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TOWARDS AN INNOVATION THEORY OF SPATIAL BROWNIAN MOTION UNDER BOUNDARY CONDITIONS

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I am honored to dedicate this paper to Professor Nicholas Vakhania on the occasion of his 70th birthday.

Abstract. Set-parametric Brownian motion $\boldsymbol{b}$ in a star-shaped set G is considered when the values of $\boldsymbol{b}$ on the boundary of G are given. Under the conditional distribution given these boundary values the process $\boldsymbol{b}$ becomes some set-parametrics Gaussian process and not Brownian motion. We define the transformation of this Gaussian process into another Brownian motion which can be considered as "martingale part" of the conditional Brownian motion $\boldsymbol{b}$ and the transformation itself can be considered as Doob–Meyer decomposition of $\boldsymbol{b}$. Some other boundary conditions and, in particular, the case of conditional Brownian motion on the unit square given its values on the whole of its boundary are considered.

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1. Introduction. A Heuristic Description of the Problem

Let G be a bounded star-shaped set in $\mathbb{R}^d$ containing some open neighborhood of 0. Suppose $b(x), x \in \mathbb{R}^d$, is Brownian motion in $\mathbb{R}^m$ and consider its restriction to G. More precisely, let $\mathcal{B}_G$ denote σ-algebra of Borel measurable subsets of G. For every $C \in \mathcal{B}_G$, define $b(C) = \int_C b(dx)$ and call $\boldsymbol{b} = \{b(C), C \in \mathcal{B}_G\}$ *set-parametric Brownian motion* on G. In other words, $\boldsymbol{b}$ is the family of 0-mean Gaussian random variables with covariance function

$$Eb(C)b(C') = \mu(C \wedge C'), \quad C, C' \in \mathcal{B}_G,$$

where $\mu(\cdot)$ denotes d-dimensional Lebesgue measure. In particular, the variance of $b(C)$ is equal to $Eb^2(C) = \mu(C)$ while if C and C' are disjoint, then $b(C)$ and $b(C')$ are independent.

Suppose the value of $b(x)$ on the boundary Γ of G is given. The conditional distribution of the process $\{b(x), x \in G\}$ given $\{b(x), x \in \Gamma\}$ is very different from the distribution of Brownian motion and can be very complicated. What we would like to obtain is the transformation of $\{b(x), x \in G\}$ into some other

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process, say, $\{w(x), x \in G\}$ which under this conditional distribution still remains Brownian motion and which can "reasonably" be called an "innovation process" of $\{b(x), x \in G\}$ given $\{b(x), x \in \Gamma\}$. We believe, however, that the boundary conditions assume much more natural form for the set parametric Brownian motion $\boldsymbol{b}$ and below we will consider this problem for $\boldsymbol{b}$ rather than for $\{b(x), x \in G\}$.

The innovation problem as it was formulated, e.g., by Crameŕ in [2] consists in the following. Let H be the Hilbert space of all Gaussian random variables ξ given on some probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$ such that $E\xi = 0$ and $E\xi^2 < \infty$ with an inner product in H defined as $\langle \xi, \xi' \rangle = E\xi\xi'$. Let $\{\xi_t, t \in [0,1]\}$ be a family of random variables from H with $\xi_0 = 0$, which can be regarded as a Gaussian process in t or, alternatively, as a curve in H. (In [2] random variables ξ are not assumed Gaussian but only of finite variances and t ranges from $-\infty$ to ∞, but these are insignificant changes as far as the aims of the present paper are concerned.) With each t we can associate the linear subspace $H_t = L\{\xi_s, 0 \le s \le t\}$ so that H_1 is the smallest linear subspace of H which contains the curve $\{\xi_t, t \in [0,1]\}$. Here we have denoted by $L\{\dots\}$ the linear space of random variables shown in the curly brackets. Assume that ξ_t is a.s. continuous in t (and $\{\Omega, \mathcal{F}, \mathbb{P}\}$ is rich enough to support it) and denote π_t projection operator in H which projects on H_t. Since ξ_t is continuous, *the chain* of orthoprojectors $\{\pi_t, 0 \le t \le 1\}$ is continuous resolution of unity, that is, it is continuous and increasing family of projectors, $\pi_t \pi_{t+\triangle} = \pi_t$, and such that π_0 is the null operator and π_1 is the identity operator on H_1. For any random variable $\zeta \in H$ the process

