^n 1 = \alpha_n$ (see (49)). $\square$

Lemma 10. $\lim_{n\to+\infty} \sqrt[n]{\alpha_n(1/2)} = r(A)$.

Proof. Let us consider the Neumann series expansion of the resolvent $R_\lambda(A) := (\lambda I - A)^{-1}$ for $0 < \lambda < r(A)$:

$$\begin{aligned}\|R_\lambda(A)\| &= \frac{1}{\lambda}\left\|\sum\nolimits_{i=0}^{\infty} A^i\lambda^{-i}\right\| \leq \frac{1}{\lambda} + \frac{1}{\lambda^2}\left\|A + \lambda^{-1}A^2 + \dots\right\| \\ &= \frac{1}{\lambda} + \frac{1}{\lambda^2}\left\|A1 + \lambda^{-1}A^2 1 + \lambda^{-2}A^3 1 + \dots\right\|.\end{aligned}$$

Here and above, $\|\cdot\|$ stands for the uniform norm in $C([0,1],\mathbb{R})$.

By virtue of Lemma 9 and Corollary 10, we have

$$\begin{aligned}\|R_\lambda(A)\| &= \frac{1}{\lambda} + \frac{1}{\lambda^2}\left\|\alpha + \frac{\alpha_2}{\lambda} + \frac{\alpha_3}{\lambda^2} + \dots\right\| \\ &= \frac{1}{\lambda} + \frac{\alpha(1/2)}{\lambda^2} + \frac{\alpha_2(1/2)}{\lambda^3} + \frac{\alpha_3(1/2)}{\lambda^4} + \dots =: \sigma_\lambda. \end{aligned} \tag{55}$$

On the other hand, $\|R_\lambda(A)\,1\| = \sigma_\lambda$, where 1 denotes the constant function with the appropriate value. Therefore, $\|R_\lambda(A)\| \geq \sigma_\lambda$ and, in view of (55), we have $\|R_\lambda(A)\| = \sigma_\lambda$. This implies that $r(A)$ coincides with the convergence radius of series (55). Lemma 10 is proved. $\square$

Corollary 12. 1° $\lim_{n\to\infty} \sqrt[n]{\alpha_n(t)/\alpha(t)} = r(A)$ *uniformly in* $t \in [0,1]$;
2° $\lim_{n\to\infty} \sqrt[n]{\alpha_n(t)} = r(A)$ *for every* $t \in [0,1]$.

Proof. By Corollary 7, $\alpha_m(t) = \alpha(t)\,R_{m-1}(\alpha(t))$ for every $t \in [0,1]$, and the constant term of the polynomial R_{m-1} satisfies the relation

$$R_{m-1}(0) = \frac{1}{2}\int_0^1 \alpha_{m-1}(t)\,\mathrm{d}t = \alpha_m(1/2) =: \nu_{0,m}.$$

Furthermore, if $R_{m-1}(\alpha) = \nu_{0,m} + \nu_{1,m}\alpha + \cdots + \nu_{m-1,m}\alpha^{m-1}$, then $\nu_{0,m} = \nu_{1,m}/2 + \cdots + \nu_{m-1,m}/2^{m-1}$. Indeed,

$$\alpha_m(t) = \alpha(t)\left[\frac{1}{2}\int_0^1 \alpha_{m-1}(t)\,\mathrm{d}t + \nu_{1,m}\alpha + \cdots + \nu_{m-1,m}\alpha^{m-1}\right].$$

Let us set $d_m(t) := \alpha_m(t)/\alpha(t)$. Since

$$d_m(0) = \lim_{t\to 0}\frac{(1-t)\int_0^t \alpha_{m-1}(s)\,\mathrm{d}s + t\int_t^1 \alpha_{m-1}(s)\,\mathrm{d}s}{2t(1-t)} = \frac{1}{2}\int_0^1 \alpha_{m-1}(t)\,\mathrm{d}t$$

and

$$d_m(1/2) = \frac{\alpha_m(1/2)}{\alpha(1/2)} = \frac{2^{-1}\int_0^1 \alpha_{m-1}(t)\,\mathrm{d}t}{1/2} = \int_0^1 \alpha_{m-1}(t)\,\mathrm{d}t,$$

we have $2d_m(0) = d_m(1/2)$. Consequently,

$$d_m(1/2) = \nu_{0,m} + \frac{1}{2}\nu_{1,m} + \cdots + \frac{1}{2^{m-1}}\nu_{m-1,m}$$

and $\nu_{0,m} = \sum_{k=1}^{m-1} 2^{-k}\nu_{k,m}$, whence

$$R_{m-1}(\alpha(t)) \leq \nu_{0,m} + \sum_{k=1}^{m-1} 2^{-k}\nu_{k,m} = 2\nu_{0,m} \qquad (t \in [0,1]).$$

