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BOUNDARY INTEGRAL EQUATIONS OF PLANE ELASTICITY IN DOMAINS WITH PEAKS

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In memoriam N. I. Muskhelishvili

Abstract. Boundary integral equations of elasticity theory in a plane domain with a peak at the boundary are considered. Solvability and uniqueness theorems as well as results on the asymptotic behaviour of solutions near the peak are obtained.

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1. Introduction

The theory of elastic potentials for domains with smooth boundaries is well developed (see the monographs [6], [17]). For domains with piecewise smooth boundaries "without zero angles" theorems on the unique solvability of integral equations of elasticity were obtained in [7] by a method which does not use Fredholm and singular integral operators theories. Solutions of integral equations are expressed by the inverse operators of auxiliary exterior and interior boundary value problems, i.e., theorems on the solvability of boundary integral equations follow from the theory of elliptic boundary value problems in domains with piecewise smooth boundaries.

We apply the same approach to integral equations of the plane elasticity theory on a contour with a peak. We also use the complex form of solutions to the elasticity equations suggested by G. V. Kolosov. This method was further developed by N. I. Muskhelishvili (see [16]).

Since even for smooth functions in the right-hand side these integral equations, in general, have no solutions in the class of summable functions, we study modified integral equations for which theorems on the unique solvability prove to be valid.

Using the same method we obtained (see [9]–[11]) asymptotic formulas for solutions of integral equations of the logarithmic potential theory near cusps on boundary curves. This approach permitted us also to find, for each integral equation, a pair of weighted L_p-spaces such that the corresponding integral operator maps one space onto another (see [12]–[15]).

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In the recent articles [1], [2], criteria of solvability in weighted L_p-spaces of boundary integral equations of the logarithmic potential theory on contours with peaks were obtained. The method used in these papers is based on reducing of boundary value problems to the Riemann–Hilbert problem for analytic functions on the unit circumference.

Here we give a brief description of the results obtained in the present paper.

Let Ω be a plane simply connected domain bounded by a closed piecewise smooth curve S with a peak at the origin O. Suppose that either Ω or its complement Ω^c is described in the Cartesian coordinates x, y near O by the inequalities $\kappa_-(x) < y < \kappa_+(x),\ 0 < x < \delta$, where $\kappa_\pm$ are C^∞-functions on $[0, \delta]$ satisfying

$$\kappa_\pm(0) = \kappa'_\pm(0) = 0 \quad \text{and} \quad \kappa''_+(0) > \kappa''_-(0).$$

In the first case we say that O is an outward peak and in the second one O is an inward peak.

We introduce the class $\mathfrak{N}_\nu$ $(\nu > -1)$ of infinitely differentiable on $S \backslash \{O\}$ vector-valued functions h admitting representations $h_\pm(x) = x^\nu q_\pm(x)$ on the arcs $S_\pm = \{(x, \kappa_\pm(x)) : \ x \in (0, \delta]\}$, where the vector-valued functions $q_\pm$ belong to $C^\infty[0, \delta]$ and satisfy $|q_+(0)| + |q_-(0)| \neq 0$. Let $\mathfrak{N}$ denote the set

$$\mathfrak{N} = \bigcup_{\nu > -1} \mathfrak{N}_\nu,$$

and let $\mathfrak{M}_\beta$ $(\beta > -1)$ be the class of differentiable vector-valued functions on $S \backslash \{O\}$ satisfying

$$\sigma^{(r)}(z) = O(x^{\beta - r}), \ \ z = x + iy = (x, y), \ \ r = 0, 1.$$

We introduce the class $\mathfrak{M}$ as

$$\mathfrak{M} = \bigcup_{\beta > -1} \mathfrak{M}_\beta.$$

For domains with an outward peak we put

$$\mathfrak{M}_{ext} = \bigcup_{\beta > -1/2} \mathfrak{M}_\beta \, .$$

We consider the interior and exterior first boundary value problems

$$\triangle^* u = \mu \triangle u + (\lambda + \mu) \nabla \mathrm{div}\, u = 0 \ \text{ in } \ \Omega, \ \ u = g \ \text{ on } \ S \,, \tag{$\mathcal{D}^+$}$$

$$\triangle^* u = 0 \ \text{ in } \ \Omega^c, \ \ u = g \ \text{ on } \ S \,, \ \ u(z) = O(1) \ \text{ as } \ |z| \to \infty \,, \tag{$\mathcal{D}^-$}$$

and the interior and exterior second boundary value problems

$$\triangle^* u = 0 \ \text{ in } \ \Omega, \ \ Tu = h \ \text{ on } \ S \,, \tag{$\mathcal{N}^+$}$$