$$w_t = \pi_t \zeta, \quad 0 \le t \le 1, \tag{1.1}$$

is a Gaussian one with independent increments,

$$Ew_t(w_{t+\triangle} - w_t) = \langle \pi_t \zeta, (\pi_{t+\triangle} - \pi_t)\zeta \rangle = \langle \zeta, \pi_t(\pi_{t+\triangle} - \pi_t)\zeta \rangle = 0.$$

With this process we can associate its own continuous chain $\{H_t^w, 0 \le t \le 1\}$. It is obvious that $H_t^w \subseteq H_t$, but the question is whether we can find thus constructed process $\{w_t, t \in [0,1]\}$ such that $H_t^w = H_t$ for all $t \in [0,1]$. This process is then called an innovation process of the process $\{\xi_t, t \in [0,1]\}$.

Alternative formulation of this problem is well known in the theory of semimartingales (see, e.g., [9, vol. 1], vol. 1). For a not necessarily Gaussian process $\{\xi_t, 0 \le t \le 1\}$, let $\mathcal{F}_t^\xi = \sigma\{\xi_s, 0 \le s \le t\}$ denote sub-σ-algebra of $\mathcal{F}$ generated by random variables $\xi_s, s \le t$. The family $\mathcal{F}^\xi = \{\mathcal{F}_t^\xi, 0 \le t \le 1\}$ is called *natural filtration* of the process $\{\xi_t, t \in [0,1]\}$. Suppose the conditional expectation value $E[\xi_{t+\triangle} | \mathcal{F}_t^\xi]$, for all $t \in [0,1]$, is absolutely continuous in Δ at $\Delta = 0$ with derivative $\alpha(t, \xi)$, that is,

$$E[\xi_{dt} | \mathcal{F}_t^\xi] = \alpha(t, \xi)dt. \tag{1.2}$$

Then the process

$$w_{dt} = \xi_{dt} - \alpha(t, \xi)dt \tag{1.3}$$

is martingale with respect to the filtration $\mathcal{F}^{\xi}$. If $\{\xi_t, t \in [0,1]\}$ is continuous and Gaussian, then $\alpha(t,\xi)$ is linear functional of $\xi_s, s \le t$, that is, $\alpha(t,\xi) \in H_t$ and the process $\{w_t, t \in [0,1]\}$ is Brownian motion. It is obvious that all w_s for $s \le t$ are $\mathcal{F}_t^{\xi}$ measurable, but it may be that $\mathcal{F}_t^w \subseteq \mathcal{F}_t^{\xi}$ with the proper inclusion only, where $\mathcal{F}_t^w = \sigma\{w(s), s \le t\}$. If, however, $\mathcal{F}_t^w = \mathcal{F}_t^{\xi}$ for all $t \in [0,1]$, then $\{w_t, t \in [0,1]\}$ is called innovation Brownian motion for ξ.

Compared to our initial problem pertaining to $\boldsymbol{b}$ the above recollection of the innovation problem raises the following questions:

1. There is no one-dimensional time associated with $\{b(x), x \in G\}$ and even the less so with $\boldsymbol{b}$, which was essential for the definition and use of the chains $\{H_t, 0 \le t \le 1\}$ and $\{H_t^w, 0 \le t \le 1\}$ and filtrations $\mathcal{F}^{\xi}$ and $\mathcal{F}^w$,
2. Even if we introduce an one-dimensional, and hence linearly ordered, time, why considerations similar to (1.1) or (1.2) with respect to this time would lead to anything meaningful for $b(x)$ in $x \in G$ or for $b(C)$ in $C \in \mathcal{B}_G$? That is, why innovations we may construct in t will be Brownian motions in C?

