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ENTROPY NUMBERS OF CERTAIN SUMMATION OPERATORS

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Abstract. Given nonnegative real sequences $a = (\alpha_k)_{k\in\mathbb{Z}}$ and $b = (\beta_k)_{k\in\mathbb{Z}}$ we study the generated summation operator

$$S_{a,b}(x) := \Big(\alpha_k \Big[\sum_{l<k} \beta_l x_l\Big]\Big)_{k\in\mathbb{Z}} , \quad x = (x_k)_{k\in\mathbb{Z}} ,$$

regarded as a mapping from $\ell_p(\mathbb{Z})$ to $\ell_q(\mathbb{Z})$. We give necessary and sufficient conditions for the boundedness of $S_{a,b}$ and prove optimal estimates for its entropy numbers relative to the summation properties of a and b. Our results are applied to the investigation of the behaviour of

$$\mathbb{P}\Big(\sum_{k\in\mathbb{Z}} \alpha_k^q |W(t_k)|^q < \varepsilon^q\Big) \qquad \text{and} \qquad \mathbb{P}\Big(\sup_{k\in\mathbb{Z}} \alpha_k |W(t_k)| < \varepsilon\Big)$$

as $\varepsilon \to 0$, where $(t_k)_{k\in\mathbb{Z}}$ is some nondecreasing sequence in $[0,\infty)$ and $(W(t))_{t\geq 0}$ denotes the Wiener process.

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1. Introduction

Volterra integral operators are known to play an important role within Functional Analysis as well as Probability Theory. A special class of interest is that of so-called weighted Volterra integral operators $T_{\rho,\psi}$ mapping $L_p(0,\infty)$ into $L_q(0,\infty)$ and defined by

$$T_{\rho,\psi} : f \mapsto \rho(s) \int_0^s \psi(t) f(t)\, dt , \tag{1.1}$$

where $\rho, \psi \geq 0$ are suitable functions on $(0,\infty)$. The main task is to describe properties of $T_{\rho,\psi}$ (boundedness, degree of compactness, etc.) in terms of certain properties of the weight functions ρ and ψ. Although several results have been proved in this direction ([12], [5], [6], [10]), interesting problems remain open. For example, we do not know of a complete characterization of ρ's and ψ's such that the approximation or entropy numbers of $T_{\rho,\psi}$ behave exactly as those of T_1 on $[0,1]$. ($T_1 = T_{1,1}$ denotes the ordinary integral operator).

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Given sequences $a = (\alpha_k)_{k\in\mathbb{Z}}$ and $b = (\beta_k)_{k\in\mathbb{Z}}$, the summation operator $S_{a,b} : \ell_p(\mathbb{Z}) \to \ell_q(\mathbb{Z})$ defined by

$$S_{a,b} : x = (x_k)_{k\in\mathbb{Z}} \mapsto \Big(\alpha_k \sum_{j<k} \beta_j x_j)_{k\in\mathbb{Z}}\Big) \tag{1.2}$$

can be viewed as a discrete counterpart to weighted Volterra operators. Thus one may expect for $S_{a,b}$ similar compactness properties (relative to a and b) as for $T_{\rho,\psi}$ (relative to ρ and ψ). This is indeed so as long as one considers upper estimates for the entropy numbers $e_n(S_{a,b})$. Here the most surprising result (Theorem 2.3) asserts that summation operators can behave as badly and irregularly as weighted integral operators. Yet the situation becomes completely different when considering lower estimates for $e_n(S_{a,b})$. The deeper reason is as follows: If $\rho\,\psi > 0$ on some interval (some set of positive measure suffices), then the entropy numbers of $T_{\rho,\psi}$ cannot tend to zero faster than those of T_1 considered on this interval. Since $e_n(T_1) \approx n^{-1}$, this order is a natural lower bound for $e_n(T_{\rho,\psi})$. In the case of summation operators such a"canonical" operator (as T_1 for $T_{\rho,\psi}$) does not exist. Hence it is not surprising that $e_n(S_{a,b})$ can tend to zero much faster than n^{-1}.

The aim of this paper is to investigate these phenomena more precisely. We state optimal conditions for a and b in order that $\sup_n n\, e_n(S_{a,b}) < \infty$. More precise statements can be formulated under some additional regularity assumptions on a and b, e.g., monotonicity or an exponential decay of the β_k's.

As is well-known, the ordinary Volterra operator T_1 is closely related to Brownian motion. More precisely, let $(\xi_k)_{k=1}^\infty$ be an i.i.d. sequence of standard normal distributed random variables and let $(f_k)_{k=1}^\infty$ be some orthonormal basis in $L_2(0,\infty)$. Then

$$W(t) = \sum_{k=1}^\infty \xi_k (T_1 f_k)(t), \qquad t \ge 0\,, \tag{1.3}$$

is a standard Wiener process over $(0,\infty)$. In this sense, summation operators $S_{a,b} : \ell_2 \to \ell_q$ generate Gaussian sequences $X = (X_k)_{k\in\mathbb{Z}} \in \ell_q(\mathbb{Z})$ with

$$X_k = \alpha_k W(t_k), \qquad k \in \mathbb{Z}, \tag{1.4}$$

where t_k's are defined via $\beta_k^2 = t_{k+1} - t_k$.

Recent results ([9], [11]) relate the entropy behaviour of an operator with estimates for the probability of small balls of the generated Gaussian random variable. We transform the entropy results proved for $S_{a,b}$ into the small ball estimates for $X = (X_k)_{k\in\mathbb{Z}}$ defined above. Hence we find sufficient conditions for the α_k's and t_k's such that

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}\Big(\|X\|_q < \varepsilon\Big) = 0. \tag{1.5}$$

These conditions turn out to be (in this language) the best possible ones.

The paper is organized as follows. In Section 2, the main results about the entropy of $S_{a,b}$ are stated. Section 3 provides basic tools and well-known facts, which are used in Section 4 to prove our results. Section 5 is devoted to the study of some examples. Finally, in Section 6 we establish some small ball estimates for Gaussian vectors as in (1.4), using our entropy results for $S_{a,b}$ proved before.

2. Notation and Main Results

For a given sequence $x = (x_k)_{k\in\mathbb{Z}} \subseteq \mathbb{R}$ and $p \in [1,\infty]$ set

$$\|x\|_p := \Big(\sum_{k=-\infty}^{\infty} |x_k|^p\Big)^{1/p} \quad \text{if } p<\infty, \quad \|x\|_\infty := \sup_{k\in\mathbb{Z}} |x_k|,$$

and let $\ell_p(\mathbb{Z}) := \{x : \|x\|_p < \infty\}$. As usual, for $p \in [1,\infty]$ the adjoint p' is given by $1/p + 1/p' = 1$. Now, let $p, q \in [1,\infty]$ be arbitrary. For nonnegative sequences $a = (\alpha_k)_{k\in\mathbb{Z}}$ and $b = (\beta_k)_{k\in\mathbb{Z}}$ satisfying

$$A_k := \Big(\sum_{l\geq k} \alpha_l^q\Big)^{1/q} < \infty \quad \text{and} \quad B_k := \Big(\sum_{l<k} \beta_l^{p'}\Big)^{1/p'} < \infty \tag{2.1}$$

for all $k \in \mathbb{Z}$ (with obvious modifications for $q=\infty$ or $p=1$), the expression

$$S_{a,b}(x) := \Big(\alpha_k\Big[\sum_{l<k} \beta_l x_l\Big]\Big)_{k\in\mathbb{Z}} \tag{2.2}$$

is well-defined for any $x = (x_k)_{k\in\mathbb{Z}} \in \ell_p(\mathbb{Z})$. Our first result is a version of the well-known Maz'ja-Rosin Theorem (see, e.g., [12], pp. 39-51), characterizing the boundedness of $S_{a,b}$ in terms of A_k and B_k given in (2.1):

Theorem 2.1. *Under the above assumptions, the operator $S_{a,b}$ is well-defined and bounded from $\ell_p(\mathbb{Z})$ to $\ell_q(\mathbb{Z})$ iff $D(a,b) < \infty$, where*

$$D(a,b) := \begin{cases} \sup_{k\in\mathbb{Z}} [A_k \cdot B_k] & \text{for } p \leq q\,, \\ \Big(\sum_{k\in\mathbb{Z}} [A_k \cdot B_k^{\frac{p'}{q'}}]^{\frac{pq}{p-q}} \beta_{k-1}^{p'}\Big)^{\frac{p-q}{pq}} & \text{for } p > q\,. \end{cases} \tag{2.3}$$

Moreover, there is a universal $C_{p,q} > 0$ with

$$\frac{1}{C_{p,q}} D(a,b) \leq \|S_{a,b} : \ell_p(\mathbb{Z}) \to \ell_q(\mathbb{Z})\| \leq C_{p,q} D(a,b). \tag{2.4}$$

Our next aim is to describe the compactness of $S_{a,b}$ in terms of entropy numbers. For the introduction of entropy numbers, let E, F be Banach spaces with unit balls B_E, B_F. If $T : E \to F$ is a linear bounded operator, we set the *(dyadic) entropy numbers* of T to be

$$e_n(T) := \inf\Big\{\varepsilon > 0 : \exists\, y_1, \ldots, y_{2^{n-1}} \in F \text{ s. t. } T(B_E) \subseteq \bigcup_{k=1}^{2^{n-1}} \Big(y_k + \varepsilon B_F\Big)\Big\},$$

where "+" means the Minkowski sum. It is well-known that T is compact iff $e_n(T) \to 0$, so that it makes sense to consider the speed of decay of $e_n(T)$ as a measure for the compactness of T.

To avoid some technical and notational problems, from now on it is assumed that $p > 1$, hence $p' < \infty$. In the following, the number $r > 0$ defined by

$$1/r := 1/q + 1/p' \tag{2.5}$$

will play a crucial role. Since $p > 1$, we always have $r < \infty$.

Later on, we will use the monotone rearrangement δ_k^* of a sequence $(\delta_k)_{k\in\mathbb{Z}}$. By this we mean that the sequence $(\delta_k^*)_{k\ge 0}$ is the nonincreasing rearrangement of $(\delta_k)_{k\ge 0}$, and that $(\delta_k^*)_{k\le -1}$ is the nondecreasing rearrangement of $(\delta_k)_{k\le -1}$ (i.e., $(\delta_{-k}^*)_{k\ge 1}$ is the nonincreasing rearrangement of $(\delta_{-k})_{k\ge 1}$).

For the nonnegative sequences a and b satisfying (2.1), we define

$$v_k := \inf\Big\{ m \in \mathbb{Z} : \sum_{l\le m} \beta_l^{p'} \ge 2^k \Big\} \tag{2.6}$$

for any $k \in \mathbb{Z}$, where $\inf \varnothing := \infty$, and with these v_k's we set

$$\delta_k(a,b) := \begin{cases} 2^{k/p'} \Big(\sum\limits_{l=v_k+1}^{v_{k+1}} \alpha_l^q \Big)^{1/q} & \text{if } v_k < \infty\,, \\ 0 & \text{for } v_k = \infty\,. \end{cases} \tag{2.7}$$

Here, as well as in the following, the empty sum has to be read as 0. With r given by (2.5) we define

$$|(a,b)|_r := \|\Big(\delta_k(a,b))_{k\in\mathbb{Z}}\|_r = \Big(\sum_{k\in\mathbb{Z}} \delta_k(a,b)^r \Big)^{1/r} \tag{2.8}$$

and

$$|(a,b)|_{r,\infty} := \sup_{k\ge 1} k^{1/r} \Big(\delta_{k-1}^*(a,b) + \delta_{-k}^*(a,b) \Big). \tag{2.9}$$

It is easy to see that

$$\|\Big(\alpha_l\, \beta_{l-1}\Big)_{l\in\mathbb{Z}}\|_r \le |(a,b)|_r \quad \text{and} \quad |(a,b)|_{r,\infty} \le c\, |(a,b)|_r\,.$$

On the other hand, there are easy examples such that $|(a,b)|_{r,\infty} < \infty$ and $\|\Big(\alpha_l\, \beta_{l-1}\Big)_{l\in\mathbb{Z}}\|_r = \infty$, and vice versa, i.e., these two expressions are not comparable.

