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ON A CHARACTERISATION OF INNER PRODUCT SPACES

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Abstract. It is well known that for the Hilbert space H the minimum value of the functional $F_\mu(f) = \int_H \|f-g\|^2 d\mu(g)$, $f \in H$, is achived at the mean of μ for any probability measure μ with strong second moment on H. We show that the validity of this property for measures on a normed space having support at three points with norm 1 and arbitrarily fixed positive weights implies the existence of an inner product that generates the norm.

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Let X be a real normed space, $\dim X \geq 2$, and μ be a Borel probability measure on X with strong second moment. Denote by F_μ the following functional on X:

$$F_\mu(f) = \int\limits_X \|f-g\|^2 d\mu(g), \quad f \in X.$$

It is easy to show that if X is an inner product space and there exists the mean m of μ (in the usual weak sense as the Pettis integral), then $F_\mu(f) \geq F_\mu(m)$ for all $f \in X$.

The problem which was brought to my attention by N. Vakhania was to find a class of probability measures as small as possible, for which this property of F_μ characterizes the inner product spaces. It is easy to see that the class of measures with supports containing two points is not a sufficient class since the minimum of F_μ is attained at the mean of μ for any such μ whatever the normed space X is. Indeed, for any normed space X let μ be a probability measure concentrated at two points $f,\ g \in X$ and let $\mu(f) = \alpha$, $\mu(g) = \beta$, $\alpha > 0$, $\beta > 0$, $\alpha+\beta = 1$. It is clear that $m = \alpha f + \beta g$ and $F_\mu(m) = \alpha\beta\|f-g\|^2$. The condition $F_\mu(h) \geq F_\mu(m),\ h \in X$, gives the inequality

$$\alpha\|f-h\|^2 + \beta\|g-h\|^2 \geq \alpha\beta\|f-g\|^2.$$

Denoting $f-h$ and $g-h$ by p and q respectively we obtain

$$\alpha\|p\|^2 + \beta\|q\|^2 \geq \alpha\beta\|p-q\|^2. \tag{1}$$

However, this inequality is true for any normed space since by the triangle inequality we have

$$\alpha\beta\|p-q\|^2 \leq \alpha\beta\|p\|^2 + 2\alpha\beta\|p\| \cdot \|q\| + \alpha\beta\|q\|^2$$

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and, using the obvious relation $2\alpha\beta\|p\|\cdot\|q\| \le \alpha^2\|p\|^2+\beta^2\|q\|^2$, we get (1).

As the referee of the present paper noticed[1], a sufficient class can be constructed using measures concentrated at three points. There are two types of results in this direction:

a) the sufficient class consists of measures $\mu=\frac{1}{3}\delta_{f_1}+\frac{1}{3}\delta_{f_2}+\frac{1}{3}\delta_{f_3}$ for all triplets $\{f_1,f_2,f_3\}$ from X (δ_p being the Dirac measure at $p\in X$) (see [1], Proposition (1.12), p. 10)

b) the sufficient class consists of measures $\mu=\alpha\delta_{f_1}+\beta\delta_{f_2}+\gamma\delta_{f_3}$ for all triplets $\{f_1,f_2,f_3\}$ with unit norms and all weights α,β,γ such that $\alpha f_1+\beta f_2+\gamma f_3=0$ (Theorem 5.3 in [2], p. 236).

The aim of this paper is to show that in fact yet a smaller class of measures can be taken.