Finally, for all $t \in [0,1]$, $\alpha_m(t)$ admits the estimates $\alpha_m(t) \leq 2\alpha(t)\,\alpha_m(1/2)$ and $\alpha_m(t) \geq \alpha(t)\,\alpha_m(1/2)$. Therefore,

$$\alpha(t)\,\alpha_m(1/2) \leq \alpha_m(t) \leq 2\alpha(t)\,\alpha_m(1/2),$$

which implies the two-sided inequality

$$\alpha_m^{1/n}(1/2) \leq d_m^{1/n}(t) \leq \sqrt[n]{2}\,\alpha_m^{1/n}(1/2).$$

By virtue of Lemma 10, we obtain $\lim_{n\to\infty}\sqrt[n]{d_m(t)} = r(A)$, and the proof is complete. $\square$

We conclude the study of sequence (49) by stating one more result, which can be regarded as a continuation of Lemma 10. For every $t \in [0,1]$ and $n \in \mathbb{N}$, we put $\varrho_n(t) := \alpha_{n+1}(t)/\alpha_n(t)$, $m_n := \min_{t\in[0,1]} \varrho_n(t)$, and $M_n := \max_{t\in[0,1]} \varrho_n(t)$.

Corollary 13. $\lim_{n\to\infty} m_n = \lim_{n\to\infty} M_n = r(A)$.

Proof. This statement follows from Lemma 10 and Corollary 12. It can also be obtained by using the theory exposed in [28, pp. 406–407] and [19], Ch. 2, taking into account the α-positivity of A. □

Note that the results obtained allow us to improve the estimate of Lemma 4 from [4]. More precisely, the following statement is true.

Corollary 14. *For all $n \geq 3$, the estimate*

$$\max_{t\in[0,1]} \frac{\alpha_{n+1}(t)}{\alpha_n(t)} \leq \frac{1}{\lambda^*} \leq 0.29372$$

holds. Furthermore, for an arbitrary $\varepsilon > 0$, there exists an $N_\varepsilon \in \mathbb{N}$ such that

$$\max_{t\in[0,1]} \frac{\alpha_{n+1}(t)}{\alpha_n(t)} \leq \frac{1}{c_\infty} + \varepsilon \tag{56}$$

whenever $n \geq N_\varepsilon$.

In relation (56), the constant $c_\infty \approx 3.4161$ is defined by Lemma 4, whereas the expression for λ^* is specified in assertion 1° of Corollary 9.

6.2. Properties of operators (14)**.** The next three lemmata concern the mutual estimates of the values of the operators A and A_N (see formulae (31) and (14)), which play a crucial rôle in establishing the solvability of equation (5) and proof of Lemma 4.

For every $x \in C(S^1)$, we put $H_N x := A_N \varphi_N x - \varphi_N A x$, where φ_N is the discretisation operator (34). Thus, $H_N : C(S^1) \to \ell^\infty_N$ is a certain linear mapping. Recall that the space ℓ^∞_N is introduced in §1.

Lemma 11. *For all $x \in C^1(S^1)$, the following estimate holds:*

$$|H_N x| \leq \frac{\max\limits_{t\in[0,1]} |x'(t)|}{N} \varphi_N \alpha. \tag{57}$$