Let us formulate now the main results of this paper. We have the following general upper estimate for the entropy numbers of $S_{a,b}$:

Theorem 2.2. *Let $S_{a,b}$ map $\ell_p(\mathbb{Z})$ into $\ell_q(\mathbb{Z})$.*

(1) *There is a numerical constant $c > 0$ such that*

$$\sup_{n\in\mathbb{N}} n\, e_n(S_{a,b}) \le c\, |(a,b)|_r. \tag{2.10}$$

(2) *Whenever the right hand side of* (2.10) *is finite, we have*

$$\lim_{n\to\infty} n\, e_n(S_{a,b}) = 0. \tag{2.11}$$

The next result shows that estimate (2.10) cannot in general be improved to an estimate neither against $|(a,b)|_{r,\infty}$ nor against $\|\Big(\alpha_l\,\beta_{l-1}\Big)_{l\in\mathbb{Z}}\|_r$:

Theorem 2.3. *For any sequence $d_k>0$, $k=1,2,\ldots$ satisfying $\sum_k d_k^q<\infty$ and $\sum_k d_k^r=\infty$ there are sequences $a=(\alpha_k)_{k\geq 1}$ and $b=(\beta_k)_{k\geq 1}$ such that:*

(1) *The operator $S_{a,b}:\ell_p\to\ell_q$ is bounded.*

(2) $\sum\limits_{k=1}^{\infty}[\alpha_{k+1}\beta_k]^r<\infty.$

(3) $d_k=\delta_k(a,b)=2^{k/p'}\Big(\sum\limits_{l=v_k+1}^{v_{k+1}}\alpha_l^q\Big)^{1/q}$ *for all $k\geq 1$*

(4) $\sup\limits_{n\in\mathbb{N}} n\,e_n(S_{a,b})=\infty.$

On the other hand, under some additional regularity assumptions about a and b assertion (2.11) holds even under weaker conditions. More precisely, the following is valid.

Proposition 2.4. *For a,b given, assume that $\alpha_k^q/\beta_k^{p'}$ is monotone near $\pm\infty$ if $q<\infty$, or that α_k is monotone near $\pm\infty$ if $q=\infty$. Then the condition $\|\Big(\alpha_l\,\beta_{l-1}\Big)_{l\in\mathbb{Z}}\|_r<\infty$ implies*

$$\lim_{n\to\infty} n\,e_n(S_{a,b})=0.$$

For sequences b which do not increase too fast (e.g., not superexponentially) at $+\infty$, and do not decrease too fast at $-\infty$, we have the following lower estimate for the entropy numbers of $S_{a,b}$.

Theorem 2.5. *Assume that there is $m\geq 1$ such that*

$$|\{k'\in\mathbb{Z}: v_{k'}=v_k\}|\leq m \tag{2.12}$$

for all $k\in\mathbb{Z}$. Then, for $\rho>0$ with $1/s:=\rho-1/p+1/q>0$ we have

$$\sup_{n\in\mathbb{N}} n^\rho\,e_n\Big(S_{a,b}:\ell_p\to\ell_q\Big)\geq c\cdot\sup_{n\in\mathbb{N}} n^{1/s}(\delta^*_{n-1}(a,b)+\delta^*_{-n}(a,b)) \tag{2.13}$$

with $c>0$ only depending on p,q,m and ρ. In particular, if $\rho=1$, then

$$\sup_{n\in\mathbb{N}} n\,e_n\Big(S_{a,b}:\ell_p\to\ell_q\Big)\geq c\,|(a,b)|_{r,\infty}\,. \tag{2.14}$$

Moreover, for ρ and s in (2.13) *we have*

$$\varliminf_{n\to\infty} n^\rho\ e_n\Big(S_{a,b}:\ell_p\to\ell_q\Big)\geq c\ \varliminf_{n\to\infty} n^{1/s}(\delta^*_{n-1}(a,b)+\delta^*_{-n}(a,b)). \tag{2.15}$$

Remark. Note that for $v_k<\infty$, $k\in\mathbb{Z}$, i.e., for $\sum_{l\in\mathbb{Z}}\beta_l^{p'}=\infty$, condition (2.12) is equivalent to

$$\beta_k\leq\gamma\Big(\sum_{l<k}\beta_l^{p'}\Big)^{1/p'},\quad k\in\mathbb{Z}\,,$$

for some $\gamma>0$.

Of course, condition (2.12) is violated provided that

(a) $\beta_l = 0$ if $l \le l_0$ for some $l_0 \in \mathbb{Z}$ or

(b) $\sum_{l\in\mathbb{Z}} \beta_l^{p'} < \infty$.

Here we have the following weaker variant of Theorem 2.5.

Proposition 2.6. *Suppose that there are some $k_0, k_1 \in \mathbb{Z}$ such that in case* (a) *condition* (2.12) *holds for all $k \ge k_0$ or in case* (b) *for all $k \le k_1$. Then if $1/s = \rho - 1/p + 1/q > 0$, from*

$$\sup_{n\in\mathbb{N}} n^\rho e_n(S_{a,b} : \ell_p \to \ell_q) < \infty$$

we conclude that

$$\sup_{n\in\mathbb{N}} n^{1/s}(\delta^*_{n-1}(a,b) + \delta^*_{-n}(a,b)) < \infty$$

and (2.15) *holds for $S_{a,b}$ as well.*

Note that in case (a), $\delta_{-n}(a,b) = 0$ for $n \ge n_0$, and in case (b), $\delta_{n-1}(a,b) = 0$ for $n \ge n_1$ with certain $n_0, n_1 \in \mathbb{Z}$. Examples (see Proposition 5.3) show that Theorem 2.5 and Proposition 2.6 become false without an additional regularity assumption.

3. Basic Tools

3.1. Connection to Integral Operators. Let $\rho, \psi : (0,\infty) \to [0,\infty)$ be measurable functions with

$$\rho \in L_q(x,\infty) \quad \text{and} \quad \psi \in L_{p'}(0,x) \tag{3.1}$$

for any $x \in (0,\infty)$. Then for $f \in L_p(0,\infty)$ the function

$$\Big(T_{\rho,\psi}(f)\Big)(s) := \rho(s)\int_0^s \psi(t)f(t)\,dt \tag{3.2}$$

is well-defined. The operator $T_{\rho,\psi}$ is called the *Volterra integral operator induced by ρ and ψ*. There is a close connection between summation and integral operators. Generally speaking, one can always consider a summation operator as “part” of a special Volterra integral operator. This allows one to use, for these operators, results established in [10].

Let us denote $\Delta_k := [2^k, 2^{k+1})$ for $k \in \mathbb{Z}$, and set

$$I_k := \Delta_{2k} = [2^{2k}, 2^{2k+1}) \quad \text{and} \quad J_k := \Delta_{2k+1} = [2^{2k+1}, 2^{2k+2}). \tag{3.3}$$

For any set $X \subseteq (0,\infty)$, let $\mathbf{1}_X$ be the indicator function of X. We denote by $\ell_0(\mathbb{Z}) := \mathbb{R}^{\mathbb{Z}}$ the space of all real sequences and set $L_0(0,\infty)$ to be the space of all equivalence classes of measurable functions. It is easily checked that for any $p \in [1,\infty]$ the mappings

$$\Phi_I^p : \ell_0(\mathbb{Z}) \to L_0(0,\infty), \quad (x_k)_{k\in\mathbb{Z}} \mapsto \sum_{k\in\mathbb{Z}} x_k \mathbf{1}_{I_k}|I_k|^{-1/p} \tag{3.4}$$

and

$$\Phi_J^p : \ell_0(\mathbb{Z}) \to L_0(0,\infty), \quad (x_k)_{k\in\mathbb{Z}} \mapsto \sum_{k\in\mathbb{Z}} x_k \mathbf{1}_{J_k} |J_k|^{-1/p} \tag{3.5}$$

induce isometric embeddings of $\ell_p(\mathbb{Z})$ into $L_p(0,\infty)$ (for $p=\infty$ let, as usual, $1/p=0$). Given now sequences $a=(\alpha_k)_{k\in\mathbb{Z}}$ and $b=(\beta_k)_{k\in\mathbb{Z}}$, we set

$$\rho := \Phi_I^q(a) \quad \text{ and } \quad \psi := \Phi_J^{p'}(b)\ , \tag{3.6}$$

i.e., we have

$$\rho = \sum_{k\in\mathbb{Z}} \alpha_k \mathbf{1}_{I_k} |I_k|^{-1/q} \quad \text{ and } \quad \psi = \sum_{k\in\mathbb{Z}} \beta_k \mathbf{1}_{J_k} |J_k|^{-1/p'}\ . \tag{3.7}$$

The connection between summation and integral operators reads as follows:

Proposition 3.1. *In the above setting, ρ and ψ defined in* (3.6) *satisfy* (3.1) *if and only if a and b satisfy* (2.1), *and it holds*

$$T_{\rho,\psi} \circ \Phi_J^p = \Phi_I^q \circ S_{a,b}. \tag{3.8}$$

Moreover, $T_{\rho,\psi}$ is bounded from L_p to L_q iff $S_{a,b}$ is so from ℓ_p to ℓ_q. In this case there is an operator $Q : L_p(0,\infty) \to \ell_p(\mathbb{Z})$ with $\|Q\| \le 1$ such that for all $f \in L_p(0,\infty)$ we have

$$\|T_{\rho,\psi}(f)\|_q = \|S_{a,b} \circ Q(f)\|_q\,. \tag{3.9}$$

In particular,

$$\|T_{\rho,\psi} : L_p \to L_q\| = \|S_{a,b} : \ell_p \to \ell_q\|\,. \tag{3.10}$$

Proof. The equivalence of (3.1) and (2.1) is clear by the definition of ρ and ψ and of Φ_I^q and $\Phi_J^{p'}$, respectively. We easily compute

$$T_{\rho,\psi}\Big(\Phi_J^p(x_k)\Big) = \sum_{k\in\mathbb{Z}} \alpha_k |I_k|^{-1/q} \Big[\sum_{l<k} \beta_l x_l\Big] \mathbf{1}_{I_k} = \Phi_I^q(S_{a,b}(x_k))\,.$$

This verifies that (3.8) holds.

If $T_{\rho,\psi}$ is bounded, then, due to (3.8), so is the operator $S_{a,b}$, and $\|S_{a,b}\| \le \|T_{\rho,\psi}\|$. Conversely, if $S_{a,b}$ is bounded and $f \in L_p((0,\infty))$, set

$$Q(f) := \Big(|J_k|^{-1/p'} \int_{J_k} f(t)\,dt\Big)_{k\in\mathbb{Z}}\,. \tag{3.11}$$

Due to Hölder's inequality, $Q(f) \in \ell_p$ with $\|Q(f)\|_p \le \|f\|_p$, and, additionally, we find

$$\Big\|T_{\rho,\psi}(f)\Big\|_q^q = \sum_{k\in\mathbb{Z}} \alpha_k^q \Big|\sum_{l<k} \beta_l |J_l|^{-1/p'} \int_{J_l} f(t)\,dt\Big|^q = \Big\|S_{a,b}(Q(f))\Big\|_q^q. \tag{3.12}$$

This proves (3.9), and implies, in particular, that $\|T_{\rho,\psi}\| \le \|S_{a,b}\|$ which completes the proof. $\square$

As a simple corollary we have

Corollary 3.2. *For a and b satisfying* (2.1), *and ρ and ψ defined as in* (3.7) *we have*

$$(1/2)\, e_n(T_{\rho,\psi}) \le e_n(S_{a,b}) \le 2\, e_n(T_{\rho,\psi}).$$

Proof. First note that

$$e_n(T) \le 2\, e_n(J \circ T)$$

for any operator $T : E \to F$ and any isometric embedding $J : F \to F_0$. Hence, by (3.8) we conclude

$$e_n(S_{a,b}) \le 2\, e_n(\Phi_I^q \circ S_{a,b}) = 2\, e_n(T_{\rho,\psi} \circ \Phi_J^p) \le 2\, e_n(T_{\rho,\psi})\, \|\Phi_J^p\| \le 2\, e_n(T_{\rho,\psi})\ .$$

On the other hand, by (3.9) we have (use Lemma 4.2 in [10], yet observe that there the entropy numbers were defined slightly different so that additional factor 2 had to be added)

$$e_n(T_{\rho,\psi}) \le 2\, e_n(S_{a,b} \circ Q) \le 2\, e_n(S_{a,b})\, \|Q\| \le 2\, e_n(S_{a,b}),$$

which completes the proof. □

For later use we cite now some of the main results in [10] about weighted Volterra integral operators. Let ρ and ψ now be arbitrary nonnegative measurable functions on $(0,\infty)$ satisfying (3.1). For $s > 0$, set

$$R(s) := \|\rho\|_{L_q(s,\infty)} \qquad \text{and} \qquad \Psi(s) := \|\psi\|_{L_{p'}(0,s)}. \tag{3.13}$$

The following version of the Maz'ja–Rosin Theorem can be found in [10], Theorem 6.1:

Theorem 3.3. *Let $1 \le p, q \le \infty$ and $\rho, \psi \ge 0$ on $(0,\infty)$. Then $T_{\rho,\psi}$ is bounded from $L_p(0,\infty)$ into $L_q(0,\infty)$ iff $\mathcal{D}(\rho,\psi) < \infty$, where*

$$\mathcal{D}(\rho,\psi) := \begin{cases} \sup\limits_{s>0}\ R(s)\Psi(s) & \text{if } p \le q, \\ \left(\int\limits_0^\infty \left[R(s) \cdot \Psi(s)^{p'/q'} \right]^{\frac{pq}{p-q}} \psi(s)^{p'}\, ds \right)^{\frac{p-q}{pq}} & \text{for } p > q. \end{cases} \tag{3.14}$$

Moreover, there are universal constants $c_{p,q}, C_{p,q} > 0$ such that

$$c_{p,q}\mathcal{D}(\rho,\psi) \le \|T_{\rho,\psi} : L_p \to L_q\| \le C_{p,q}\mathcal{D}(\rho,\psi)\ . \tag{3.15}$$

Now we turn to upper estimates for the entropy numbers of $T_{\rho,\psi}$. Therefore set

$$u_k := \inf\Big\{ s > 0 : \int\limits_0^s \psi(t)^{p'}\, dt \ge 2^k \Big\}, \qquad k \in \mathbb{Z}\,, \tag{3.16}$$

and define

$$\delta_k(\rho,\psi) := 2^{k/p'} \Big(\int\limits_{u_k}^{u_{k+1}} \psi(t)^q\, dt \Big)^{1/q}. \tag{3.17}$$

Similarly to the setting for a and b, with r given by (2.5) define

$$|(\rho,\psi)|_r := \|\Big(\delta_k(\rho,\psi)\Big)_{k\in\mathbb{Z}}\|_r = \Big(\sum_{k\in\mathbb{Z}}\delta_k(a,b)^r\Big)^{1/r} \tag{3.18}$$

and

$$|(\rho,\psi)|_{r,\infty} := \sup_{k\geq 1} k^{1/r}\Big(\delta_k^*(a,b)+\delta_{-k+1}^*(a,b)\Big). \tag{3.19}$$

With this notation we have ([10], Theorem 4.6):

Theorem 3.4. *Let $p>1$ and $1\leq q\leq\infty$. Then for all ρ,ψ as above, the following statements are valid.*

(1) *For $T_{\rho,\psi}$ as in* (3.2), *we have* $\sup_n n\ e_n(T_{\rho,\psi}) \leq C\,|(\rho,\psi)|_r$.

(2) *Whenever $|(\rho,\psi)|_r<\infty$, we even have*

$$\varlimsup_{n\to\infty} n\ e_n(T_{\rho,\psi}) \leq C\,\|\rho\,\psi\|_r\ . \tag{3.20}$$

Remarks. (1) There exist functions ρ and ψ (cf. [10], Theorem 2.3) with $\|\rho\,\psi\|_r<\infty$ such that $\varlimsup_{n\to\infty} n\,e_n(T_{\rho,\psi})=\infty$. But note that since these ρ's and ψ's are not of form (3.7) with suitable sequences a and b, hence they cannot be used to construct similar examples for summation operators. However to prove Theorem 2.3 the general ideas may be taken over in order.

(2) In [10] lower estimates for $e_n(T_{\rho,\psi})$ were also proved. But unfortunately all these lower bounds turn out to be zero in the case of disjointly supported functions ρ and ψ. So they do not provide any information about lower bounds for $e_n(S_{a,b})$.

3.2. Entropy Numbers of Diagonal Operators. For a given sequence $\sigma=(\sigma_k)_{k\geq 1}$ of nonnegative numbers, one defines the diagonal operator

$$D_\sigma(x) := \Big(\sigma_k x_k)_{k\in\mathbb{N}}\,, \quad x=(x_k)_{k\in\mathbb{N}}\ . \tag{3.21}$$

Considered as a mapping from $\ell_p(\mathbb{N})$ to $\ell_q(\mathbb{N})$, the entropy of these operators can be estimated by means of σ_k's. As an example, we have the following special case of general results in [1] or [13].

Theorem 3.5. *Let $p,q\in[1,\infty]$ be arbitrary, and let $\sigma=(\sigma_k)_{k\geq 0}$ be a non-negative bounded sequence of real numbers. Let $\rho>0$ satisfy $1/s:=\rho-1/p+1/q>0$. Then there are constants $c_1,c_2>0$ depending on p,q and ρ only such that*

$$c_1\sup_{n\in\mathbb{N}} n^{1/s}\sigma_n^* \leq \sup_{n\in\mathbb{N}} n^\rho\, e_n\Big(D_\sigma:\ell_p(\mathbb{N})\to\ell_q(\mathbb{N})\Big) \leq c_2\sup_{n\in\mathbb{N}} n^{1/s}\sigma_n^*. \tag{3.22}$$

For the lower bound, one even has

$$n^{1/q-1/p}\sigma_n^* \leq 6\,e_n(D_\sigma). \tag{3.23}$$

Let us draw an easy conclusion out of this:

Corollary 3.6. *In the above setting,*

$$c_1 \varliminf_{n\to\infty} n^{1/s}\sigma_n^* \le \varliminf_{n\to\infty} n^\rho\, e_n(D_\sigma) \tag{3.24}$$

as well as

$$\varlimsup_{n\to\infty} n^\rho\;\; e_n(D_\sigma) \le 2\, c_2 \varlimsup_{n\to\infty} n^{1/s}\sigma_n^* \tag{3.25}$$

holds.

Proof. Without loss of generality, assume that σ_n's are already nonincreasing, e.g., $\sigma_n = \sigma_n^*$. Then estimate (3.24) is an immediate consequence of (3.23).

To prove (3.25), fix a number $N \in \mathbb{N}$ and set

$$D^N((x_i)_{i=1}^\infty) := (\sigma_1 x_1, \dots, \sigma_N x_N, 0, \dots), \quad D_0^N := D_\sigma - D^N.$$

Because of $e_{2n-1}(D_\sigma) \le e_n(D^N) + e_n(D_0^N)$ and the exponential decay of $e_n(D^N)$ as $n \to \infty$ (cf. [2], 1.3.36), we get

$$\varlimsup_{n\to\infty} n^\rho e_n(D_\sigma) \le 2 \varlimsup_{n\to\infty} n^\rho e_n(D_0^N) \le 2 \sup_{n\in\mathbb{N}} n^\rho e_n(D_0^N). \tag{3.26}$$

Now D_0^N is not a diagonal operator with nonincreasing entries, but this difficulty can be easily overcome. For $x = (x_i)_{i\in\mathbb{N}}$ let

$$S_N^-(x) := (x_{i+N})_{i\in\mathbb{N}} \quad \text{and} \quad S_N^+(x) := (0, \dots, 0, x_1, \dots)$$

be the operators of shifting N times to the right or to the left in $\ell_p(\mathbb{N})$ or $\ell_q(\mathbb{N})$, respectively. If

$$D_1^N((x_i)_{i=1}^\infty) := (x_1\sigma_{N+1}, x_2\sigma_{N+2}, \dots)$$

is the diagonal operator defined by $(\sigma_{N+i})_{i\ge 1}$, we clearly have

$$D_0^N = S_N^+ \circ D_1^N \circ S_N^- \quad \text{and} \quad D_1^N = S_N^- \circ D_0^N \circ S_N^+,$$

and therefore (note that both shift operators have norm one) it holds by the multiplicity of the entropy numbers

$$e_n(D_0^N) = e_n(D_1^N).$$

But for D_1^N Theorem 3.5 applies and leads to

$$\sup_{n\in\mathbb{N}} n^\rho e_n(D_1^N) \le c_2 \sup_{n\in\mathbb{N}} n^{1/s}\sigma_{n+N}$$

with $s > 0$ defined in this theorem. Since $n^{1/s} \le (n+N)^{1/s}$, this yields

$$\sup_{n\in\mathbb{N}} n^\rho e_n(D_0^N) \le c_2 \sup_{n\in\mathbb{N}} (n+N)^{1/s}\sigma_{n+N} = c_2 \sup_{m>N} m^{1/s}\sigma_m. \tag{3.27}$$

Now we combine (3.26) and (3.27) to find that

$$\varlimsup_{n\to\infty} n^\rho e_n(D_\sigma) \le 2c_2 \sup_{n>N} n^{1/s}\sigma_n \tag{3.28}$$

holds for all $N \in \mathbb{N}$. Taking the infimum over the right side of (3.28), we end up with

$$\overline{\lim_{n\to\infty}}\, n^{\rho} e_n(D_\sigma) \le 2c_2 \inf_{N\in\mathbb{N}} \Big(\sup_{n>N} n^{1/s}\sigma_n \Big) = 2c_2 \,\overline{\lim_{n\to\infty}}\, n^{1/s}\sigma_n,$$

which proves (3.25). □

It is of more use for us to consider diagonal operators mapping $\ell_p(\mathbb{Z})$ into $\ell_q(\mathbb{Z})$, generated, say, by a sequence $(\sigma_k)_{k\in\mathbb{Z}}$. However these operators are isomorphic to the product of two diagonal operators generated by $(\sigma_{k-1})_{k\ge 1}$ and $(\sigma_{-k})_{k\ge 1}$. It is then quite easy to establish

Corollary 3.7. *Let $p, q \in [1,\infty]$ be arbitrary, and let $\sigma = (\sigma_k)_{k\in\mathbb{Z}}$ be a nonnegative bounded sequence of real numbers. For given $\rho > 0$ assume $1/s := \rho - 1/p + 1/q > 0$. Then there are constants $c_1, c_2 > 0$ depending only on p, q and ρ such that*

$$c_1 \sup_{n\in\mathbb{N}} n^{1/s}(\sigma^*_{n-1} + \sigma^*_{-n}) \le \sup_{n\in\mathbb{N}} n^{\rho} e_n(D_\sigma) \le c_2 \sup_{n\in\mathbb{N}} n^{1/s}(\sigma^*_{n-1} + \sigma^*_{-n}) \quad (3.29)$$

and, moreover,

$$\underline{\lim_{n\to\infty}}\, n^{\rho} e_n(D_\sigma) \ge c_1 \,\underline{\lim_{n\to\infty}}\, n^{1/s}(\sigma^*_{n-1} + \sigma^*_{-n})$$

as well as

$$\overline{\lim_{n\to\infty}}\, n^{\rho} e_n(D_\sigma) \le c_2 \,\overline{\lim_{n\to\infty}}\, n^{1/s}(\sigma^*_{n-1} + \sigma^*_{-n}).$$

4. Proof of the Results

Let us start with the proof of Theorem 2.1.

Proof of Theorem 2.1. Let $a = (\alpha_k)_{k\in\mathbb{Z}}$ and $b = (\beta_k)_{k\in\mathbb{Z}}$ be given, satisfying (2.1), and define ρ and ψ by means of (3.7). Then $R(s)$ and $\Psi(s)$ are given by (3.13). Because of Proposition 3.1 and Theorem 3.3 we have the estimate

$$\frac{1}{C_{p,q}}\mathcal{D}(\rho,\psi) \le \|S_{a,b} : \ell_p(\mathbb{Z}) \to \ell_q(\mathbb{Z})\| \le C_{pq,}\mathcal{D}(\rho,\psi)$$

with $\mathcal{D}(\rho,\psi)$ given in (3.14). So it remains to show that with $D(a,b)$ from (2.3) we have

$$c_1\mathcal{D}(\rho,\psi) \le D(a,b) \le c_2\mathcal{D}(\rho,\psi)\,. \quad (4.1)$$

Let us treat the case $p \le q$ first. Take any $s \in I_k$ for some $k \in \mathbb{Z}$. Then on the one hand we have $R(s) \le A_k$, while, on the other hand, $\Psi(s) = B_k$. Now assume $s \in J_k$ for some $k \in \mathbb{Z}$. Then $R(s) = A_{k+1}$, while $\Psi(s) \le B_{k+1}$. Combining both cases, we find

$$\sup_{s>0}\; R(s)\cdot\Psi(s) \le \sup_{k\in\mathbb{Z}}\, [A_k \cdot B_k]\,. \quad (4.2)$$

Since $A_k \cdot B_k = R(2^{2k}) \cdot \Psi(2^{2k})$, we even have equality in (4.2), which proves (4.1) for $p \le q$.