Theorem. *Let X be a real normed space, $\dim X\ge 2$, $S(X)$ be the set of points of norm one, α,β,γ be arbitrarily fixed positive numbers and δ_p be the Dirac measure at $p\in X$. The following propositions are equivalent:*

(i) *X is an inner-product space.*

(ii) *For any two points f,g from $S(X)$ and the point $h=0$, the mean of the measure $\mu=\frac{1}{\alpha+\beta+\gamma}(\alpha\delta_f+\beta\delta_g+\gamma\delta_0)$ is a point of a local minimum for the functional*

$$F_\mu(t)=\alpha\|t-f\|^2+\beta\|t-g\|^2+\gamma\|t\|^2,\ t\in X.$$

(iii) *For any three points f,g,h from $S(X)$ the mean of $\mu=\frac{1}{\alpha+\beta+\gamma}(\alpha\delta_f+\beta\delta_g+\gamma\delta_h)$ is a point of a local minimum for the functional*

$$F_\mu(t)=\alpha\|t-f\|^2+\beta\|t-g\|^2+\gamma\|t-h\|^2,\ t\in X.$$

According to the well-known von Neumann–Jordan criterion it is enough to prove the Theorem for the case $\dim X=2$. Thus we should prove that the surface $S(X)$ of the unit ball in $(R^2,\|\cdot\|)$ is an ellipse. It is clear that we may assume $\alpha+\beta+\gamma=1$.

The proof of the Theorem is based on the following auxiliary results.

Lemma 1. *Let α,β,γ be given positive reals with $\alpha+\beta+\gamma=1$. For any two noncollinear A and B from $S(X)$ there exist:*

(i) *A_1 and B_1 on $S(X)$ such that*

$$\frac{A_1-M}{\|A_1-M\|}=A,\qquad \frac{B_1-M}{\|B_1-M\|}=B,$$

where $M=\alpha A_1+\beta B_1$.

(ii) *$\overline{A}_1,\overline{B}_1,\overline{C}_1$ on $S(X)$ such that*

$$\frac{\overline{A}_1-\overline{M}}{\|\overline{A}_1-\overline{M}\|}=A,\qquad \frac{\overline{B}_1-\overline{M}}{\|\overline{B}_1-\overline{M}\|}=B,$$

where $\overline{M}=\alpha\overline{A}_1+\beta\overline{B}_1+\gamma\overline{C}_1$.

[1] The author takes this opportunity to express his deep gratitude to the referee for his comments including this information.

Proof. (i). Denote by $S'(X)$ the part of $S(X)$ which is inside the smaller angle generated by the vectors A and B. Let C be any point of $S'(X)$. It is clear that for all u, $0<u<1$, there exist A_u and B_u on $S'(X)$ such that for some $u_1>0$, $u_2>0$ we have $A_u-uC=u_1A$ and $B_u-uC=u_2B$. Since $M_u=uC$ is inside the triangle A_uOB_u, where O denotes the zero vector, we have

$$M_u=\alpha_uA_u+\beta_uB_u$$

for some $\alpha_u>0$, $\beta_u>0$, $\gamma_u>0$, $\alpha_u+\beta_u+\gamma_u=1$. Therefore we have to prove that for some $C\in S'(X)$ and $u>0$ there exist α_u,β_u and γ_u such that $\alpha_u=\alpha$, $\beta_u=\beta$, $\gamma_u=\gamma$. It is clear that $\frac{\|M_u\|}{\|O_u-M_u\|}=\frac{1-\gamma_u}{\gamma_u}$ where O_u is the intersection of the lines (A_uB_u) and (OC). Consider the function $\varphi(u)=\frac{\|M_u\|}{\|O_u-M_u\|}=\frac{1-\gamma_u}{\gamma_u}$. Since $S(X)$ is a continuous curve, the function φ defined on the interval $(0,1)$ is continuous and $\lim\limits_{u\to1}\varphi(u)=+\infty$, $\lim\limits_{u\to0}\varphi(u)=0$. Therefore there exists u_C such that $\varphi(u_C)=\frac{\|M_{u_C}\|}{\|M_{u_C}-O_{u_C}\|}=\frac{1-\gamma}{\gamma}$. Now we consider the following two continuous functions on $S'(X)$:

$$\psi_1(C)=\frac{\|A_{u_C}-M_{u_C}\|}{\|A'_{u_C}-M_{u_C}\|}=\frac{1-\alpha_{u_C}}{\alpha_{u_C}},\qquad \psi_2(c)=\frac{\|B_{u_C}-M_{u_C}\|}{\|B'_{u_C}-M_{u_C}\|}=\frac{1-\beta_{u_C}}{\beta_{u_C}},$$

where A'_{u_C} (B'_{u_C}) denotes the intersection of the lines $(A_{u_C}M_{u_C})$ and (OB_{u_C}) $((B_{u_C}M_{u_C})$ and $(OA_{u_C}))$. Obviously, $\lim\limits_{C\to B}\psi_1(C)=+\infty$, $\lim\limits_{C\to A}\psi_2(C)=+\infty$. Since $\gamma_{u_C}=\gamma$ and $\alpha_{u_C}+\beta_{u_C}+\gamma=1$, we get $\frac{1}{1+\psi_1(C)}+\frac{1}{1+\psi_2(C)}+\gamma=1$. This equality gives $\lim\limits_{C\to A}\psi_1(C)=\frac{\gamma}{1-\gamma}$. As ψ_1 receives all values from the interval $(\frac{\gamma}{1-\gamma},+\infty)$, the inequality $\frac{1-\alpha}{\alpha}>\frac{\gamma}{1-\gamma}$ shows the existence of a point $C_1\in S'(X)$ such that $\psi_1(C_1)=\frac{1-\alpha}{\alpha}$, $\psi_2(C_1)=\frac{1-\beta}{\beta}$. For such C_1 we have $\alpha_{u_{C_1}}=\alpha$, $\beta_{u_{C_1}}=\beta$, $\gamma_{u_{C_1}}=\gamma$ which proves the statement (i).

(ii). Now we consider the same points A_1,B_1 as in (i) and the other point A_2 of the intersection $S(X)\cap(A_1M)$.

Let M_1 be a point on the line (A_1A_2) which is inside the unit ball $B(X)$. Let now B_2 be the point on $S(X)$ for which $B_2-M_1=uB$, $u>0$. Denote by C_2 the point $\frac{1}{\gamma}(M_1-\alpha A_1-\beta B_2)$. Since $\frac{A_1-M_1}{\|A'_1-M_1\|}=\frac{1-\alpha}{\alpha}$, where $A'_1=(B_2C_2)\cap(A_1A_2)$, A'_1 is outside of $B(X)$ if $\|M_1-A_2\|$ is small enough and hence C_2 is outside of $B(X)$ as well. Therefore there exists a point M_1 on (A_1A_2) such that the points $B_2=M_1+uB$, $u>0$, and $C_2=\frac{1}{\gamma}(M_1-\alpha A_1-\beta B_2)$ are on $S(X)$ and the proof is complete. □

Lemma 2. *There exists an ellipse which is inside the unit ball $B(X)$ and touches $S(X)$ at four points at least.*

Proof. It is easy to show that an ellipse of maximum area inside $B(X)$ touches $S(X)$ at four points at least (this argument seems to be used frequently, see, e.g., [3], p. 322). □

Lemma 3. *Let φ and ψ be two functions defined on the interval $I = (a-\varepsilon, a+\varepsilon)$, $\varepsilon > 0$, such that $\psi(x) \geq \varphi(x)$, $\forall x \in I$, $\psi(a) = \varphi(a)$ and the derivatives $\varphi'(a)$, $\psi'_-(a)$, $\psi'_+(a)$ exist. If $\psi'_-(a) \geq \psi'_+(a)$, then $\psi'_-(a) = \psi'_+(a) = \varphi'(a)$.*

Proof.