Proof. Let $x \in C^1(S^1)$ be arbitrary. Then

$$(H_N x)(n) = \left(1 - nN^{-1}\right) \sum_{i=0}^{n-1} \Big[N^{-1} x(i/N) - \int_{\frac{i}{N}}^{\frac{i+1}{N}} x(s)\,\mathrm{d}s \Big]$$
$$+ nN^{-1} \sum_{i=n}^{N-1} \Big[N^{-1} x(i/N) - \int_{\frac{i}{N}}^{\frac{i+1}{N}} x(s)\,\mathrm{d}s \Big]$$

and, by the Lagrange mean value theorem,

$$|(H_N x)(n)| \le \left(1 - nN^{-1}\right) \sum_{i=0}^{n-1} \left| N^{-1} x(i/N) - \int_{\frac{i}{N}}^{\frac{i+1}{N}} x(s)\,\mathrm{d}s \right|$$

$$+ nN^{-1} \sum_{i=n}^{N-1} \left| N^{-1} x(i/N) - \int_{\frac{i}{N}}^{\frac{i+1}{N}} x(s)\,\mathrm{d}s \right|$$

$$= N^{-2} \left(1 - nN^{-1}\right) \sum_{i=0}^{n-1} |x'(\eta_i)| + nN^{-3} \sum_{i=n}^{N-1} |x'(\eta_i)|,$$

where $\eta_i \in [i/N, (i+1)/N]$. Thus,

$$\begin{aligned} |(H_N x)(n)| &\le \frac{1}{N^2} \max_{t \in [0,1]} |x'(t)| \left[n \left(1 - nN^{-1}\right) + nN^{-1}(N-n) \right] \\ &= \frac{1}{N} \max_{t \in [0,1]} |x'(t)| \cdot \alpha(n/N). \end{aligned} \tag{58}$$

In view of the definition (34) of the operator φ_N, (58) yields (57). □

Lemma 12. $A_N \varphi_N \alpha_3 \le \lambda^* \varphi_N \alpha_3 + N^{-1} \max_{t \in [0,1]} |\alpha_3'(t)| \varphi_N \alpha$.

Proof. By virtue of Corollary 9, we have $A\alpha_3 \le \lambda^* \alpha_3$ and $\varphi_N A \alpha_3 \le \lambda^* \varphi_N \alpha_3$. Therefore,

$$A_N \varphi_N \alpha_3 \le \lambda^* \varphi_N \alpha_3 + A_N \varphi_N \alpha_3 - \varphi_N A \alpha_3$$

and, in view of Lemma 11, the desired inequality follows. □

Lemma 13. $A_N \varphi_N \alpha \le \frac{5}{\sqrt{2}} \varphi_N \alpha_3 + \frac{1}{2N} \varphi_N \alpha$.

Proof. For all $n = 0, 1, \ldots, N$, we have

$$(A_N \varphi_N \alpha)(n) = \frac{(\varphi_N \alpha)(n)}{6} \left[1 + 2(\varphi_N \alpha)(n) + \frac{6n}{N^2} - \frac{3}{N} - \frac{2}{N^2} \right] \tag{59}$$

$$\begin{aligned} &\le \frac{(\varphi_N \alpha)(n)}{6} \left[1 + 2(\varphi_N \alpha)(n) + \left| \frac{6n}{N^2} - \frac{3}{N} \right| \right] \\ &\le \frac{(\varphi_N \alpha)(n)}{6} \left[1 + 2(\varphi_N \alpha)(n) + \frac{3}{N} \right] \\ &= (\varphi_N \alpha_2)(n) + \frac{1}{2N} (\varphi_N \alpha)(n). \end{aligned}$$

(Equality (59) can be found, e.g., in [2, p. 180]). Thus, $A_N \varphi_N \alpha \le \varphi_N \alpha_2 + \frac{1}{2N} \varphi_N \alpha$ and, in view of assertion 4° of Corollary 9, we obtain the required inequality $A_N \varphi_N \alpha \le \frac{5}{\sqrt{2}} \varphi_N \alpha_3 + \frac{1}{2N} \varphi_N \alpha$. □

The last statement of this paper, Lemma 14, establishes the relation between the spectral radius of operator (14) and the non-negative roots of a certain algebraic equation.