Let us now assume $p > q$. First note that the integral in the definition of $\mathcal{D}(\rho, \psi)$ has only to be taken over $\bigcup_{k\in\mathbb{Z}} J_k$. But if $s \in J_{k-1}$, then $R(s) = A_k$ while $\Psi(s) \le B_k$, hence (recall $\psi(s) = \beta_{k-1}|J_{k-1}|^{-1/p'}$)

$$\mathcal{D}(\rho, \psi) \le D(a, b) .$$

Conversely, whenever $s \in [2^{2k+3/2}, 2^{2k+2}] \subseteq J_k$, we have $R(s) = A_{k+1}$, while

$$\Psi(s) \ge \left(B_k^{p'} + 2^{-1}\beta_k^{p'}\right)^{1/p'} \ge 2^{-1/p'} B_{k+1} .$$

Consequently, we obtain

$$\left(\int\limits_{J_k} \left[R(s) \cdot \Psi(s)^{p'/q'}\right]^{\frac{pq}{p-q}} \psi(s)^{p'}\, ds\right)^{\frac{p-q}{pq}} \ge 2^{-1/q'} \left[(A_{k+1} \cdot B_{k+1}^{p'/q})^{\frac{pq}{p-q}} \beta_k^{p'}\right]^{\frac{p-q}{pq}} ,$$

yielding

$$D(a, b) \le 2^{1/q'}\, \mathcal{D}(\rho, \psi) ,$$

which proves (4.2) in this case as well. □

Proof of Theorem 2.2. Let a, b be given, and ρ, ψ be defined via (3.7). In view of Proposition 3.1 and Theorem 3.4 we have to show only that

$$\delta_k(a, b) = \delta_k(\rho, \psi), \tag{4.3}$$

where δ_k's are defined in (2.7) and (3.17), respectively. But for v_k as in (2.6) and u_k as in (3.16), we have $u_k \in \bar{J}_{v_k} = [2^{2k+1}, 2^{2k+2}]$. Inserting this into the definition of δ_k's yields (4.3), and hence Theorem 2.2. □

Proof of Theorem 2.3. We treat the case $q < \infty$ first. Let d_k be given with

$$\sum_{k\in\mathbb{N}} d_k^q < \infty \quad \text{and} \quad \sum_{k\in\mathbb{N}} d_k^r = \infty. \tag{4.4}$$

We assume $0 < d_k \le 1$. For any $k \in \mathbb{N}$, we can find $s_k \in \mathbb{N}$ such that

$$s_k \le \frac{1}{d_k^q} \le 2s_k. \tag{4.5}$$

According to our assumptions, it is possible to find a partition $(K_m)_{m\in\mathbb{N}}$ of $\mathbb{N}$ with $\inf K_{m+1} = \sup K_m + 1$, and

$$\sum_{k\in K_m} d_k^r \ge m^r. \tag{4.6}$$

Set $\nu_m := \max\limits_{k\in K_m} d_k^{-1}$. Then for any $k \in K_m$ there are $n_k \in \mathbb{N}$ such that

$$(\nu_m d_k)^r \le n_k \le 2(\nu_m d_k)^r. \tag{4.7}$$

Further we define

$$\gamma_{k,j} := \sum_{l<k} (1 + 2s_l)\, n_k \;+\; (j - 1)(1 + 2s_k) \tag{4.8}$$

for $k \in \mathbb{N}$ and $j = 1, \ldots, n_k + 1$. In particular, $\gamma_{1,1} = 0$ and $\gamma_{k,n_k+1} = \gamma_{k+1,1}$. Now we are ready to define a partition of $\mathbb{N}$ suitable for our purposes: For any $k \in \mathbb{N}$ and $j = 1, \ldots, n_k$ set

$$A_{k,j}^+ := \{\gamma_{k,j} + 1, \ldots, \gamma_{k,j} + s_k\}, \tag{4.9}$$

as well as

$$I_{k,j} := \{\gamma_{k,j} + s_k + 1\} \tag{4.10}$$

and

$$A_{k,j}^- := \{\gamma_{k,j} + s_k + 2, \ldots, \gamma_{k,j+1}\}. \tag{4.11}$$

Clearly, $|A_{k,j}^+| = |A_{k,j}^-| = s_k$, and

$$A_{k,1}^+ \prec I_{k,1} \prec A_{k,1}^- \prec A_{k,2}^+ \prec \cdots \prec A_{k,n_k}^- \prec A_{k+1,1}^+ \prec \cdots, \tag{4.12}$$

where $\prec$ denotes the natural ordering of intervals in $\mathbb{N}$. Denoting

$$\mathcal{U}_k := \bigcup_{j=1}^{n_k} \Big(A_{k,j}^+ \cup I_{k,j} \cup A_{k,j}^- \Big) = \{\gamma_{k,1} + 1, \ldots, \gamma_{k+1,1}\}, \tag{4.13}$$

it is clear that $\mathbb{N} = \bigcup_{k \geq 1} \mathcal{U}_k$. Let us define $\beta_0 := 1$, and

$$\beta_l := \frac{2^{(k-1)/p'}}{n_k^{1/p'}(1 + 2s_k)^{1/p'}} \qquad \text{for } l \in \mathcal{U}_k\,. \tag{4.14}$$

This way we ensure (note that $|\mathcal{U}_k| = n_k(1 + 2s_k)$) that for v_k's defined in (2.6) we have $v_k = \gamma_{k,1}$ for $k \geq 1$ and $v_k = 0$ if $k \leq 0$ so that

$$\mathcal{U}_k = \{v_k + 1, \ldots, v_{k+1}\}, \qquad k \geq 1. \tag{4.15}$$

Now, let us set

$$\alpha_l := \begin{cases} \dfrac{2^{-k/p'} d_k}{n_k^{1/q}} & \text{if } l \in I_{k,j} \ \text{ for some } k \in \mathbb{N},\ j = 1, \ldots, n_k, \\ 0 & \text{otherwise.} \end{cases} \tag{4.16}$$

We can easily compute that

$$\sum_{l=v_k+1}^{v_{k+1}} a_l^q = \sum_{l=1}^{n_k} \frac{2^{-kq/p'} d_k^q}{n_k} = 2^{-kq/p'} d_k^q. \tag{4.17}$$

In particular, we know $a = (\alpha_l)_{l \in \mathbb{Z}} \in \ell_q(\mathbb{Z})$, and $\delta_k(a, b) = d_k$ and hence (3) holds.

If $l \in I_{k,j}$, then $l - 1 \in \mathcal{U}_k$, which implies

$$\begin{aligned} \sum_{l \in \mathbb{N}} \Big(\alpha_l \beta_{l-1} \Big)^r &= \sum_{k \in \mathbb{N}} \sum_{j=1}^{n_k} \Big(\frac{2^{-k/p'} d_k}{n_k^{1/q}} \cdot \frac{2^{(k-1)/p'}}{n_k^{1/p'}(1 + 2s_k)^{1/p'}} \Big)^r \\ &\leq \sum_{k \in \mathbb{N}} n_k \frac{d_k^r}{n_k (1 + 2s_k)^{r/p'}} \leq \sum_{k \in \mathbb{N}} \Big(d_k / (2s_k)^{1/p'} \Big)^r. \end{aligned}$$

Because of the definition of s_k we assured $2s_k \geq d_k^{-q}$, and hence we can estimate further

$$\sum_{l\in\mathbb{N}}\Big(\alpha_l\beta_{l-1}\Big)^r \leq \sum_{k\in\mathbb{N}} d_k^{(1+q/p')r} = \sum_{k\in\mathbb{N}} d_k^q < \infty, \tag{4.18}$$

i.e., (2) is valid. Using Theorem 2.1 one easily checks that $S_{a,b}$ is bounded, i.e.,(1) is satisfied as well.

So it remains to verify (4). To do so, let us fix $k \in \mathbb{N}$ and $j \in \{1,\dots,n_k\}$. Then consider the vector $x^{k,j} = (x_l^{k,j})_{l\in\mathbb{Z}}$ given by

$$x_l^{k,j} := \begin{cases} s_k^{-1/p} & \text{if } l \in A_{k,j}^+, \\ -s_k^{-1/p} & \text{for } l \in A_{k,j}^-, \\ 0 & \text{otherwise}. \end{cases} \tag{4.19}$$

Recall that $|A_{k,j}^+| = |A_{k,j}^-| = s_k$, so $\|x^{k,j}\|_p \leq 2$. Setting $y^{k,j} := S_{a,b}(x^{k,j})$, we have

$$y_l^{k,j} = \begin{cases} s_k^{1/p'} \dfrac{d_k}{n_k^{1/r}(1+2s_k)^{1/p'}} & \text{if } l \in I_{k,j} \\ 0 & \text{otherwise}. \end{cases} \tag{4.20}$$

For $m \in \mathbb{N}$, set $N_m := \sum_{k\in K_m} n_k$. Let us denote by $e_{k,j}$ the standard basis of $\ell_p^{N_m}$ or $\ell_q^{N_m}$, respectively, where k runs through K_m and j through $1,\dots,n_k$. Then we define two mappings, an injection

$$X_m : \ell_p^{N_m} \to \ell_p(\mathbb{Z}), \quad e_{k,j} \mapsto x^{k,j}, \tag{4.21}$$

and a quasi-projection

$$Y_m : \ell_q(\mathbb{Z}) \to \ell_q^{N_m}, \quad e_l \mapsto \begin{cases} e_{k,j} & \text{if } l \in I_{k,j}, \\ 0 & \text{otherwise}. \end{cases} \tag{4.22}$$

We have $\|X_m\| \leq 2$ and $\|Y_m\| \leq 1$. Additionally,

$$Y_m \circ S_{a,b} \circ X_m = D_{\sigma^m}, \tag{4.23}$$

where D_{σ^m} is the diagonal operator generated by the sequence

$$\sigma_{k,j}^m = s_k^{1/p'} \frac{d_k}{n_k^{1/r}(1+2s_k)^{1/p'}} \geq 3^{-1/p'} \frac{d_k}{n_k^{1/r}} \geq 1/6\ \nu_m^{-1}. \tag{4.24}$$

(Here we have used the definition of n_k.) It is well-known (cf. [13] or (3.23)) that

$$Ne_N\Big(id : \ell_p^N \to \ell_q^N\Big) \geq (1/6)\, N^{1/r}, \tag{4.25}$$

so, putting things together and using (4.23), (4.24) and (4.25) we get

$$N_m e_{N_m}(S_{a,b}) \geq 1/6\ \nu_m^{-1} N_m e_{N_m}\Big(id : \ell_p^{N_m} \to \ell_q^{N_m}\Big) \geq c\, \nu_m^{-1} N_m^{1/r}. \tag{4.26}$$

But due to the definition of n_k we have

$$N_m \geq \nu_m^r \sum_{k \in K_m} d_k^r, \tag{4.27}$$

which, inserted in (4.26), yields, with regard to (4.6),

$$N_m e_{N_m}(S_{a,b}) \geq c\Big(\sum_{k \in K_m} d_k^r \Big)^{1/r} \geq c\, m, \tag{4.28}$$

so that $\overline{\lim}_{n\to\infty} n\, e_n(S_{a,b}) = \infty$.

Let now $q = \infty$, so $r = p'$. This means we have $\sup_n d_n < \infty$ while $\sum_n d_n^{p'} = \infty$. We adapt the construction above, changing only the definition of variables where q is involved. So define $s_k := 2^k$, and set $\alpha_l := 2^{-k/p'} d_k$ for $l \in I_{k,j}$, while all other variables are defined as above (they have different values now, though). It is trivial to see that $\alpha_l \in \ell_\infty$ and that $\delta_k(a,b) = d_k$. Further, we estimate

$$\sum_{l \in \mathbb{Z}} [\alpha_l \beta_{l-1}]^{p'} = \sum_{k \in \mathbb{N}} \sum_{j=1}^{n_k} \frac{d_k^{p'}}{n_k(1+2s_k)^{p'}} \leq \Big(\sup_{n \in \mathbb{N}} d_n^{p'} \Big) \sum_{k \in \mathbb{N}} 2^{-kp'} \ < \ \infty\,.$$

As above, Theorem 2.1 shows the boundedness of $S_{a,b}$. The construction of X_m and Y_m remains exactly the same, and since (4.25) remains valid for $q = \infty$, the arguments given above apply also here, so (4.28) holds in this case, too. $\square$

Next we want to prove Proposition 2.4. For this it clearly suffices to show the following:

Lemma 4.1. *Assume $\alpha_l^q / \beta_{l-1}^{p'}$ to be monotone near $\pm\infty$ if $q < \infty$, or α_k to be monotone near $\pm\infty$ for $q = \infty$. Then $\big(\alpha_l\, \beta_{l-1}\big)_{l \in \mathbb{Z}} \in \ell_r$ always implies $|(a,b)|_r < \infty$.*

Remark. It is notable that in the setting of Lemma 4.1, one cannot establish the estimate $|(a,b)|_r \leq c\, \|\big(\alpha_l\, \beta_{l-1}\big)_l\|_r$, as simple examples show.