$$\varphi'(a) = \lim_{u\to 0,\ u>0} \frac{\varphi(a)-\varphi(a-u)}{u} \geq \lim_{u\to 0,\ u>0} \frac{\psi(a)-\psi(a-u)}{u} = \psi'_-(a)$$
$$\geq \psi'_+(a) = \lim_{u\to 0,\ u>0} \frac{\psi(a+u)-\psi(a)}{u} \geq \lim_{u\to 0,\ u>0} \frac{\varphi(a+u)-\varphi(a)}{u} = \varphi'(a)$$

which proves the lemma. □

Proof of the Theorem. Let E be the ellipse from Lemma 2 and A', B' be the points of the intersection $S(X) \cap E$, $A' \neq B'$, $A' \neq -B'$. Apply an affine transformation T that carries E into the unit circle of $(R^2, \|\cdot\|_2)$, $\|\cdot\|_2$ being the usual l_2 norm. Let XOY be an orthogonal Cartesian system on R^2 such that $T(A') = (-1, 0)$. Denote $(-1, 0)$ by A, and $T(B')$ by $B = (b_1, b_2)$. Obviously, $b_1^2 + b_2^2 = 1$ and $b_2 \neq 0$. By Lemma 1 there exist the points $\overline{A}_1$, $\overline{B}_1$, $\overline{C}_1$ from $T(S(X))$ for which the following equalities hold:

$$\frac{\overline{A}_1 - \overline{M}}{\|\overline{A}_1 - \overline{M}\|} = A, \qquad \frac{\overline{B}_1 - \overline{M}}{\|\overline{B}_1 - \overline{M}\|} = B, \tag{2}$$

where

$$\overline{M} = \alpha\overline{A}_1 + \beta\overline{B}_1 + \gamma\overline{C}_1 . \tag{3}$$

Denote now

$$(x', y') = \frac{\overline{C}_1 - \overline{M}}{\|\overline{C}_1 - \overline{M}\|}. \tag{4}$$

Since $\beta > 0$ relation (2) and (3) show that $y' \neq 0$.

Let M_ε be the point $M_\varepsilon = (a\varepsilon, \varepsilon)$, $a = \frac{x'}{y'}$. Introduce the notation:

$$\overline{A}_1 - \overline{M} - M_\varepsilon = (x_1 - x_0 - a\varepsilon,\ y_1 - y_0 - \varepsilon) := (m_1 - a\varepsilon,\ n_1 - \varepsilon),$$
$$\overline{B}_1 - \overline{M} - M_\varepsilon = (x_2 - x_0 - a\varepsilon,\ y_2 - y_0 - \varepsilon) := (m_2 - a\varepsilon,\ n_2 - \varepsilon),$$
$$\overline{C}_1 - \overline{M} - M_\varepsilon = (x_3 - x_0 - a\varepsilon,\ y_3 - y_0 - \varepsilon) := (m_3 - a\varepsilon,\ n_3 - \varepsilon).$$

It is clear that $n_1 = 0$, $m_1 \neq 0$, $n_2 \neq 0$, $n_3 \neq 0$ and $\|\overline{A}_1 - \overline{M}\| = -m_1$, $\|\overline{B}_1 - \overline{M}\| = \frac{n_2}{b_2}$, $\|\overline{C}_1 - \overline{M}\| = \frac{n_3}{y'}$. Since $a = \frac{x'}{y'} = \frac{x_3 - x_0}{y_3 - y_0} = \frac{m_3}{n_3}$ we get $\overline{C}_1 - \overline{M} - M_\varepsilon = (m_3 - a\varepsilon,\ n_3 - \varepsilon) = (m_3 - \frac{m_3}{n_3}\varepsilon,\ n_3 - \varepsilon) = \frac{n_3 - \varepsilon}{y'}(x', y')$ and hence

$$\|\overline{C}_1 - \overline{M} - M_\varepsilon\| = \frac{n_3 - \varepsilon}{y'}. \tag{5}$$

We are going to estimate the norms of the two other vectors. First we consider the case $\varepsilon > 0$. Without loss of generality we may assume that $b_2 < 0$. Consider the two lines $(L_1) : y = -ux - u$ and $(L_2) : y = (-b - \omega(u))(x - b_1) + b_2$ where $u > 0$, $b = b_1/b_2$ and ω is a positive continuous function defined on $[0, \infty)$ such that