These are very clear and natural concerns and indeed in probability theory serious efforts were spent in different directions on the theory of semimartingales with "truly" multidimensional time – see, e.g., the well known papers Cairoli and Walsh [1], Wong and Zakai [10], Hajek and Wong [4] and Hajek [3] among others.

However, parallel to this development, in the papers of Khmaladze ([5], [6], [7]) alternative approach to the innovation theory for processes with multidimensional time was developed. Its basic idea can be outlined as follows. Let $\{\eta(z), z \in \mathcal{Z}\}$ be a random process indexed by the parameter z from some linear space $\mathcal{Z}$ (or a subset of this space). Let also $\{\pi_t, 0 \le t \le 1\}$ be continuous resolution of unity acting in $\mathcal{Z}$, and, using it, construct the process $\boldsymbol{\eta_\pi} = \{\eta(\pi_t z), z \in \mathcal{Z}, t \in [0,1]\}$ and let $\mathcal{F}_t^{\eta} = \sigma\{\eta(\pi_t z), z \in \mathcal{Z}\}$. Then the process $\boldsymbol{w_\pi} = \{w_t(z), z \in \mathcal{Z}, t \in [0,1]\}$ defined as

$$w_{dt}(z) = \eta(\pi_{dt} z) - E[\eta(\pi_{dt} z) | \mathcal{F}_t^{\eta}] \tag{1.4}$$

"usually" becomes Brownian motion not only in t but also in z. It also is in one-to-one correspondence with $\boldsymbol{\eta_\pi}$ and, for this reason, can be viewed as an innovation process of $\{\eta(z), z \in \mathcal{Z}\}$ with respect to $\mathcal{F}_\pi^{\eta} = \{\mathcal{F}_t^{\eta}, 0 \le t \le 1\}$.

In our present context of Brownian motion, conditional on the boundary, we recall that, as it follows from [7], this is exactly the case for Brownian motion $\{b(x), x \in [0,1]^d\}$ given $b(\mathbf{1}_d)$, where $\mathbf{1}_d = (1, \ldots, 1) \in \mathbb{R}^d$. It also follows that the innovation process for this conditional process is the same as the innovation process for the Brownian bridge $\{v(x), x \in [0,1]^d\}$ defined as $v(x) = b(x) - \mu([0,x]) b(\mathbf{1}_d)$. It is well known that the latter process is the limit in distribution of the uniform empirical process on $[0,1]^d$ and appears in large number of statistical problems. As it follows from [8], innovation process (1.3) can be constructed for conditional Brownian motion $\{b(x), x \in [0,1]^d\}$ under the condition $\{b(\mathbf{1}_k, x''), x'' \in [0,1]^{d-k}\}, 1 \le k < d$, which is the same as the innovation for the Kiefer process $\{u(x), x \in [0,1]^d\}$, where $u(x) = b(x) - \mu([0,x'] \times [0,1]^{d-k}) b(\mathbf{1}_k, x''), x = (x', x''), x' \in [0,1]^k$. The Kiefer process, say, for $k = d-1$ is the limit in distribution of the partial sum process $u_n(x', x'') = n^{-1/2} \sum_{i \le nx''} [I_{\{U_i \le x'\}} - \mu([0,(x',1)])$ based on independent uniform random variables $\{U_i\}_{i=1}^n$ in $[0,1]^{d-1}$ and appears in large number of sequential problems of statistics. Intuitively speaking, although in the latter case the distribution of the process $\{b(x), x \in [0,1]^d\}$ has essentially higher order of degeneracy, only under the condition $b(\mathbf{1}_d)$, in x'' it is still free and remains Brownian motion. Our theorem of the next section shows that this "freedom" in "some direction" is not necessary and the approach can go through for the process conditioned over the whole boundary.