Lemma 14. *For every $N \in \mathbb{N}$, the number $r(A_N) = 1/c_N$ is equal to the greatest positive root of the polynomial*

$$p_N(\lambda) := N^2\lambda^{N-1} - (N-1)\lambda^{N-2} - \sum_{i=1}^{N-2} i\lambda^{i-1} \prod_{\nu=i+1}^{N-1} \left(\lambda + \frac{N-2\nu}{N^2}\right). \tag{60}$$

Proof. Since A_N leaves invariant the cone $\mathbb{R}_+^N$ of non-negative N-dimensional real vectors, the well-known Perron–Frobenius theorem [20] guarantees that its maximal, in modulus, eigen-value $\lambda_N := r_\sigma(A_N)$ is positive, and a non-negative eigen-vector $\vec{x} = (x_0, x_1, \dots, x_{N-1}) \in \mathbb{R}_+^N \setminus \{0\}$ corresponds to it. In other words,

$$\frac{1}{N}\left(1 - \frac{n}{N}\right)\sum_{i=0}^{n-1} x_i + \frac{n}{N^2}\sum_{i=n}^{N-1} x_i = \lambda_N x_n, \tag{61}$$

where $n = 1, 2, \dots, N-1$ and $x_0 = 0$.

For $n = 0, 1, \dots, N-1$, we set $X_n := \sum_{\nu=0}^{n} x_\nu$. Relation (61) then rewrites as

$$\frac{1}{N}\left(1 - \frac{n}{N}\right)X_{n-1} + \frac{n}{N^2}\left(X_{N-1} - X_{n-1}\right) = \lambda_N\left(X_n - X_{n-1}\right)$$

for $n = 1, 2, \dots, N-1$. (Obviously, $X_0 = 0$.) After rearranging the summands, we obtain

$$\lambda_N X_n = \left(\frac{N-2n}{N^2} + \lambda_N\right)X_{n-1} + \frac{n}{N^2}X_{N-1}$$

or, which is the same,

$$X_{n-1} = \frac{\lambda_N X_n - \frac{n}{N^2}X_{N-1}}{\lambda_N + \frac{N-2n}{N^2}} \qquad (n = 1, 2, \dots, N-1). \tag{62}$$

When $n = 1$, relation (62) means that

$$\lambda_N X_1 = \frac{1}{N^2}X_{N-1}. \tag{63}$$

Let us now fix some $k \in \{2, 3, \dots, N-1\}$ and consider the quantity X_{N-k}. Relation (62) yields

$$X_{N-k} = \frac{\lambda_N}{\lambda_N + \frac{N-2(N-k+1)}{N^2}}X_{N-k+1} - \frac{1}{\lambda_N + \frac{N-2(N-k+1)}{N^2}}\frac{N-k+1}{N^2}X_{N-1}.$$

Continuing sequentially, we obtain

$$\begin{aligned} X_{N-k} = {} & \frac{\lambda_N^2}{\left(\lambda_N + \frac{N-2(N-k+1)}{N^2}\right)\left(\lambda_N + \frac{N-2(N-k+2)}{N^2}\right)}X_{N-k+2} \\ & - \frac{\lambda_N}{\left(\lambda_N + \frac{N-2(N-k+1)}{N^2}\right)\left(\lambda_N + \frac{N-2(N-k+2)}{N^2}\right)}\frac{N-k+2}{N^2}X_{N-1} \\ & - \frac{1}{\lambda_N + \frac{N-2(N-k+1)}{N^2}}\frac{N-k+1}{N^2}X_{N-1} \end{aligned}$$

and

$$X_{N-k} = \Bigg[\frac{\lambda_N^{k-1}}{\prod_{\nu=1}^{k-1}\left(\lambda_N + \frac{N-2(N-k+\nu)}{N^2}\right)} - \frac{\lambda^{k-2}}{\prod_{\nu=1}^{k-1}\left(\lambda_N + \frac{N-2(N-k+\nu)}{N^2}\right)}\frac{N-k+(k-1)}{N^2} - \frac{\lambda_N^{k-3}}{\prod_{\nu=1}^{k-2}\left(\lambda_N + \frac{N-2(N-k+\nu)}{N^2}\right)}\frac{N-k+(k-2)}{N^2} - \dots - \frac{1}{\lambda_N + \frac{N-2(N-k+1)}{N^2}}\frac{N-k+1}{N^2}\Bigg] X_{N-1},$$

which can be rewritten as

$$\frac{X_{N-k}}{X_{N-1}} = \frac{\lambda_N^{k-1}}{\prod\limits_{\nu=1}^{k-1}\left(\lambda_N + \frac{2(k-\nu)-N}{N^2}\right)} - \frac{1}{N^2}\sum_{i=1}^{k-1}\frac{\lambda_N^{i-1}\left(N-k+i\right)}{\prod\limits_{\nu=1}^{i}\left(\lambda_N + \frac{N-2(N-k+\nu)}{N^2}\right)}\,. \tag{64}$$