$\lim_{u\to\infty} \omega(u) = 0$. By Lemma 3 there exist the tangents to $T(S(X))$ at the points A and B and they are expressed by the equations $x = -1$, $y = -b(x - b_1) + b_2$, respectively. The line (L_1) passes the point A and is different from the tangent at A. Therefore (L_1) intersects $T(S(X))$ at some other point $A_u \neq A$. By the convexity of the unit ball the segment $\overline{A_u A} = \{vA + (1 - v)A_u,\ 0 \leq v \leq 1\}$ is inside $T(B(X))$. Let $(\overline{x}, \overline{y})$ be the point of intersection of the lines $\{v(\overline{A}_1 - \overline{M} - M_\varepsilon),\ v \in R\}$ and (L_1), i.e., $\overline{x} = \frac{-u(m_1 - a\varepsilon)}{(m_1 - a\varepsilon)u - \varepsilon}$. If ε is small enough, then the point $(\overline{x}, \overline{y})$ is on the segment $\overline{A_u A}$ and therefore we get the inequality

$$\|\overline{A}_1 - \overline{M} - M_\varepsilon\| \leq \frac{\|\overline{A}_1 - \overline{M} - M_\varepsilon\|_2}{\|(\overline{x}, \overline{y})\|_2} = \frac{m_1 - a\varepsilon}{\overline{x}} = -m_1 + a\varepsilon + \varepsilon/u \qquad (6)$$

for all $\varepsilon,\ 0 < \varepsilon < \varepsilon'_u,\ \varepsilon'_u > 0$.

Now we consider the intersection $(\overline{\overline{x}}, \overline{\overline{y}})$ of the lines $\{v(\overline{B}_1 - \overline{M} - M_\varepsilon),\ v \in R\}$ and (L_2). We get $\overline{\overline{x}} = \frac{(m_2 - a\varepsilon)(b_1 b + b_2 + b_1\omega(u))}{n_2 - \varepsilon + (m_2 - a\varepsilon)(b + \omega(u))}$. The same arguments show that there exists $\varepsilon''_u > 0$ such that

$$\|\overline{B}_1 - \overline{M} - M_\varepsilon\| \leq \frac{m_2 - a\varepsilon}{\overline{\overline{x}}} = \frac{n_2 - \varepsilon + (m_2 - a\varepsilon)(b + \omega(u))}{b_1 b + b_2 + b_1\omega(u)}$$

for all $\varepsilon,\ 0 < \varepsilon < \varepsilon''_u$.

Since $\frac{m_2}{n_2} = \frac{b_1}{b_2} = b$, we get

$$\|\overline{B}_1 - \overline{M} - M_\varepsilon\| \leq \frac{n_2}{b_2} - \frac{1 + ab + a\omega(u)}{(1 + b^2 + b\omega(u))b_2} \cdot \varepsilon. \qquad (7)$$

By the property of the functional F_μ, there exists $\varepsilon' > 0$ such that $F(\overline{M}) \leq F(\overline{M} + M_\varepsilon)$ for all $\varepsilon,\ 0 < \varepsilon < \varepsilon'$. If $\varepsilon < \min(\varepsilon', \varepsilon'_u, \varepsilon''_u) = \varepsilon_u$, we obtain, using relations (5), (6) and (7),

$$F(\overline{M}) = \alpha m_1^2 + \beta \frac{n_2^2}{b_2^2} + \gamma \frac{n_3^2}{y'^2}$$
$$\leq \alpha \left(m_1 - \left(a + \frac{1}{u}\right)\varepsilon\right)^2 + \beta\left(\frac{n_2}{b_2} - \frac{1 + ab + a\omega(u)}{(1 + b^2 + b\omega(u))b_2}\varepsilon\right)^2 + \gamma\left(\frac{n_3 - \varepsilon}{y'}\right)^2,$$

i.e., $0 \leq 2h_u\varepsilon + h'_u\varepsilon^2$, where

$$h_u = -\alpha m_1\left(a + \frac{1}{u}\right) - \beta\frac{n_2(1 + ab + a\omega(u))}{b_2^2(1 + b^2 + b\omega(u))} - \gamma\frac{n_3}{y'^2}.$$