2. Exact Formulation and the Basic Result

Let $\{A_t, 0 \le t \le 1\}$ be any collection of Borel subsets of G such that $0 \subseteq A_t \subseteq A_{t'}$ for $t \le t'$, $\mu(A_0) = 0, \mu(A_1) = \mu(G)$ and $\mu(A_t)$ is absolutely continuous in t with positive density. We call such family of subsets *a scanning family.* One natural scanning family for star-shaped G can be constructed as follows. Let $\mathbb{S}^{d-1}$ be unit sphere in $\mathbb{R}^d$, and for $x \in \mathbb{S}^{d-1}$ let $\rho(x,G) = \max\{t : tx \in G\}$ be the so-called radial function of G. Then take as A_t the set $A_t' = \{y : y = \tau\rho(x,G)x,\ 0 \le \tau \le t\}$.

For $x \in \mathbb{S}^{d-1}$ let $\alpha(x)$ be the ray which passes through x and for a subset $S \subseteq \mathbb{S}^{d-1}$ let $\alpha(S)$ be the solid angle passing through S so that $\alpha(dx)$ is solid angle passing through differential $dx \subset \mathbb{S}^{d-1}$. Most of the time we will use the notation $A_{t,S} = A_t \cap \alpha(S)$, $A_{dt} = A_{t+dt} \setminus A_t, A_{dt,dx} = A_{dt} \cap \alpha(dx)$, etc. Also denote $\Gamma_S = G \cap \alpha(S), \Gamma_{dx} = G \cap \alpha(dx)$, etc., even though these are not subsets of the boundary Γ. The measure $\mu(A_{t,S})$ is absolutely continuous with respect to $\mu(\Gamma_S)$ and the Radon–Nikodým derivative

$$F(t|x) = \frac{\mu(A_{t,dx})}{\mu(\Gamma_{dx})}$$

is absolutely continuous distribution function in t. Let us define now the process on $\mathcal{B}_G$ as

$$\begin{aligned} v(C) &= \int \mathbf{1}_{\{tx \in C\}} [b(A_{dt,dx}) - F(dt|x) b(\Gamma_{dx})] \\ &= b(C) - \int\limits_{\mathbb{S}^{d-1}} F(C|x) b(\Gamma_{dx}). \end{aligned} \tag{2.1}$$

Call $\boldsymbol{v} = \{v(C), C \in \mathcal{B}_G\}$ the *total Brownian bridge* on G. It follows from the definition of $\boldsymbol{v}$ that $v(\Gamma_S) = 0$ for any $S \subseteq \mathbb{S}^{d-1}$ and in this sense we say that $\boldsymbol{v}$ is 0 on the boundary Γ.

Remark. With the choice of a scanning family as $\{A_t' 0 \le t \le 1\}$ and/or defining the process on $[0,1]\times\mathbb{S}^{d-1}$ as

$$u(t,S)=b(A_{t,S})-\int\limits_{x\in S}F(t|x)b(\Gamma_{dx}),$$

we could reduce our considerations to the unit ball $\mathbb{B}^d$. We, however, will stay with general G and the general scanning family.

Consider now Brownian motion $\boldsymbol{b}$ and its conditional distribution given $\{b(\Gamma_S),\ S\subseteq\mathbb{S}^{d-1}\}$. Obviously under this conditional distribution the process $\boldsymbol{b}$ is not Brownian motion any more and Theorem 1 below describes $\boldsymbol{b}$ as the sum of new set-parametric Brownian motion and another "smooth" set-parametric process. In our view, this representation can be considered as a generalisation of Doob–Meyer decomposition for $\boldsymbol{b}$ given $\{b(\Gamma_S),S\subseteq\mathbb{S}^{d-1}\}$ and, as we will see below, also as that for the total Brownian bridge.