Proof of Lemma 4.1. Let us first assume $q < \infty$. Since all possible situations can be treated similarly, we treat, e.g., the case where $\alpha_l^q/\beta_{l-1}^{p'}$ is monotonically increasing near ∞, say, for $k > k_0$. So let $k > k_0$ be given with $v_{k+2} < \infty$ and $v_k \neq v_{k+1}$. Then, $\alpha_l^q \leq \beta_{l-1}^{p'} \alpha_{v_{k+1}}^q (\beta_{v_{k+1}-1})^{-p'}$ holds for any $l \in \{v_k+1, \ldots, v_{k+1}\}$ and hence

$$\begin{aligned} \delta_k(a,b)^r = 2^{kr/p'} \Big(\sum_{l=v_k+1}^{v_{k+1}} \alpha_l^q \Big)^{r/q} \ &\leq\ 2^{kr/p'} \Big(\sum_{l=v_k}^{v_{k+1}-1} \beta_l^{p'} \Big)^{r/q} \frac{\alpha_{v_{k+1}}^r}{\beta_{v_{k+1}-1}^{rp'/q}} \\ &\leq\ 2^{k+1} \frac{\alpha_{v_{k+1}}^r}{\beta_{v_{k+1}-1}^{rp'/q}}\,. \end{aligned} \tag{4.29}$$

On the other hand, for any $l \in \{v_{k+1}+1, \dots, v_{k+2}+1\}$ we have of course $\alpha_l^q \geq \beta_{l-1}^{p'} \alpha_{v_{k+1}+1}^q (\beta_{v_{k+1}})^{-p'}$, which implies

$$\sum_{l=v_{k+1}+1}^{v_{k+2}+1} \alpha_l^r \beta_{l-1}^r \geq \Big(\sum_{l=v_{k+1}}^{v_{k+2}} \beta_l^{p'} \Big) \frac{\alpha_{v_{k+1}+1}^r}{\beta_{v_{k+1}}^{rp'/q}} \geq 2^{k+1} \frac{\alpha_{v_{k+1}}^r}{\beta_{v_{k+1}-1}^{rp'/q}} . \tag{4.30}$$

We combine (4.29) and (4.30) to see that

$$\delta_k(a,b)^r \leq \sum_{l=v_{k+1}+1}^{v_{k+2}+1} \alpha_l^r \beta_{l-1}^r \tag{4.31}$$

holds for all $k > k_0$ with $v_{k+2} < \infty$ and $v_k < v_{k+1}$. In order to prove that $\sum_{k>k_0} \delta_k(a,b)^r < \infty$, in view of (4.31) it suffices to show that with $K := \{k \in \mathbb{Z} : k > k_0 \text{ and } v_k < v_{k+1} \leq v_{k+2} < \infty\}$ we have

$$\sum_{k\in K} \sum_{l=v_{k+1}+1}^{v_{k+2}+1} \alpha_l^r \beta_{l-1}^r \leq 2 \sum_{l\in\mathbb{Z}} \alpha_l^r \beta_{l-1}^r. \tag{4.32}$$

This takes place if every summand appearing in the right-hand sum appears at most two times in the sums on the left-hand side. Thus we have to show only that for any $l \in \mathbb{Z}$ the set

$$M_l := \Big\{ k \in \mathbb{Z} : v_k \neq v_{k+1},\ v_{k+1}+1 \leq l \leq v_{k+2}+1 \Big\}$$

consists of at most two elements. So assume that $k_1, k_2, k_3 \in M_l$ with $k_1 < k_2 < k_3$. Since $v_{k_3} < v_{k_3+1}$, we get

$$v_{k_1+2} \leq v_{k_2+1} \leq v_{k_3} < v_{k_3+1} . \tag{4.33}$$

But by the definition of M_l we must have $v_{k_3+1} \leq l-1 \leq v_{k_1+2}$, which is a contradiction. So we know $|M_l| \leq 2$, and (4.32) is shown.

Let us now consider the case $q = \infty$, so $r = p'$. Again, we demonstrate only the summability in $+\infty$, provided that α_k is monotonically increasing for $k > k_0$. The arguments above have only to be modified slightly: For any $k > k_0$ with $v_k < v_{k+1}$ we have

$$\delta_k(a,b)^{p'} = 2^k \sup_{v_k < l \leq v_{k+1}} \alpha_l^{p'} = 2^k \alpha_{v_{k+1}}^{p'},$$

while

$$\sum_{l=v_{k+1}+1}^{v_{k+2}+1} \alpha_l^{p'} \beta_{l-1}^{p'} \geq \alpha_{v_{k+1}+1}^{p'} \Big(\sum_{l=v_{k+1}}^{v_{k+2}} \beta_l^{p'} \Big) \geq \alpha_{v_{k+1}}^{p'} 2^{k+1}.$$

So we conclude that it suffices to verify (4.32) for the case $q = \infty$, which is proved completely analogously as above for the case $q < \infty$. $\square$

Next we prove Theorem 2.5. To do so we construct suitable "embeddings" of diagonal operators. Let $m \in \mathbb{N}$ be a number for which the regularity condition (2.12) holds. For $l \in \mathbb{Z}$ we set

$$l + (2m+4)\,\mathbb{Z} \ := \ \{l + (2m+4)k : k \in \mathbb{Z}\} \tag{4.34}$$

as usual. If $l \in \{0, \dots, 2m+3\}$, then we define

$$\pi_l : \mathbb{Z} \to l + (2m+4)\,\mathbb{Z}, \qquad \pi_l(k) := l + (2m+4)k\,. \tag{4.35}$$

Now let D^l be a diagonal operator generated by the sequence

$$\sigma_k^l := \delta_{\pi_l(k)}(a,b) = 2^{\pi_l(k)/p'} \Big(\sum_{j=v_{\pi_l(k)}+1}^{v_{\pi_l(k)+1}} \alpha_j^q \Big)^{1/q}. \tag{4.36}$$

With these operators, we have

Proposition 4.2. *For any $l \in \{0, \dots, 2m+3\}$, there is a bounded linear operator $X^l : \ell_p(\mathbb{Z}) \to \ell_p(\mathbb{Z})$ with $\|X^l\| \le C_m$ and*

$$\|D^l(x)\| \le 2\|S_{a,b} \circ X^l(x)\| \tag{4.37}$$

for all $x \in \ell_p(\mathbb{Z})$.

Before we prove Proposition 4.2, let us mention a useful lemma:

Lemma 4.3. *Assume $v_k \ne v_{k+1}$. Then*

$$2^{k-1} \le \sum_{j=v_{k-1}}^{v_{k+1}-1} \beta_l^{p'} \le 2^{k+1}. \tag{4.38}$$

Proof. The upper estimate is trivial. For the lower estimate, combine

$$\sum_{l<v_{k+1}} \beta_l^{p'} \ge \sum_{l \le v_k} \beta_l^{p'} \ge 2^k \quad \text{with} \quad \sum_{l<v_{k-1}} \beta_l^{p'} < 2^{k-1}. \quad \square$$

Proof of Proposition 4.2. We prove the proposition for the case $l = 0$, the other cases can be treated analogously. Let $(e_k)_{k\in\mathbb{Z}}$ be the standard base of $\ell_p(\mathbb{Z})$. We define the mapping X^0 by setting $X^0(e_k) := x^k$, where x^k's are defined below. The construction guarantees that the vectors $y^k := S_{a,b}(x^k)$ are disjointly supported, that $\|x^k\|_p \le c$, and moreover $\|y^k\| \ge 1/2\sigma_k^0$ with σ_k^0 defined in (4.36) for $l = 0$. From this, one can directly conclude the assertion. So let $k \in \mathbb{Z}$ be given, and set $k_0 := (2m+4)k = \pi_0(k)$. If $v_{k_0} = v_{k_0+1}$, then $\sigma_k^0 = \delta_{k_0}(a,b) = 0$, and we set $x^k := 0$. Otherwise, we distinguish two cases:

Case 1*: $v_{k_0-1} \ne v_{k_0}$.*

Then we fix $k_1 \in \{k_0+m+2, k_0+2m+2\}$ such that $v_{k_1} \ne v_{k_1+1}$ (this is possible due to our assumption (2.12)). Since $k_1 - 1 > k_0 + 1 + m$, (2.12) also shows that $v_{k_1-1} > v_{k_0+1}$. Now we define $x^k = (x_j^k)_{j\in\mathbb{Z}}$ via

$$x_j^k := 2^{-k_0/p} \cdot \begin{cases} \beta_j^{p'-1} & \text{if } j \in \{v_{k_0-2}, \dots, v_{k_0} - 1\}, \\ -\mathcal{C}_k\ \beta_j^{p'-1} & \text{for } j \in \{v_{k_1-1}, \dots, v_{k_1+1} - 1\}, \\ 0 & \text{otherwise}, \end{cases} \tag{4.39}$$

where

$$\mathcal{C}_k := \Big(\sum_{j=v_{k_0-2}}^{v_{k_0}-1} \beta_l^{p'} \Big) \Big(\sum_{j=v_{k_1-1}}^{v_{k_1+1}-1} \beta_l^{p'} \Big)^{-1}. \tag{4.40}$$

Lemma 4.3 tells us that $\mathcal{C}_k \in [2^{-2m-4}, 2^{-m+1}]$. It is easy to obtain

$$\|x^k\|_p^p = 2^{-k_0}\Big[\sum_{j=v_{k_0-2}}^{v_{k_0}-1} \beta_j^{p'} + \mathcal{C}_k^p \sum_{j=v_{k_1-1}}^{v_{k_1+1}-1} \beta_j^{p'}\Big] \le 1 + 2^{-k_0+1}2^{k_0+2m+1} \le 2^{2m+2}.$$

Let $y^k = (y_l^k)_{l\in\mathbb{Z}} := S_{a,b}(x^k)$. We will not calculate all values of y_l^k's, since it is only important to note that $y_l^k \ge 0$, and

$$y_l^k = \begin{cases} 0 & \text{if } l < v_{k_0-2}, \\ 2^{-k_0/p}\alpha_l \sum\limits_{j=v_{k_0-2}}^{v_{k_0}-1} \beta_j^{p'} & \text{when } l \in \{v_{k_0}+1, \dots, v_{k_0+1}\}, \\ 0 & \text{for } l \ge v_{k_1+1}. \end{cases} \tag{4.41}$$

Case 2: $v_{k_0} = v_{k_0-1}$.

In this case, because of $v_{k_0} < v_{k_0+1}$, we know that

$$2^{k_0-1} \le \beta_{v_{k_0}}^{p'} \le 2^{k_0+1}. \tag{4.42}$$

Let k_1 be fixed like in case 1, and set

$$x_j^k := 2^{-k_0/p}\begin{cases} \beta_{v_{k_0}}^{p'-1} & \text{for } j = v_{k_0}, \\ -\mathcal{C}_k\ \beta_j^{p'-1} & \text{for } j \in \{v_{k_1-1}, \dots, v_{k_1+1}-1\}, \\ 0 & \text{otherwise}, \end{cases} \tag{4.43}$$

where this time

$$\mathcal{C}_k := \beta_{v_{k_0}}^{p'}\Big(\sum_{j=v_{k_1-1}}^{v_{k_1+1}-1} \beta_l^{p'}\Big)^{-1}. \tag{4.44}$$

Because of (4.42) and Lemma 4.3, we know that $\mathcal{C}_k \in [2^{-2m-4}, 2^{-m+1}]$. Again, it is easy to obtain $\|x^k\|_p \le 2^{(2m+2)/p}$. Setting $y^k := S_{a,b}(x^k)$, we conclude that

$$y_l^k = \begin{cases} 0 & \text{if } l < v_{k_0}, \\ 2^{-k_0/p}\alpha_l\beta_{v_{k_0}} & \text{when } l \in \{v_{k_0}+1, \dots , v_{k_0+1}\}, \\ 0 & \text{for } l \ge v_{k_1+1}. \end{cases} \tag{4.45}$$

So in both cases we have defined our vectors x^k. Since $k_1 \le k_0 + 2m + 2 \le \pi_0(k+1) - 2$, we easily see that the y^k 's are as disjointly supported as the x^k 's are with

$$\|y^k\|_q \ge 2^{k_0/p'-1}\Big(\sum_{j=v_{k_0}+1}^{v_{k_0+1}} \alpha_j^q\Big)^{1/q} = 1/2\sigma_k^0. \tag{4.46}$$

Taking into account $y^k = S_{a,b} \circ X^0(e_k)$, we conclude that (4.37) holds. $\square$

Before we can finally prove Theorem 2.5, a technical difficulty has to be fixed. Note that for a subsequence $\sigma_{\pi(n)}$ of a sequence σ_n, the sequence $(\sigma_{\pi(n)})_n^*$ cannot, in general, be estimated from below by σ_n^*, e.g. $(\sigma_{\pi(n)})_n^*$ could be constantly 0 and σ_n^* be 1. In particular, for the sequences σ^l defined in (4.36), $(\sigma^l)_n^*$ cannot be estimated from below by $\delta_n^*(a,b)$, but fortunately this is possible for the maximum:

Lemma 4.4. *Let $(\sigma_n)_{n\geq 0}$ be a nonnegative sequence tending to zero, and let $(\sigma_{\pi_1(n)})_{n\in\mathbb{N}}, \dots, (\sigma_{\pi_N(n)})_{n\in\mathbb{N}}$ be partial sequences of $(\sigma_n)_{n\in\mathbb{N}}$, where the π_j's satisfy $\bigcup_{j=1}^N \pi_j(\mathbb{N}) = \mathbb{N}$. Then for any $s > 0$ we have*

$$\sup_{n\in\mathbb{N}} n^{1/s}\sigma_n^* \leq 2N^{1/s} \max_{j=1,\dots,N} \sup_{n\in\mathbb{N}} n^{1/s}(\sigma_{\pi_j(n)})_n^*, \tag{4.47}$$

and for all $k \in \mathbb{N}$

$$(Nk)^{1/s}\sigma_{kN}^* \leq 2\, N^{1/s}k^{1/s} \max_{j=1,\dots,N}(\sigma_{\pi_j(n)})_k^* \tag{4.48}$$

holds.