The latter is true, because $X_{N-1} > 0$ (otherwise $\vec{x} = 0$, which is impossible). Inserting $k = N-1$ into (64), we obtain

$$\frac{X_1}{X_{N-1}} = \frac{\lambda^{N-2}}{\prod\limits_{\nu=1}^{N-2}\left(\lambda + \frac{N-2(\nu+1)}{N^2}\right)} - \frac{1}{N^2}\sum_{i=1}^{N-2}\frac{\lambda^{i}\left(i+1\right)}{\prod\limits_{\nu=1}^{i}\left(\lambda + \frac{N-2(\nu+1)}{N^2}\right)}\,. \tag{65}$$

Finally, combining (65) and (63), we arrive at the relation

$$\frac{1}{N^2\lambda_N} = \frac{\lambda_N^{N-2}}{\prod\limits_{\nu=1}^{N-2}\left(\lambda_N + \frac{N-2(\nu+1)}{N^2}\right)} - \frac{1}{N^2}\sum_{i=1}^{N-2}\frac{\lambda_N^{i-1}\left(i+1\right)}{\prod\limits_{\nu=1}^{i}\left(\lambda_N + \frac{N-2(\nu+1)}{N^2}\right)}, \tag{66}$$

which can be regarded as an equation with respect to λ_N.

Equation (66) is, obviously, an algebraic one. It can be brought to the form

$$N^2\lambda_N^{N-1} - (N-1)\,\lambda_N^{N-2} - \sum_{i=1}^{N-2} i\lambda_N^{i-1}\prod_{\nu=i}^{N-2}\left(\lambda_N + \frac{N-2\,(\nu+1)}{N^2}\right) = 0. \tag{67}$$

The left-hand side of (67) is a polynomial of degree $N-1$ in λ_N, which in fact coincides with (60). Note also that (67) is satisfied by all the eigen-values of A_N corresponding to non-negative eigen-vectors. Consequently, $r\,(A_N)$ is the greatest root of this equation. Lemma is proved. □

Thus, we can compute c_N, which is necessary for the efficient application of Theorems 2 and 3, as the maximal positive root of polynomial (60), bypassing definition (19). For this purpose, one can use, e.g., Bernoulli's method [30, 31], which is applicable in view of the properties of operator (14) and does not involve operations with matrices of high dimension. The coefficients of polynomial (60) can be found analytically by the use of the Viète theorem and the formulae for sums like $\sum_{\nu=\nu_1}^{\nu_2} \nu^p$, where $p \in \mathbb{N}$ is fixed. These questions are not considered here.

In conclusion, we note that the techniques developed in this paper are likely to be of use when studying other similar problems.

Acknowledgements

The authors acknowledge the partial support of OMFB, grant #UK-3/99 (A. N. Ronto, M. Rontó, and A. M. Samoilenko), OTKA, grant #T031961 (M. Rontó), and FONDECYT, grant #8990013 (S. I. Trofimchuk).