It is interesting and, for some applications important, to consider boundary condition of a somewhat more general form. Namely, we want to be able to assume that $b(\Gamma_S)$ may not be known on the total $\mathbb{S}^{d-1}$ but only on its subset S_0 and also not on the whole σ-algebra of Borel measurable subsets of $\mathbb{S}^{d-1}$ but on a "rougher" σ-algebra which may contain "blobs" or "splashes" on $\mathbb{S}^{d-1}$ as atoms. Therefore introduce σ-algebra $\mathcal{S}$ generated by *some* collection of Borel measurable subsets of some S_0. For example, $\mathcal{S}$ may consist only of S_0 and ϕ. Concerning S_0, we assume that $\mu_{d-1}(S_0)>0$, where $\mu_{d-1}(\cdot)$ is an area measure on $\mathbb{S}^{d-1}$. Now consider $\{b(\Gamma_S),S\in\mathcal{S}\}$ as the boundary condition. As we have said, we will write down what seems to us the Doob–Meyer decomposition of $\boldsymbol{b}$ under the condition $\{b(\Gamma_S),S\in\mathcal{S}\}$. However, let us first define Brownian bridge similar to (2.1).

The measure $\mu(C_S),S\in\mathcal{S}$, is absolutely continuous with respect to the measure $\mu(\Gamma_S)$, $S\in\mathcal{S}$, and corresponding Radon-Nikodým derivative

$$F(C|s)=\frac{\mu(C_{ds})}{\mu(\Gamma_{ds})},$$

where s is an atom of $\mathcal{S}$, is absolutely continuous distribution in C. For any set $C\in\mathcal{B}_G$ define

$$v(C)=b(C)-\int\limits_{S_0}F(C|s)b(\Gamma_{ds}). \tag{2.2}$$

The process $v_S=\{v(C),C\in\mathcal{S}\}$ can be naturally called the $\mathcal{S}$-Brownian bridge. It is convenient to list some of its properties:

a) for $S\in\mathcal{S}$, $v(\Gamma_S)=0$,
b) if $C\cap\alpha(S_0))=\phi$, then $v(C)=b(C)$,
c) $v_{\mathcal{S}}$ and $\{b(\Gamma_S),S\in\mathcal{S}\}$ are independent processes,
d) the covariance function of $v_{\mathcal{S}}$ is

$$Ev(C)v(D)=\mu(C\wedge D)-\int\limits_{S_0}F(C|s)F(D|s)\mu(\Gamma_{ds}).$$

Now we formulate our theorem. Let

$$\mathcal{F}_t = \sigma\{b(C \cap A_t), C \in \mathcal{B}_G\} \vee \sigma\{b(\Gamma_S), S \in \mathcal{S}\}$$

be σ-algebra generated by $\boldsymbol{b}$ on the subsets of A_t and by the boundary condition. It is clear that $\mathcal{F}_0 = \sigma\{b(\Gamma_S), S \in \mathcal{S}\}$. The filtration $\{\mathcal{F}_t, 0 \le t \le 1\}$ describes the "history" of $\boldsymbol{b}$ up to the moment t under the boundary conditions. For any $C \in \mathcal{B}_G$ and $t \in [0,1]$, define

$$w(C,t) = b(C \cap A_t) - \int_0^t \int_{S_0} \frac{b(\Gamma_{ds}) - b(A_{\tau,ds})}{1 - F(A_\tau|s)} F(C \cap A_{d\tau}|s). \tag{2.3}$$

Theorem 1. *The process on* $[0,1] \times \mathcal{B}_G$ *defined by* (2.3) *has the following properties:*

(1) *For any* $t \in [0,1]$, *the process in* $C, \boldsymbol{w}_t = \{w(C,t), C \in \mathcal{B}_G\}$ *is* $\mathcal{F}_t$-*measurable.*

(2) *For any* $C \in \mathcal{B}_G$, $\quad w(C,t) = w(C \cap A_t, 1)$.

(3) *Under the conditional distribution of* $\boldsymbol{b}$ *given* $\{b(\Gamma_S), S \in \mathcal{S}\}$ *the process* $\boldsymbol{w} \equiv \boldsymbol{w}_1$ *is Brownian motion on* $\mathcal{B}_G$.