Proof. For $x \geq 0$ denote $\lfloor x\rfloor := \sup\{n \in \mathbb{N} : n \leq x\}$. Given $n \in \mathbb{N}$, there are $k_1, \dots, k_n \in \mathbb{N}$ with $\sigma_{k_i} \geq \sigma_n^*/2$. For $l := \lfloor n/N\rfloor + 1$ we conclude that there is $j \in \mathbb{N}$ such that $|\pi_j(\mathbb{N}) \cap \{k_1, \dots, k_n\}| \geq l$. Consequently, for this j we have $(\sigma_{\pi_j(n)})_l^* \geq \sigma_n^*/2$, and since $n/l \leq N$, we find

$$n^{1/s}\sigma_n^* \leq 2\,(n/l)^{1/s}l^{1/s}(\sigma_{\pi_j(n)})_l^* \leq 2N^{1/s}l^{1/s}(\sigma_{\pi_j(n)})_n^*\,,$$

which implies (4.47). The second assertion is analogous. $\square$

Proof of Theorem 2.5. Using Lemma 4.2 from [10], it is easy to deduce from Proposition 4.2 and Corollary 3.7 that

$$\begin{aligned}\sup_{n\in\mathbb{N}} n^\rho e_n(S_{a,b}) &\geq 1/4 \max_{l\leq 2m+3} \sup_{n\in\mathbb{N}} n^\rho e_n(D^l)\\ &\geq 1/4 \max_{l\leq 2m+3} \sup_{n\in\mathbb{N}} n^{1/s}((\sigma^l)_n^* + (\sigma^l)_{-n+1}^*)\end{aligned} \tag{4.49}$$

with D^l and σ^l defined in (4.36). On the other hand, by Lemma 4.4 with $\delta_n := \delta_n(a,b)$ we get

$$\begin{aligned}\sup_{n\in\mathbb{N}} n^{1/s}(\delta_{n-1}^* + \delta_{-n}^*) &\leq \sup_{n\in\mathbb{N}} n^{1/s}\delta_{n-1}^* + \sup_{n\in\mathbb{N}} n^{1/s}\delta_{-n}^*\\ &\leq C \max_{l=0,\dots,2m+3}\Big(\sup_{n\in\mathbb{N}} n^{1/s}(\sigma^l)_{n-1}^*\Big)\\ &+ C \max_{l=0,\dots,2m+3}\Big(\sup_{n\in\mathbb{N}} n^{1/s}(\sigma^l)_{-n}^*\Big)\\ &\leq 2\,C \max_{l=0,\dots,2m+3}\left(\sup_{n\in\mathbb{N}} n^{1/s}\Big((\sigma^l)_{n-1}^* + (\sigma^l)_{-n}^*\Big)\right).\end{aligned}$$

Combined with (4.49), this shows the first assertion.

The last estimate is based on the fact that for $a_n, b_n \geq 0$ we have

$$\sup_n(a_n + b_n) \leq \sup_n a_n + \sup_n b_n \leq 2\sup_n(a_n + b_n).$$

Such estimates cannot be established for the infimum. So for estimate (2.15) we have to be a little more careful. First we define two subspaces of $\ell_p(\mathbb{Z})$, namely

$$\ell_p^+ := \left\{ (x_k)_{k\in\mathbb{Z}} \in \ell_p(\mathbb{Z}) : x_k = 0 \text{ for } k < 0 \right\}, \tag{4.50}$$

as well as

$$\ell_p^- := \left\{ (x_k)_{k\in\mathbb{Z}} \in \ell_p(\mathbb{Z}) : x_k = 0 \text{ if } k \geq 0,\ \sum_{j<0} \beta_j x_j = 0 \right\}. \tag{4.51}$$

The corresponding restrictions of $S_{a,b}$ are denoted analogously, e.g.,

$$S_{a,b}^+ := \Big(S_{a,b}\Big)_{|\ell_p^+}, \qquad S_{a,b}^- := \Big(S_{a,b}\Big)_{|\ell_p^-}. \tag{4.52}$$

We observe that $\operatorname{Im}(S_{a,b}^+) \cap \operatorname{Im}(S_{a,b}^-) = \{0\}$. It is easy to deduce that

$$e_n(S_{a,b}^+) + e_n(S_{a,b}^-) \leq C_{p,q}\, e_n(S_{a,b}). \tag{4.53}$$

A careful inspection of the proof of Proposition 4.2 reveals that $X^l(e_k) \in \ell_p^+$ if $k \geq 0$, and that $X^l(e_k) \in \ell_p^-$ if $k < -1$. As above, this yields

$$e_n(S_{a,b}^+) \geq C \max_{0\leq l\leq 2m+3} e_n\Big(D_{(\sigma_k^l)_{k\geq 0}}\Big)$$

and

$$e_n(S_{a,b}^-) \geq C' \max_{0\leq l\leq 2m+3} e_n\Big(D_{(\sigma_{-k}^l)_{k\geq 2}}\Big).$$

From this we easily derive with the help of (3.23) and (4.48) that

$$n^\rho e_n(S_{a,b}^+) \geq C\, n^{1/s} \delta^*_{(2m+4)(n-1)} \tag{4.54}$$

and

$$n^\rho e_n(S_{a,b}^-) \geq C'\, n^{1/s} \Big((\delta_{-k})_{k\geq 2m+4} \Big)^*_{(2m+4)n}. \tag{4.55}$$

But it is not hard to see that $\Big((\delta_{-k})_{k\geq 2m+4} \Big)^*_{(2m+4)n} \geq \delta^*_{-2(2m+4)n}$, which allows to continue the estimate (4.55) into

$$n^\rho e_n(S_{a,b}^-) \geq C'\, n^{1/s} \delta^*_{-2(2m+4)n}. \tag{4.56}$$

Combined with (4.54) and (4.53), this reveals

$$n^\rho e_n(S_{a,b}) \geq C''\, n^{1/s} \Big(\delta^*_{2(2m+4)n-1} + \delta^*_{-2(2m+4)n} \Big).$$

From this we are directly led to (2.15). $\square$

5. Examples and Special Cases

At first we will study polynomial growth of the α_k's and β_k's: Let us assume that for $\alpha, \beta \in \mathbb{R}$ we have $\alpha_k = k^{-\alpha}$ and $\beta_k = k^{-\beta}$ if $k \geq 1$, and $\alpha_k = 0 = \beta_k$ otherwise. First of all, (2.1) is satisfied iff $\alpha > 1/q$ for $q < \infty$, and iff $\alpha \geq 0$ when $q = \infty$. For A_k and B_k given by (2.1) we find that then $A_k \approx k^{-\alpha+1/q}$ while

$$B_k \approx \begin{cases} k^{-\beta+1/p'}, & \beta < 1/p', \\ \log k, & \beta = 1/p', \\ c, & \beta > 1/p'. \end{cases} \tag{5.1}$$

Assume first $p \leq q$. Then Theorem 2.1 tells us that $S_{a,b}$ is bounded iff $A_k \cdot B_k$ is so; by treating the three cases given in (5.1) separately and regarding $1/p' \geq 1/r - \alpha$ one can easily check that $S_{a,b}$ is bounded from ℓ_p to ℓ_q iff $\alpha + \beta \geq 1/r$. For $p > q$, the condition in Theorem 2.1 seems to be more complicated; however, using (5.1) one can verify that in this case we can estimate

$$[A_k \cdot B_k^{p'/q'}]^{\frac{pq}{p-q}} \beta_{k-1}^{p'} \approx \begin{cases} k^{\frac{pq}{p-q}(1-\alpha-\beta)}, & \beta < 1/p', \\ k^{\frac{pq}{p-q}(1/q-\alpha)-1} (\log k)^{\frac{pq}{q'(p-1)}}, & \beta = 1/p', \\ k^{\frac{pq}{p-q}(1/q-\alpha)-\beta p'}, & \beta > 1/p'. \end{cases} \tag{5.2}$$

For $\beta \geq 1/p'$ this is always summable, while for $\beta < 1/p'$ we infer that $S_{a,b}$ is bounded from ℓ_p to ℓ_q iff $\frac{pq}{p-q}(1-\alpha-\beta) < -1$, which is equivalent to $\alpha+\beta > 1/r$. Since $1/p' > 1/r - \alpha$ (we have $q < \infty$ in this case), we find that for $p > q$ the operator $S_{a,b}$ is bounded iff $\alpha + \beta > 1/r$. Next we wish to calculate $\delta_k(a, b)$. For this purpose we assume from now on that $\beta < 1/p'$. Then it is easy to see that for v_k as in (2.6) we have $v_k \approx 2^{k/(1-\beta p')}$, and consequently,

$$\delta_k(a, b) \approx 2^{k/p'} \left(\sum_{j=2^{k/(1-\beta p')}}^{2^{(k+1)/(1-\beta p')}} j^{-\alpha q} \right)^{1/q} \approx 2^{k((1/q-\alpha)/(1-\beta p')+1/p')}.$$

Since it is easily verified that $(1/q - \alpha)/(1 - \beta p') < -1/p'$ if and only if $\beta > 1/r - \alpha$, we derive that $S_{a,b}$ is compact iff $\beta > 1/r - \alpha$, and in this case, we have an exponential decrease of $\delta_k(a, b)$. Through easy direct calculations one verifies that $\delta_k(a, b)$ decrease even more rapidly for $\beta \geq 1/p'$. Summing up our results and using Theorem 2.2, we verify the following proposition:

Proposition 5.1. *For $\alpha, \beta \in \mathbb{R}$, define $S_{\alpha,\beta} : \ell_p(\mathbb{N}) \to \ell_q(\mathbb{N})$ by*

$$S_{\alpha,\beta} : x = (x_k)_{k\in\mathbb{N}} \mapsto \left(k^{-\alpha} \sum_{l=1}^{k-1} l^{-\beta} x_l \right)_{k\in\mathbb{N}}.$$

Then

(i) *$S_{\alpha,\beta}$ is bounded from ℓ_p to ℓ_q iff*

$$\begin{cases} \alpha \geq 0 \quad \text{and} \quad \alpha+\beta \geq 1/r & \text{if } q=\infty, \\ \alpha > 1/q \quad \text{and} \quad \alpha+\beta \geq 1/r & \text{if } p \leq q < \infty, \\ \alpha > 1/q \quad \text{and} \quad \alpha+\beta > 1/r & \text{for } p > q; \end{cases}$$

and

(ii) *$S_{\alpha,\beta}$ is compact iff $\alpha > 1/q$ ($\alpha \geq 0$, if $q = \infty$), and $\alpha+\beta > 1/r$. In this case, we have $\lim_{n\to\infty} n e_n(S_{\alpha,\beta}) = 0$.*

As a second class of interest we investigate the case that β_k's grow exponentially:

Proposition 5.2. *Let $a = (\alpha_k)_{k\in\mathbb{Z}}$ be arbitrary, and let $\beta_k = 2^{k/p'}$, $k \in \mathbb{Z}$. Then we have*

$$c_1 |(a,b)|_{r,\infty} \leq \sup_{n\in\mathbb{N}} n\ e_n(S_{a,b}) \leq c_2 |(a,b)|_{r,\infty} \tag{5.3}$$

and

$$c_1 \varliminf_{n\to\infty} n^{1/r}\tau_n \leq \varliminf_{n\to\infty} n\, e_n(S_{a,b}) \leq \varlimsup_{n\to\infty} n\, e_n(S_{a,b}) \leq c_2 \varlimsup_{n\to\infty} n^{1/r}\tau_n, \tag{5.4}$$

with $\tau_n := \delta^*_{n-1}(a,b) + \delta^*_{-n}(a,b)$.