Since $\varepsilon > 0$, we have

$$h_u \geq -\frac{\varepsilon h'_u}{2}$$

for all $\varepsilon,\ 0 < \varepsilon < \varepsilon_u$, i.e. $h_u \geq 0$ for all $u > 0$. Let $\overline{h} = -\alpha m_1 a - \frac{\beta n_2(1+ab)}{b_2^2(1+b^2)} - \gamma\frac{n_3}{y'^2}$. We have $\overline{h} = \lim_{u\to\infty} h_u \geq 0$. Passing now to the case $\varepsilon < 0$, we consider the two lines $y = ux + u$ and $y = (-b + \omega(u))(x - b_1) + b_2$. Using the same arguments as for the case $\varepsilon > 0$, we can derive the inequality $\overline{h} \leq 0$. Therefore $\overline{h} = 0$, which gives

$$y'^2 = -\frac{\gamma n_3}{\alpha a m_1 + \beta(1+ab)n_2}.$$

Using the relations $x_0 = x_2 + b(y_1 - y_2)$, $y_0 = y_1$, $x_3 = -\frac{\alpha}{\gamma}x_1 + \frac{1-\beta}{\gamma}x_2 + \frac{b}{\gamma}(y_1 - y_2)$, $y_3 = \frac{1-\alpha}{\gamma}y_1 - \frac{\beta}{\gamma}y_2$ which follow from (2), (3), (4), we get $\alpha m_1 = \beta a n_2 - \beta b n_2$ and $n_3 = -\frac{\beta}{\gamma}n_2$, i.e.,

$$x'^2 + y'^2 = (1+a^2)y'^2 = \frac{\beta n_2(1+a^2)}{\beta a^2 n_2 - \beta a b n_2 + \beta n_2 + \beta a b n_2} = 1.$$

Denote by $arc(A,B)$ the part of the circle $T(E)$ which is inside the smaller angle generated by the vectors A and B. As we have just proved, if $T(S(X))$ and $T(E)$ coincide at two points A and B they coincide at one more point $C \in arc(A,B)$. Continuing this process, we see that $T(S(X))$ and $arc(A,B)$ coincide on a dense set of points. Hence $arc(A,B) \subset T(S(X))$ and by the symmetry argument $arc(-A,-B) \subset T(S(X))$. The same reasoning for the points A and $-B$ shows that $arc(A,-B) \subset T(S(X))$ and therefore $arc(-A,B) \subset T(S(X))$ as well. The proof of statement (ii) is complete. Statement (i) can be proved similarily. $\square$

Remarks: 1. In the Theorem we can replace the unit sphere $S(X)$ by any sphere with center at $\overline{x}$ and radius R. Moreover, in the case $\dim X = 2$, $S(X)$ and its center can be replaced by any continuous convex closed curve S on R^2 and any point from the area which is bounded by S.

2. The Theorem holds true for measures concentrated at n points of $S(X)$, $n \geq 3$, with any fixed positive weights $\alpha_1, \alpha_2, \dots, \alpha_n$.

3. The complex and quaternion versions of the Theorem are easily derived from the real one.

References

1. D. Amir, Characterization of inner product spaces. *Birkhäuser Verlag, Basel*, 1986.
2. R. Durier, Optimal locations and inner products. *J. Math. Anal. Appl.* **207**(1991), 220–230.
3. M. Day, Some characterization of inner-product spaces. *Trans. Amer. Math. Soc.* **62**(1947), 320–327.

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