$$\triangle^* u = 0 \ \text{ in } \ \Omega^c, \ \ Tu = h \ \text{ on } \ S \,, \ \ u(z) = o(1) \ \text{ as } \ |z| \to \infty \,, \tag{$\mathcal{N}^-$}$$

for the displacement $u = (u_1, u_2)$. Here $T(\partial_\zeta, n_\zeta)$ is the traction operator

$$T(\partial_\zeta, n_\zeta)\, u = 2\mu \partial u / \partial n + \lambda n \ \mathrm{div}\ u + \mu [n, \ \mathrm{rot}\ u],$$

where $n = (n_\xi, n_\eta)$ is the outward normal to the boundary S at the point $\zeta = (\xi, \eta)$, and λ, μ are the Lamé coefficients. Henceforth we shall not distinguish a displacement $u = (u_1, u_2)$ and a complex displacement $u = u_1 + iu_2$.

A classical method for solving the first and second boundary value problems of elasticity theory consists in representing their solutions in the form of the double-layer potential

$$W\sigma(z) = \int_S \{T(\partial_\zeta, n_\zeta)\, \Gamma\, (z,\zeta)\}^* \sigma(\zeta) ds_\zeta\,,$$

and the simple-layer potential

$$V\tau(z) = \int_S \Gamma(z,\zeta)\tau(\zeta) ds_\zeta\,, \quad z = (x,y) \in\ \Omega \ \text{ or } \ \Omega^c\,,$$

where * denotes the passage to the transposed matrix and Γ is the Kelvin–Somigliana tensor

$$\Gamma(z,\zeta) = \frac{\lambda+3\mu}{4\pi\mu(\lambda+2\mu)} \left\{ \log \frac{1}{|z-\zeta|} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\lambda+\mu}{\lambda+3\mu} \frac{1}{|z-\zeta|^2} \begin{pmatrix} (x-\xi)^2 & (x-\xi)(y-\eta) \\ (x-\xi)(y-\eta) & (y-\eta)^2 \end{pmatrix} \right\}.$$

For the problems $\mathcal{D}^+$ and $\mathcal{N}^-$ the densities of the corresponding potentials can be found from the systems of boundary integral equations

$$-2^{-1}\sigma + W\sigma = g \tag{1}$$

and

$$-2^{-1}\tau + TV\tau = h\,. \tag{2}$$

Under certain general conditions on g in (1) there exist solutions u^+ and u^- of the problems $\mathcal{D}^+$ and $\mathcal{D}^-$ in Ω and Ω^c with the boundary data g satisfying

$$g(z) = \lim_{\varepsilon\to 0} \int_{\{S:|\zeta|>\varepsilon\}} \Gamma(z,\zeta) \Big(T(\partial_\zeta, n_\zeta)u^+(\zeta) - T(\partial_\zeta, n_\zeta)u^-(\zeta) \Big) ds_\zeta + u^-(\infty) \tag{3}$$

on $S \setminus \{O\}$. Let v^- denote a solution of $\mathcal{N}^-$ in Ω^c, vanishing at infinity, with the boundary data $T\,u^+$ on $S \setminus \{O\}$. We can choose v^- so that, for $w = v^- - u^- + u^-(\infty)$ on $z \in S \setminus \{O\}$, the equality

$$w(z) - 2\lim_{\varepsilon\to 0} \int_{\{S:|\zeta|>\varepsilon\}} \{T(\partial_\zeta, n_\zeta)\, \Gamma\, (z,\zeta)\}^* w(\zeta) ds_\zeta = -2\varphi(z) + 2u^-(\infty) \tag{4}$$

holds. Solutions of equations (1) and (2) are constructed by means of (3) and (4). So, the function

$$\sigma = v^- - g$$

is a solution of (1). A solution of (2) can be obtained as follows. Let us introduce the solution v^- of $\mathcal{N}^-$ in Ω^c with the boundary data h, vanishing at infinity, and the solution u^+ of $\mathcal{D}^+$ in Ω equal to v^- on $S \setminus \{O\}$. Under sufficiently general assumptions on h we can select v^- and u^+ so that the density

$$\tau = T\,u^+ - h$$

satisfies (2).