Proof. Note first that $\delta_k(a,b) = 2^{k/p'}\alpha_k$ for $k \in \mathbb{Z}$. Let $V : \ell_p(\mathbb{Z}) \to \ell_p(\mathbb{Z})$ denote an auxiliary summation operator defined by

$$V((x_k)_{k\in\mathbb{Z}}) := \left(2^{-k/p'} \sum_{j<k} x_j 2^{j/p'}\right)_{k\in\mathbb{Z}}.$$

Using Theorem 2.1 one easily checks that V is bounded. We set now $D : \ell_p(\mathbb{Z}) \to \ell_q(\mathbb{Z})$ to be the diagonal operator generated by the sequence $\delta_k(a,b)$. Then it is trivial to obtain $S_{a,b} = D \circ V$, yielding $e_n(S_{a,b}) \leq C e_n(D)$. On the other hand, let $R : \ell_p(\mathbb{Z}) \to \ell_p(\mathbb{Z})$ be given by $R(e_k) := 2e_{k-1} - e_k$. We find that $D = S_{a,b} \circ R$, and thus $e_n(D) \leq C' e_n(S_{a,b})$. Now we have to estimate $e_n(D)$ only, which has already been done in Corollary 3.7. □

Next we study the case of a rapid (i.e., superexponential) increase of β_k:

Proposition 5.3. *Set $\beta_k = 2^{2^k}$ for $k \geq 0$ and $\beta_k = 0$ for $k < 0$. Then for any sequence $a = (\alpha_k)_{k\in\mathbb{Z}}$ we have*

$$|(a,b)|_{r,\infty} \geq c \sup_{l\in\mathbb{N}} l^{1/r} (\alpha_l\ \beta_l)^*_l\,, \tag{5.5}$$

while

$$\sup_{l\in\mathbb{N}} l\, e_l(S_{a,b}) \leq C \sup_{l\in\mathbb{N}} l^{1/r} (\alpha_l\ \beta_{l-1})^*_l\,. \tag{5.6}$$

In particular, if we set $\alpha_l := 2^{-l}\ \beta^{-1}_{l-1}$ whenever $l \geq 0$, and $\alpha_l = 0$ otherwise, then $|(a,b)|_{r,\infty} = \infty$ while $\sup_n n\ e_n(S_{a,b}) < \infty$.

Proof. It is easy to see that $v_k = \lfloor \log_2 k \rfloor$ for $k \geq k_0$, where as usual

$$\lfloor x \rfloor := \inf\{n \in \mathbb{N} : n \geq x\} .$$

In particular, $v_k \neq v_{k+1}$ iff $k = 2^l - 1$ for some $l \geq 1$. So for $k = 2^l - 1$ we derive

$$\delta_k(a,b) = 2^{k/p'} \alpha_{v_{k+1}} = 2^{(2^l-1)/p'} \alpha_l \geq c_p \ \beta_l \ \alpha_l. \tag{5.7}$$

Since $k^{1/r} \geq l^{1/r}$, this implies (5.5).

On the other hand, let $V : \ell_p(\mathbb{Z}) \to \ell_p(\mathbb{Z})$ be defined by

$$V\Big((x_l)_{l \in \mathbb{Z}}\Big) := \Big(\beta_{l-1}^{-1} \sum_{j<l} \beta_j x_j\Big)_{l \in \mathbb{Z}} . \tag{5.8}$$

Then we have by Theorem 2.1 that $\|V\| \leq C_p$ and, additionally,

$$S_{a,b} = D_{\alpha_l \cdot \beta_{l-1}} \circ V. \tag{5.9}$$

Thus $e_n(S_{a,b}) \leq C_p \, e_n(D_{(\alpha_l \cdot \beta_{l-1})})$ and so (5.6) follows by virtue of Theorem 3.5. □

Finally, we will estimate the first entropy numbers of finite-dimensional standard summation operators. Estimates for the approximation, Gelfand and Kolmogorov numbers of such operators can be found in [8]. Let $N \geq 2$ be arbitrary, and set $\lambda := \min\{1, 1/r\}$. Then for the standard summation operator $S_1^N : \ell_p^N \to \ell_q^N$ with

$$S_1^N((x_i)_{i=1}^N) := \Big(\sum_{j<i} x_j\Big)_{i=1}^N$$

we have

Proposition 5.4. *There are universal constants $c_0, c_1, c_2 > 0$ such that the estimate*

$$c_1 \frac{N^{1/r}}{n} \leq e_n(S_1^N : \ell_p^N \to \ell_q^N) \leq c_2 \frac{N^{1/r}}{n} \tag{5.10}$$

holds for all $n \leq c_0 N^{\lambda}$.

Proof. We start with the upper estimate. Therefore choose $m \in \mathbb{N}$ such that

$$2^m \leq N < 2^{m+1}.$$

v_k's are given now by

$$v_k = \begin{cases} 1 & \text{for } k \leq 0 \\ 2^k & \text{if } 1 \leq k \leq m \\ \infty & \text{when } k > m. \end{cases} \tag{5.11}$$

Thus $\delta_k(a,b) = 2^{k/p' + k/q} = 2^{k/r}$ for $k = 0, \ldots, m-1$, and for $k = m$ we have $\delta_m(a,b) = 2^{m/p'}(N - 2^m)^{1/q} \leq 2^{m/r}$. By Theorem 2.2 we obtain

$$\sup_{n \in \mathbb{N}} n \ e_n(S_1^N) \leq c \left(\sum_{k=0}^m \delta_k^r\right)^{1/r} \leq c \ 2^{m/r} \leq c' N^{1/r}.$$

For the lower estimate, set

$$I_j := \Big[\frac{j-1}{N}, \frac{j}{N}\Big], \qquad 1 \le j \le N,$$

and denote by P_N the conditional expectation operator mapping from $L_p[0,1]$ to $L_p[0,1]$ generated by the partition $I_1, \dots I_N$ of $[0,1]$, i.e.,

$$P_N(f) := \sum_{j=1}^N \mathbf{1}_{I_j} |I_j|^{-1} \int\limits_{I_j} f(t)\, dt.$$

It is easily checked that $\|P_N\| = 1$. Further we define φ_p to be an embedding of ℓ_p^N into $L_p[0,1]$ with

$$\varphi_p((x_j)) := \sum_{j=1}^N x_j \mathbf{1}_{I_j} |I_j|^{-1/p}.$$

Recalling $|I_j| = N^{-1}$, we derive easily

$$\Big(\varphi_q \circ (N^{-1/r} S_1^N) \circ \varphi_p^{-1} \circ P_N\Big) f = \sum_{k=1}^N \mathbf{1}_{I_k} \Big(\sum_{j<k} \int\limits_{I_j} f\, dt\Big). \tag{5.12}$$

Next define $R_N : L_p[0,1] \to L_q[0,1]$ via

$$(R_N f)(s) := \sum_{k=1}^N \Big(\int\limits_{\frac{k-1}{N}}^{s} f(t)\, dt\Big)\, \mathbf{1}_{I_k}(s).$$

Using (5.12), we find

$$T_1 = R_N + \Big(\varphi_q \circ (N^{-1/r} S_1^N) \circ \varphi_p^{-1} \circ P_N\Big). \tag{5.13}$$

We derive an upper bound for $\|R_N\|$ by estimating (we assume $q < \infty$; the case $q = \infty$ may be treated in the same way)

$$\begin{aligned}
\|R_N f\|_q &= \Big(\sum_{k=1}^N \int\limits_{I_k} \Big| \int\limits_{\frac{k-1}{N}}^{s} f(t)\, dt\Big|^q ds\Big)^{1/q} \\
&\le \Big(\sum_{k=1}^N \int\limits_{I_k} \Big(s - \frac{k-1}{N}\Big)^{q/p'} ds\, \Big(\int\limits_{I_k} |f(t)|^p\, dt\Big)^{q/p}\Big)^{1/q} \\
&\le N^{-1/r} \Big(\sum_{k=1}^N \Big(\int\limits_{I_k} |f(t)|^p\, dt\Big)^{q/p}\Big)^{1/q}
\end{aligned}$$

The last sum can either be estimated by $\|f\|_p$ when $p \ge q$, or by $N^{1/q-1/p}\|f\|_p$ if $p \le q$. This shows that $\|R_N\| \le N^{-\lambda}$.

From now on, $d_1, d_2, \dots$ shall denote fixed constants (that is, the value of d_i remains the same at each occurrence). Because of the additivity of the entropy numbers (see, e.g., [2]) and equality (5.13) we come to

$$e_n(T_1) \leq \|R\| + e_n(\varphi_q \circ (N^{-1/r} S_1^N) \circ \varphi_p^{-1} \circ P_N) \leq d_1 N^{-\lambda} + d_2 N^{-1/r} e_n(S_1^N).$$

Since it is well-known that $e_n(T_1) \geq d_3/n$ (cf. [7]), this reveals

$$d_2 N^{-1/r} e_n(S_1^N) \geq \frac{d_3}{n} - \frac{d_1}{N^\lambda} \tag{5.14}$$

for all $n, N \in \mathbb{N}$. Let $c_0 := d_3(2d_1)^{-1}$, then (5.14) provides that for $n \leq c_0 N^\lambda$ we have

$$N^{-1/r} e_n(S_1^N) \geq \frac{d_3}{2d_2 n}. \qquad \square$$

For $p < q$, Proposition 5.4 is somewhat unsatisfying, because we can estimate $e_n(S_1^N)$ only for $n \leq c_0 N^{1/r}$. Recall that in this case, $1/r < 1$. We leave this as an open problem.

Problem.[1] Find optimal estimates for $e_n(S_1^N : \ell_p^N \to \ell_q^N)$ when $p < q$ and $c_0 N^{1/r} \leq n \leq N$.

6. Some Probabilistic Applications

Let T be an operator from a (separable) Hilbert space H into a Banach space E, and let $(\xi_k)_{k=1}^\infty$ be an i.i.d. sequence of $\mathcal{N}(0,1)$-distributed random variables. Then the behaviour of $e_n(T)$ as $n \to \infty$ is known to be closely related with the properties of the E-valued Gaussian random variable

$$X := \sum_{k=1}^\infty \xi_k \cdot T f_k\,, \tag{6.1}$$

where $(f_k)_{k=1}^\infty$ denotes some orthonormal basis in H. For example, in [4] it was proved that $\sum_{n=1}^\infty n^{-1/2} e_n(T) < \infty$ implies the a.s. convergence of (6.1) in E, while, conversely, the a.s. convergence of (6.1) yields $\sup_n n^{1/2} e_n(T) < \infty$ (cf. [14]). A faster decay of $e_n(T)$ as $n \to \infty$ is reflected by a "better" small ball behaviour of X. More precisely the following holds (cf. [9], [11] and [3]):

Proposition 6.1. *Let $0 < \alpha < 2$ and $T : H \to E$ generating X via* (6.1). *Then the following statements are equivalent:*

(i) $\overline{\lim\limits_{n\to\infty}}\, n^{1/\alpha} e_n(T) := c_\alpha(T) < \infty$.

(ii) $\underline{\lim\limits_{\varepsilon\to 0}}\, \varepsilon^{\frac{2\alpha}{2-\alpha}} \log \mathbb{P}\Big(\|X\|_E < \varepsilon\Big) := -d_\alpha(X) > -\infty$.

[1] Recently M. Lifshits kindly showed us a proof of the following fact: For any $h > 0$ and any pair $(p,q) \neq (1,\infty)$ there is $c(p,q,h) > 0$ such that $e_n(S^N) \geq c(p,q,h) n^{-1} N^{1/r}$, $n \geq (1-h)N$. Since the upper estimate is valid for all $n \in \mathbb{N}$, this almost answers the problem.