(4) *Given* $\{b(\Gamma_S), S \in \mathcal{S}\}$, *there is one-to-one correspondence between the processes* $\boldsymbol{w}$ *and* $\boldsymbol{b}$. *Moreover, given* $\{b(\Gamma_S), S \in \mathcal{S}\}$, *there is one-to-one correspondence between* $\boldsymbol{w}_t$ *and the restriction* $\boldsymbol{b}_t$ *of* $\boldsymbol{b}$ *on* A_t.

The following corollary describes the process $\boldsymbol{w}_t$ in a form analogous to (1.1).

Corollary 1. *Define the set-parametric process* $\boldsymbol{\zeta} = \{\zeta(C), C \in \mathcal{B}_G\}$ *as*

$$\zeta(C) = b(C) - \int_0^1 \int_{S_0} \frac{b(\Gamma_{ds}) - b(A_{\tau,ds})}{1 - F(A_\tau|s)} F(C \cap A_{d\tau}|s).$$

Then

$$\boldsymbol{w}_t = E[\boldsymbol{\zeta}|\mathcal{F}_t].$$

The next corollary reformulates (2.3) for the function parametric Brownian motion in $L_2(G)$. Denote $b(f) = \int_0^1 \int_{S_0} f(tx) b(A_{dt,ds})$, and let $w(f)$ be defined similarly. The family $\mathbf{b} = \{b(f), f \in L_2(G)\}$ of 0-mean Gaussian random variables with covariance $Eb(f)b(h) = \langle f, h \rangle$ is called (function-parametric) Brownian motion on $L_2(G)$. Denote by $\mathcal{V}$ a linear operator in $L_2(G)$:

$$\mathcal{V}f(tx) = \int_0^t \frac{f_S(t'x)}{1 - F(t'|s)} F(dt'|s),$$

where f_S is conditional expectation of the function f given the σ-algebra $\{\Gamma_S, S \in \mathcal{S}\}$.

Corollary 2. *The process on* $L_2(G)$ *defined as*

$$w(f) = b(f) - b(\mathcal{V}f)$$

is Brownian motion on $L_2(G)$*. This process and the function-parametcir process* $\mathbf{b}_\Gamma = \{\int_{S_0} g(s) b(\Gamma_{ds}\}$ *are independent.*

Proof of Theorem 1. Property (1) is true because $b(\Gamma_{ds})$ is $\mathcal{F}_t$-measurable and so is $b(A_{\tau,ds})$ for $\tau \leq t$.

Property (2) follows from the fact that $F(C \wedge A_t \wedge A_{d\tau}|s) = 0$ for $\tau \geq t$ and hence, the integration over τ in $w(C \wedge A_t, 1)$ will stop at t.

To prove property (3) we notice that $w(C,t)$ can be rewritten as the transformation of $v_{\mathcal{S}}$ only:

$$w(C,t) = v(C \wedge A_t) + \int_0^t \int_{S_0} \frac{v(A_{\tau,ds})}{1 - F(A_\tau|s)} F(C \cap A_{d\tau}|s). \tag{2.4}$$

Since the distribution of $v_{\mathcal{S}}$ does not depend on $\{b(\Gamma_S), S \in \mathcal{S}\}$ (see (c) above), the distribution of $\boldsymbol{w}$ under conditional distribution of $\boldsymbol{b}$ given $\mathcal{F}_0$ does not depend on $\mathcal{F}_0$ either and, hence, coincides with its unconditional distribution. Therefore we can derive it from (2.3) assuming $\boldsymbol{b}$ is Brownian motion. Being linear transformation of $\boldsymbol{b}$ the process $\boldsymbol{w}$ is certainly Gaussian, and we only need to prove that

$$Ew(C,1)w(D,1) = \mu(C \wedge D). \tag{2.5}$$

Then property 2) insures that for $w(C,t)$ and $w(D,t')$, $t \leq t'$, we will have