Inward peak. In fact, the integral equation (1), in general, has no solutions in $\mathfrak{M}$ even if $g \in \mathfrak{N}$ vanishes on $S_\pm$. However, for a function from $\mathfrak{N}_\nu$ with $\nu > 3$ the solvability of (1) can be attained by changing the equation in the following way. A solution u of the problem $\mathcal{D}^+$ is sought as the sum of the double-layer potential with density σ and the linear combination of explicitly given functions $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ with unknown real coefficients

$$u(z) = W\sigma(z) + c_1\mathcal{A}_1(z) + c_2\mathcal{A}_2(z) + c_3\mathcal{A}_3(z).$$

The functions $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ are given by

$$\begin{aligned}\mathcal{A}_1(z) &= \frac{i}{2\mu}[2\kappa \operatorname{Im} z^{1/2} - \overline{z^{-1/2}} \operatorname{Im} z] + \\ &+ i\frac{(\kappa-1)(\alpha_+ - \alpha_-)}{8\pi\kappa\mu}[2\kappa \operatorname{Im}(z^{3/2}\log z) - 3\overline{z^{1/2}\log z} \operatorname{Im} z - \\ &- 2\overline{z^{1/2}} \operatorname{Im} z] - i\frac{(\kappa-1)\alpha_+}{2\mu}\overline{z^{3/2}},\end{aligned}$$

$$\begin{aligned}\mathcal{A}_2(z) &= -\frac{Q}{2\mu}[2\kappa \operatorname{Im} z^{1/2} + \overline{z^{-1/2}} \operatorname{Im} z] - \\ &- Q\frac{(\alpha_+ - \alpha_-)(\kappa+1)}{8\pi\mu\kappa}[2\kappa \operatorname{Im}(z^{3/2}\log z) + 3\overline{z^{1/2}\log z} \operatorname{Im} z + \\ &+ 2\overline{z^{1/2}} \operatorname{Im} z] + \frac{i}{2\mu}[2\kappa \operatorname{Im} z^{3/2} - 3\overline{z^{1/2}} \operatorname{Im} z] + Q\frac{(\kappa+1)\alpha_+}{2\mu}\overline{z^{3/2}},\\ \mathcal{A}_3(z) &= -\frac{\kappa+1}{\mu} \operatorname{Im} z - \frac{(\alpha_+ - \alpha_-)(\kappa+1)}{4\pi\mu\kappa}[2\kappa \operatorname{Im}(z^2\log z) + \\ &+ 4\overline{z\log z} \operatorname{Im} z + 2\overline{z} \operatorname{Im} z] + \frac{(\kappa+1)\alpha_+}{\mu}\overline{z^2},\end{aligned}$$

where

$$\kappa = (\lambda + 3\mu)/(\lambda + \mu) \ \text{ and } \ Q = [(\alpha_+ + \alpha_-) - (\alpha_+ - \alpha_-)/2\kappa + 2\alpha_-]/2\,.$$

Here and in the sequel by symbols $z^\nu(\log z)^k$ we mean the branch of the analytic function taking real values on the upper boundary of the slit along the positive part of the real axis. By the limit relation for the double-layer potential we obtain

$$-2^{-1}\sigma + W\sigma + c_1\mathcal{A}_1 + c_2\mathcal{A}_2 + c_3\mathcal{A}_3 = g \tag{5}$$

for the pair (σ, c), where $c = (c_1, c_2, c_3)$.

We prove the uniqueness assertion for equation (5) in the class of pairs $\{\sigma, c\}$ with $\sigma \in \mathfrak{M}$ and $c \in \mathbf{R}^3$ in Theorem 5. The solvability of (5) with the right-hand side $g \in \mathfrak{N}_\nu,\ \nu > 3$, in $\mathfrak{M} \times \mathbf{R}^3$ is proved in Theorem 6. Moreover, in the same theorem we derive the following asymptotic formula for σ near the peak:

$$\sigma(z) = \Big(\alpha(\log x)^2 + \beta \log x + \gamma\Big)\, x^{-1/2} + O(x^{-\varepsilon}), \quad z \in S,$$

with positive ε.

A solution v of the problem $\mathcal{N}^-$ with the boundary data h from $\mathfrak{N}_\nu,\ \nu > 3$, is sought in the form of the simple-layer potential $V\tau$. The density τ satisfies the system of integral equations (2) on $S \backslash \{O\}$. In Theorems 7 and 8 we prove that if h has the zero mean value on S, then equation (2) has the unique solution τ in the class $\mathfrak{M}$ and this solution admits the following representation on the arcs $S_\pm$:

$$\tau_\pm(z) = \alpha_\pm x^{-1/2} + O(1)\,.$$

Outward peak. We represent a solution u of the problem $\mathcal{D}^+$ as the double-layer potential $W\sigma$. The density σ is found from the system of integral equations (1). It is proved that the kernel of the integral operator in (1) is two-dimensional in the class $\mathfrak{M}$. Solutions of the homogeneous system of integral equations (1) are functions obtained as restrictions to S of solutions to the homogeneous problem $\mathcal{N}^-$. Near the peak these displacements have the estimate $O(r^{-1/2})$ with r being the distance to the peak. So $\mathfrak{M}_{ext}$ is the uniqueness class for equation (1). The situation where (1) has at least two solutions is considered in Theorem 9.