Moreover,

$$d_\alpha(X) \le c_0 \cdot c_\alpha(T)^{\frac{2\alpha}{2-\alpha}} \tag{6.2}$$

for some universal $c_0 > 0$. *Conversely, if* T *satisfies*

$$\varliminf_{n\to\infty} n^{1/\alpha} e_n(T) := \kappa_\alpha(T) > 0\,,$$

then

$$\varlimsup_{\varepsilon\to 0} \varepsilon^{\frac{2\alpha}{2-\alpha}} \log \mathbb{P}\Big(\|X\|_E < \varepsilon) := -\lambda_\alpha(X) < 0$$

and

$$\lambda_\alpha(X) \ge c_0'\, \kappa_\alpha(T)^{\frac{2\alpha}{2-\alpha}}\ .$$

We want to apply these relations between the entropy and the small ball behaviour to the summation operator $S_{a,b} : \ell_2(\mathbb{Z}) \to \ell_q(\mathbb{Z}),\ 1 \le q \le \infty$. To do so, let $W = (W(t))_{t\ge 0}$ be a standard Wiener process over $[0,\infty)$. Taking the natural unit vector basis $(e_k)_{k\in\mathbb{Z}}$ in $\ell_2(\mathbb{Z})$ leads to

$$X = \sum_{k=-\infty}^{\infty} \xi_k S_{a,b}(e_k), \tag{6.3}$$

where the $\ell_q(\mathbb{Z})$-valued Gaussian random variable $X = (X_k)_{k\in\mathbb{Z}}$ is defined by

$$X_k := \alpha_k W(t_k) \tag{6.4}$$

with $t_k := \sum_{j<k} \beta_j^2$ being an increasing sequence of nonnegative numbers such that $\lim_{k\to-\infty} t_k = 0$. In our situation we have $r = \frac{2q}{2+q}$ and v_k's given in (2.6) can be calculated by

$$v_k = \sup\{m \in \mathbb{Z} : t_m < 2^k\}. \tag{6.5}$$

With this notation the following is valid:

Theorem 6.2. *Let* $(\alpha_k)_{k\in\mathbb{Z}}$ *be a sequence of nonnegative numbers satisfying* (2.1) *and let* $(t_k)_{k\in\mathbb{Z}}$ *be a nondecreasing sequence with* $\lim_{k\to-\infty} t_k = 0$. *For* $X = (X_k)_{k\in\mathbb{Z}}$ *defined by* (6.4) *and* v_k*'s as in* (6.5) *we have the following.*

(i) *If*

$$\sum_{k=-\infty}^{\infty} 2^{\frac{kq}{2+q}} \cdot \Big(\sum_{l=v_k+1}^{v_{k+1}} \alpha_l^q\Big)^{\frac{2}{2+q}} < \infty, \tag{6.6}$$

then

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}(\|X\|_q < \varepsilon) = 0. \tag{6.7}$$

(ii) *Assume that* $[t_{k+1} - t_k]^{-1}\alpha_k^q$ *is monotone near* $\pm\infty$. *Then* (6.7) *holds provided that*

$$\sum_{k=-\infty}^{\infty} [t_k - t_{k-1}]^{\frac{q}{2+q}} \cdot \alpha_k^{\frac{2q}{2+q}} < \infty\,. \tag{6.8}$$

Proof. Combining Proposition 6.1 for $\alpha = 1$ with Theorem 2.2, one easily sees that (6.6) implies

$$\varliminf_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}(\|X\|_q < \varepsilon) = -d_1(X) > -\infty. \tag{6.9}$$

Yet by the second part of Theorem 2.2 and estimate (6.2) we even have $d_1(X) = 0$ which completes the proof of (i). Assertion (ii) is a direct consequence of Proposition 2.4. Recall that $\beta_k^2 = t_{k+1} - t_k$. $\square$

Remarks. (1) Theorem 2.3 shows that, in general, condition (6.6) cannot be replaced by the weaker assumption (6.8). More precisely, for each $q \in [1, \infty]$ there are α_k's and t_k's satisfying (6.8), yet

$$\varliminf_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}(\|X\|_q < \varepsilon) = -\infty\,.$$

(2) In the case $q = \infty$ Theorem 6.2 reads as follows. Suppose that

$$\sum_{k=-\infty}^{\infty} 2^k \sup_{v_k < l \le v_{k+1}} \alpha_l^2 < \infty. \tag{6.10}$$

Then

$$\lim_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}(\sup_{k \in \mathbb{Z}} \alpha_k |W(t_k)| < \varepsilon) = 0. \tag{6.11}$$

Again, (6.11) becomes wrong under a weaker condition

$$\sum_{k=-\infty}^{\infty} [t_k - t_{k-1}] \cdot \alpha_k^2 < \infty\,. \tag{6.12}$$

Recall that (6.12) suffices for α_k's monotone near $\pm\infty$.

Observe that Theorem 6.2 provides us with sufficient conditions for

$$-\log \mathbb{P}(\|X\|_q < \varepsilon) \preceq \varepsilon^{-\alpha} \tag{6.13}$$

in the case $\alpha = 2$. Of course, similar conditions would also be of interest for α's different of 2, yet we could not handle this problem by our methods. Recall that we have derived upper bounds for the entropy of summation operators by the corresponding results about integration operators. And there only the case $\alpha = 2$ has been treated. But for t_k's not increasing too fast we can prove necessary conditions for (6.13) and all $\alpha > 0$. To be more precise, suppose that t_k's satisfy

$$\sup \left\{ \frac{t_{k+1}}{t_k} \ :\ t_k > 0 \right\} < \infty\,. \tag{6.14}$$

Then the following holds.

Proposition 6.3. *Let $X = (X_k)_{k\in\mathbb{Z}}$ be a Gaussian sequence defined by* (6.4) *with t_k's satisfying* (6.14)*. If*

$$\varliminf_{\varepsilon\to 0} \varepsilon^\alpha \log \mathbb{P}(\|X\|_q < \varepsilon) > -\infty \tag{6.15}$$

for some $\alpha > 0$, then necessarily

$$\sup_{n\geq 1} n^{1/\alpha+1/q} \left(\delta^*_{n-1} + \delta^*_{-n}\right) < \infty \tag{6.16}$$

with

$$\delta_n = 2^{n/2} \left(\sum_{l=v_k+1}^{v_{k+1}} \alpha_l^q\right)^{1/q}, \quad n \in \mathbb{Z}, \tag{6.17}$$

and v_k's defined by (6.5)*.*

Proof. Using Proposition 6.1, we see that (6.15) implies

$$\sup_{n\geq 1} n^\rho e_n(S_{a,b}) < \infty, \tag{6.18}$$

where, as before, $\beta_k^2 = t_{k+1} - t_k$ and $\rho = 1/\alpha + 1/2$. Let us first treat the case $t_k > 0$ for all $k \in \mathbb{Z}$ and $\sup_k t_k = \infty$. Then condition (6.14) implies

$$\sup_{k\in\mathbb{Z}} |\{k' : v_k = v_{k'}\}| < \infty \tag{6.19}$$

(in fact, here (6.14) and (6.19) are even equivalent), i.e., condition (2.12) is satisfied. Thus Theorem 2.5 applies and by (6.16) we obtain

$$\sup_{n\geq 1} n^{1/s} \left(\delta^*_{n-1} + \delta^*_{-n}\right) < \infty,$$

where

$$1/s = \rho - 1/2 + 1/q = 1/\alpha + 1/q .$$

This completes the proof in this case.

Next assume $t_k > 0$ and $\sup_k t_k < \infty$. Then by (6.14) there is $k_0 \in \mathbb{N}$ such that

$$\sup_{k\leq k_0} |\{k' : v_k = v_{k'}\}| < \infty .$$

Here Proposition 2.6 applies and completes the proof as before.

If we have $\sup_k t_k < \infty$ and $t_k = 0$ for $k \leq \tilde{k}$,

$$|\{n \in \mathbb{Z} : \delta_n \neq 0\}| < \infty ,$$

so there is nothing to prove.

Finally, if $t_k = 0$ for $k \leq \tilde{k}$ and $\sup_k t_k = \infty$, this time (6.14) yields

$$\sup_{k\geq k_0} |\{k' : v_k = v_{k'}\}| < \infty$$

for a certain $k_0 \in \mathbb{N}$, and we may proceed as in the preceding case. $\square$

Next we ask for conditions which ensure $-\log \mathbb{P}(\|X\|_q < \varepsilon) \succeq \varepsilon^{-\alpha}$ for some $\alpha > 0$.

Proposition 6.4. *Let $(t_k)_{k\in\mathbb{Z}}$ and X be as in Proposition* 6.3. *If for some $\alpha > 0$ and δ_n's defined by* (6.17)

$$\underline{\lim}_{n\to\infty} n^{1/\alpha+1/q}\left(\delta^*_{n-1}+\delta^*_{-n}\right) > 0\ ,$$

then necessarily

$$\overline{\lim}_{\varepsilon\to 0} \varepsilon^{\alpha} \log \mathbb{P}(\|X\|_q < \varepsilon) < 0\ .$$

Proof. This can be proved exactly as Proposition 6.3 by using Theorem 2.5, Proposition 2.6 and Proposition 6.1. □

Let us illustrate the preceding results by an example. We define random variables $(Y_k)_{k\geq 1}$ by

$$Y_k := \left(\frac{1}{2^k}\sum_{l=2^k}^{2^{k+1}-1}\left|\frac{W(l)}{l^{1/2}}\right|^q\right)^{1/q}, \quad q<\infty\,,$$

and

$$Y_k := \sup_{2^k\leq l<2^{k+1}}\left|\frac{W(l)}{l^{1/2}}\right|, \quad q=\infty\,.$$

Proposition 6.5. *Let $(\gamma_k)_{k\geq 1}$ be a sequence of nonnegative real numbers, and for $q\in[1,\infty]$, let $1/r = 1/2+1/q$.*

(i) *If $\|(\gamma_k)_{k\in\mathbb{N}}\|_r < \infty$, then $\lim\limits_{\varepsilon\to 0}\varepsilon^2 \log\mathbb{P}\Big(\sum\limits_{k=1}^{\infty}\gamma_k^q Y_k^q < \varepsilon^q\Big) = 0$.*

(ii) *For γ_k decreasing and $-\log\mathbb{P}\Big(\sum\limits_{k=1}^{\infty}\gamma_k^q Y_k^q < \varepsilon^q\Big) \preceq \varepsilon^{-\alpha}$ we necessarily have $\sup\limits_{k\geq 1} k^{1/\alpha+1/q}\gamma_k < \infty$.*

(iii) *Suppose that γ_k's are decreasing with $\gamma_k \succeq k^{-1/\alpha-1/q}$ for a certain $\alpha>0$. Then it follows $-\log\mathbb{P}\Big(\sum\limits_{k=1}^{\infty}\gamma_k^q Y_k^q < \varepsilon^q\Big) \succeq \varepsilon^{-\alpha}$.*

In all three cases, for $q=\infty$, one has to read $\mathbb{P}\Big(\sup\limits_{k\geq 1}\gamma_k Y_k < \varepsilon\Big)$ everywhere.

Proof. We set

$$\alpha_l = 2^{-k/q}\,\gamma_k\, l^{-1/2}\,, \quad 2^k\leq l<2^{k+1}$$

for any $k\geq 0$, and $\alpha_l = 0$ if $l<0$. Then it is easy to obtain

$$\|(\gamma_k Y_k)_{k\in\mathbb{Z}}\|_q = \|X\|_q \tag{6.20}$$

with $X = (X_l)_{l\in\mathbb{Z}} = (\alpha_l\, W(l))_{l\in\mathbb{Z}}$. Let b be given by $\beta_k = 1$ if $k\geq 0$, $\beta_k = 0$ otherwise. Then $t_l = l$ for $l\geq 0$, and $t_l = 0$ otherwise. For v_k's defined in (2.6) we have $v_k = 0$ for $k\leq 0$ and $v_k = 2^k-1$ for $k\geq 1$. Thus

$$\delta_k(a,b) = 2^{k/2-k/q}\gamma_k\left(\sum_{l=2^k}^{2^{k+1}-1} l^{-q/2}\right)^{1/q}$$

if $q < \infty$ and

$$\delta_k(a,b) = 2^{k/2}\gamma_k\Big(\sup_{2^k \le l < 2^{k+1}} l^{-1/2}\Big) = \gamma_k$$

for $q = \infty$. In any case we obtain

$$2^{-1/2}\gamma_k \le \delta_k(a,b) \le \gamma_k \ , \tag{6.21}$$

so that $(\gamma_k)_{k\in\mathbb{N}} \in \ell_r(\mathbb{N})$ iff $(\delta_k)_{k\in\mathbb{N}} \in \ell_r(\mathbb{N})$, and (i) follows from Theorem 6.2. Conversely, if γ_k's are decreasing, by (6.21) we have $\sup_{k\ge1} k^{1/\alpha+1/q}\gamma_k < \infty$ iff the same holds for δ_k^*'s. So (ii) is a consequence of Proposition 6.3 and (6.20). Similarly, assertion (iii) is proved. □

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