References

1. A. M. Samoilenko and N. I. Ronto, Numerical-analytic methods of investigating periodic solutions. (Translated from the Russian) *Mir, Moscow,* 1980.
2. M. Kwapisz, Some existence and uniqueness results for boundary value problems for difference equations. *Appl. Anal.* **37**(1990), 169–182.
3. M. Kwapisz, On modification of Samoilenko's numerical-analytic method of solving boundary value problems for difference equations. *Appl. Math.* **38**(1993), 133–144.
4. M. Ronto and J. Mészáros, Some remarks concerning the convergence of the numerical-analytic method of successive approximation. *Ukr. Math. J.* **48**(1996), No. 1, 101–107.
5. A. M. Samoilenko and N. I. Ronto, Numerical-analytic methods in the theory of boundary value problems for ordinary differential equations. (Russian) *Naukova Dumka, Kiev,* 1992.
6. A. M. Samoilenko and V. N. Laptinskii, On estimates for periodic solutions of differential equations. (Russian) *Dokl. Akad. Nauk Ukrain. SSR, Ser.* A (1982), No. 1, 30–32.
7. N. A. Evkhuta and P. P. Zabreiko, The Poincaré method and Samoilenko method for the construction of periodic solutions to ordinary differential equations. *Math. Nachr.* **153**(1991), 85–99.
8. E. P. Trofimchuk, Integral operators of the method of successive periodic approximations. (Russian) *Mat. Fiz. Nelinein. Mekh.* (1990) No. 13(47), 31–36.
9. M. Rontó and S. I. Trofimchuk, Numerical-analytic method for non-linear differential equations. *Publ. Univ. Miskolc, Ser. D. Natur. Sc. Math.* **38**(1998), 97–116.
10. A. M. Samoilenko, Numerical-analytic method of investigating countable systems of periodic differential equations. *Mat. Fizika* **2**(1966), 115–132.
11. J. A. Yorke, Periods of periodic solutions and the Lipschitz constant. *Proc. Amer. Math. Soc.* **22**(1969), 509–512.
12. A. Lasota and J. Yorke, Bounds for periodic solutions of differential equations in Banach spaces. *J. Differential Equations* **10**(1971), 83–91.
13. S. Busenberg, D. Fisher, and M. Martelli, Better bounds for periodic orbits of differential equations in Banach spaces. *Proc. Amer. Math. Soc.* **86**(1986), 376–378.
14. S. Busenberg and M. Martelli, Bounds for the period of periodic orbits of dynamical systems. *J. Differential Equations* **67**(1987), 359–371.
15. D. Fisher, Strict bounds for the period of periodic orbits of difference equations. *J. Math. Anal. Appl.* **138**(1989), 166–177.
16. S. Busenberg, D. Fisher, and M. Martelli, Minimal periods of discrete and smooth orbits. *Amer. Math. Monthly* **96**(1989), 5–17.

17. W. Słomzyński, Bounds for periodic solutions of difference and differential equations. *Ann. Soc. Math. Polonae, Ser. Comment. Math.* **26**(1986), No. 2, 325–330.

18. A. N. Ronto, On the periods of periodic motions in autonomous systems. (Russian) *Ukrain. Mat. Zh.* **53**(2001), No. 1, 94–112.

19. M. A. Krasnosel'skii, Positive solutions of operator equations. *Noordhoff, Groningen,* 1964.

20. M. G. Krein and M. A. Rutman, Linear operators leaving invariant a cone in a Banach space. *Amer. Math. Soc. Transl.* 26, *Amer. Math. Soc., Providence, R.I.,* 1950.

21. G. Birkhoff, Lattice theory. *Amer. Math. Soc. Colloq. Publ.* 25, *Amer. Math. Soc., Providence, R.I.,* 1948.

22. M. A. Krasnosel'skii, G. M. Vainikko, P. P. Zabreiko, Ya. B. Rutitskii, and V. Ya. Stetsenko, Approximate solution of operator equations. *Noordhoff, Groningen,* 1972.

23. F. Riesz and B. Sz.-Nagy, Leçons d'analyse fonctionelle. *Akadémiai Kiadó, Budapest,* 1972.

24. I. A. Bakhtin, M. A. Krasnosel'skii, and V. Ya. Stetsenko, On the continuity of linear positive operators. (Russian) *Sibirsk. Mat. Zh.* **3**(1962), No. 1, 156–160.

25. R. Nussbaum, Some asymptotic fixed point theorems. *Trans. Amer. Math. Soc.* **171**(1972), 349–375.

26. L. V. Kantorovich and G. P. Akilov, Functional analysis, 2nd rev. ed. (Russian) *Nauka, Moscow,* 1977.

27. G. M. Vainikko, Compact approximation of operators and approximate solution of equations. (Russian) *Tartu. Gos. Univ., Tartu,* 1970.

28. Functional analysis (S. G. Krein, ed.), 2nd rev. aug. ed. (Russian) *Nauka, Moscow,* 1972.

29. E. Kamke, Differentialgleichungen. Lösungsmethoden und Lösungen. I. *Akademische Verlagsgesellschaft, Leipzig,* 1944.

30. B. P. Demidovich and I. A. Maron, Fundamentals of computational mathematics. (Russian) *Fizmatgiz, Moscow,* 1963.

31. F. B. Hildebrand, Introduction to numerical analysis, 2nd ed. *McGraw-Hill, New York,* 1974.

(Received 20.03.2000)

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