$$Ew(C,t) \cdot w(D,t') = \mu(C \cap D \cap A_t).$$

To prove (2.5) we need to show that

$$\begin{aligned} I_1 &= Eb(D) \int_0^1 \int_{S_0} \frac{b(\Gamma_{ds}) - b(A_{\tau,ds})}{1 - F(A_\tau|s)} F(C \cap A_{d\tau}|s) \\ &= \int_0^1 \int_{S_0} \frac{\mu(D_{ds}) - \mu(D_{ds} \cap A_\tau)}{1 - F(A_\tau|s)} F(C \cap A_{d\tau}|s) \\ &= \int_0^1 \int_{S_0} \frac{F(D|s) - F(D \cap A_\tau|s)}{1 - F(A_\tau|s)} F(C \cap A_{d\tau}|s) \mu(\Gamma_{ds}) \end{aligned} \tag{2.6}$$

plus similar integral, denoted by I_2, with D and C interchanging places, is equal to

$$\int_0^1\int_0^1\int_{S_0}\int_{S_0} E\frac{b(\Gamma_{ds})-b(A_{\tau,ds})}{1-F(A_\tau|s)}\cdot\frac{b(\Gamma_{ds'})-b(A_{\tau'ds'})}{1-F(A_{\tau'}|s')}F(C\cap A_{d\tau}|s)F(D\cap A_{d\tau'}|s')$$

$$=\int_0^1\int_0^1\int_{S_0}\frac{\mu(\Gamma_{ds})-\mu(A_{\tau\vee\tau',ds})}{[1-F(A_\tau|s)][1-F(A_{\tau'}|s)]}F(C\cap A_{d\tau}|s)F(D\cap A_{d\tau'}|s)$$

$$=\int_0^1\int_0^1\int_{S_0}\frac{1-F(A_{\tau\vee\tau'}|s)}{[1-F(A_\tau|s)][1-F(A_{\tau'}|s)]}F(C\cap A_{d\tau}|s)F(D\cap A_{d\tau'}|s)\mu(\Gamma_{ds}).$$

However, integration over $\tau'\geq\tau$ leads to

$$\int_0^1\int_{S_0}\frac{1}{1-F(A_\tau|s)}F(C\cap A_{d\tau}|s)\int_\tau^1 F(D\cap A_{d\tau'}|s)\cdot\mu(\Gamma_{ds})$$

$$=\int_0^1\int_{s_0}\frac{F(D|s)-F(D\cap A_\tau|s)}{1-F(A_\tau|s)}F(C\cap A_{d\tau}|s)\mu(\Gamma_{ds})$$

which is equal to (2.6). Similarly, integration over $\tau'<\tau$ leads to I_2 and hence (3) is proved. We only remark that changes of order of integration which we used can be easily justified using a simple truncation technique in the neighborhood of 1 (cf. [7], Theorem 3.13) and thus we skip it here.

To prove property (4), we need to show that $\boldsymbol{v}$ can be expressed through $\boldsymbol{w}$ in a unique way. Consider $w(C)\equiv w(C,1)$ with $C=A_{t,ds}$. We get

$$w(A_{t,ds})=v(A_{t,ds})+\int_0^t\frac{v(A_{\tau,ds})}{1-F(A_\tau|s)}F(A_{d\tau}|s)$$

because $F(A_{t,ds}\cap A_{d\tau}|s')=F(A_{d\tau,ds}|s)\cdot I(s=s',\tau\leq t)$. The above equation has a unique solution

$$v(A_{t,ds})=[1-F(A_t|s)]\int_0^t\frac{w(A_{d\tau,ds})}{1-F(A_\tau|s)}$$

which being substituted in (2.4) with $t=1$ leads to

$$v(C)=w(C)-\int_0^1\int_0^t\int_{S_0}\frac{w(A_{d\tau,ds})}{1-F(A_\tau|s)}F(A_{dt}|s).\quad\square$$

The reader may already have noticed that what was essential for Theorem 1 to hold was the structure of the boundary condition. It was essential that the sets in the σ-algebra $\mathcal{S}$ were cylindrical with respect to t. If we think about our Brownian motion restricted to G as a process which "evolves from 0 within G" and if we consider it as a random measure rather than a random function in x, then it may not, perhaps, be so natural to associate with a point of the boundary the value of our measure on the rectangle $[0,x]$ and to speak about boundary condition $\{b(x), x \in \Gamma\}$. The boundary condition $\{b(\Gamma_S), S \in \mathcal{S}\}$ we have used can, perhaps, be thought of as more natural and adequate. It is natural to have Brownian motion on the boundary Γ as the boundary condition, which the latter is but the former is not.

Another feature of conditioning on the boundary is well illustrated by the following example. Take $d = 2$ for simplicity and consider Brownian motion on the square $[0,1]^2$. Consider two sets of boundary conditions $\{b(1,\tau), \tau \in [0,1]\}$ and $\{b(t,1), t \in [0,1]\}$. For each of these conditions we can easily obtain scanning family and construct corresponding innovation process for Brownian motion thus conditioned. In particular, we can choose $A_t^{(1)} = \{(t',\tau') : t' \leq t\}$ and consider σ-algebras $\mathcal{F}_t^{(1)} = \sigma\{b(t',\tau'), (t',\tau') \in A_t^{(1)}\} \vee \sigma\{b(1,\tau), \tau \in [0,1]\}$, which will lead to an innovation process of the form

$$w^{(1)}(t,\tau) = b(t,\tau) - \int_0^t \frac{b(1,\tau) - b(t',\tau)}{1-t'} dt', \quad (t,\tau) \in [0,1]^2.$$

(This is also the innovation process for the Kiefer process defined as $v^{(1)}(t,\tau) = b(t,\tau) - tb(1,\tau)$, $(t,\tau) \in [0,1]^2$.) Similarly, in the second case we can choose $A_t^{(2)} = \{(t',\tau') : \tau' \leq \tau\}$ and derive corresponding $w^{(2)}$. However, it is interesting to consider Brownian motion $\{b(t,\tau), (t,\tau) \in [0,1]^2\}$ under both conditions simultaneously. In this case we do not have one scanning family which will serve the purpose, but we can apply Theorem 1 twice consecutively. What can be said then is this:

Theorem 2. *Let the process* $\mathbf{w}_{[0,1]^2} = \{w(t,\tau), (t,\tau) \in [0,1]^2\}$ *be defined by either of the equations:*

$$\begin{aligned} w(dt,d\tau) &= w^{(1)}(dt,d\tau) - E[w^{(1)}(dt,d\tau) | \mathcal{F}_\tau^{(2)}] \\ &= w^{(2)}(dt,d\tau) - E[w^{(2)}(dt,d\tau) | \mathcal{F}_t^{(1)}] \end{aligned}$$

or, in explicit form

$$\begin{aligned} w(t,\tau) = b(t,\tau) &- \int_0^t \frac{b(1,\tau) - b(t',\tau)}{1-t'} dt' - \int_0^\tau \frac{b(t,1) - b(t,\tau')}{1-\tau'} d\tau' \\ &+ \int_0^t \int_0^\tau \frac{b(1,1) - b(t',1) - b(1,\tau') + b(t'\tau'}{(1-t')(1-\tau')} dt' d\tau'. \end{aligned}$$

Then:

(1) *The distribution of this process under the boundary condition* $\{b(1,\tau), \tau \in [0,1]\}$ *and* $\{b(t,1), t \in [0,1]\}$ *does not depend on this condition and is the distribution of Brownian bridge on* $[0,1]^2$.

(2) *There is one-to-one correspondence between processes* $\mathbf{w}_{[0,1]^2}$ *and* $\mathbf{v} = \{v(t,\tau)\}$, *where*

$$v(t,\tau) = b(t,\tau) - tb(1,\tau) - \tau b(t,1) + t\tau b(1,1)$$

is the projection of $\{b(t,\tau),(t,\tau) \in [0,1]^2$ *on the boundary of* $[0,1]^2$.

Let us leave the proof of this theorem to the reader.

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