The non-homogeneous integral equation (1) is studied in Theorem 10. We show that the solvability in $\mathfrak{M}$ holds for all functions g from the class $\mathfrak{N}_\nu,\ \nu > 0$. One of the solutions of (1) has the representations on $S_\pm$:

$$\begin{aligned} \sigma(x) &= \beta_\pm x^{\nu-1} + O(1) && \text{for } \nu \neq 1/2, \\ \sigma(x) &= \beta_\pm x^{-1/2}\log x + O(1) && \text{for } \nu = 1/2. \end{aligned}$$

The integral equation (2) is uniquely solvable in the class $\mathfrak{M}$ if the right-hand side $h \in \mathfrak{N}_\nu$ with $\nu > 0$ satisfies

$$\int_S h\, ds = 0, \qquad \int_S h\zeta\, ds = 0,$$

where ζ is any solution of the homogeneous equation (1) in the class $\mathfrak{M}$. In order to remove the orthogonality condition we are looking for a solution v of the problem $\mathcal{N}^-$ with the boundary data h from $\mathfrak{N}$ as the sum of the simple-layer potential $V\tau$ and the linear combination of functions $\varrho_1(z),\ \varrho_2(z)$ with unknown coefficients

$$v(z) = V\tau(z) + t_1\varrho_1(z) + t_2\varrho_2(z).$$

The functions $\varrho_k(z)$, $k = 1, 2$, are defined by complex stress functions (complex potentials) $\varphi_k(z)$, $\psi_k(z)$:

$$\varrho_k(z) = \frac{1}{2\mu}\left[\kappa\varphi_k(z) - z\overline{\varphi_k'(z)} - \overline{\psi_k(z)}\right],$$

$$\varphi_1(z) = \left(\frac{zz_0}{z - z_0}\right)^{1/2}, \quad \psi_1(z) = -\frac{3}{2}\left(\frac{zz_0}{z - z_0}\right)^{1/2},$$

$$\varphi_2(z) = i\left(\frac{zz_0}{z - z_0}\right)^{1/2}, \quad \psi_2(z) = \frac{i}{2}\left(\frac{zz_0}{z - z_0}\right)^{1/2},$$

where z_0 is a fixed point in Ω. The boundary equation

$$-2^{-1}\tau + TV\tau + t_1 T\varrho_1 + t_2 T\varrho_2 = h \tag{6}$$

is considered with respect to the pair (τ, t), where τ is the density of the simple-layer potential and $t = (t_1, t_2)$ is a vector in $\mathbf{R}^2$. In Theorems 11 and 12 we prove the existence and uniqueness of the solution of (6), respectively. In Theorem 12 we also study the asymptotic behaviour of solutions. We prove that for $h \in \mathfrak{N}_\nu$ with $0 < \nu < 1$ the density τ has the following representations on the arcs $S_\pm$:

$$\begin{aligned}
&\tau(x) = \beta_\pm x^{\nu-1} + O(x^{-1/2}) && \text{for } 0 < \nu < 1/2,\\
&\tau(x) = \gamma_\pm x^{-1/2}\log x + \beta_\pm x^{-1/2} + O(\log x) && \text{for } \nu = 1/2,\\
&\tau(x) = \gamma_\pm x^{-1/2} + \beta_\pm x^{\nu-1} + O(\log x) && \text{for } 1/2 < \nu < 1.
\end{aligned}$$

Assertions on the asymptotics of solutions to problems $\mathcal{D}^+$ and $\mathcal{N}^-$ are collected in Theorems 1–4.

2. Boundary Value Problems of Elasticity

We represent densities of integral equations of elasticity theory by means of solutions of certain auxiliary interior and exterior boundary value problems. The auxiliary results concerning such problems are collected in this section.

2.1. Asymptotic behaviour of solutions to the problem $\mathcal{D}^+$. We introduce some notation to be used in the proof of the following theorem and elsewhere.

Let $\beta > 0$. As in [5], by $W^l_{2,\beta}(G)$ we denote the weighted Sobolev space with the norm

$$\left(\sum_{k=0}^{\ell}\int_G |\nabla^k(e^{\beta t}f)|^2 dt du\right)^{1/2},$$

where ∇^k is the vector of all derivatives of order k. By $\mathring{W}^l_{2,\beta}(G)$ we mean the completion of C_0^∞ in the $W^l_{2,\beta}(G)$-norm and let $W^0_{2,\beta} = L_{2,\beta}$.

Theorem 1. *Let Ω have an outward peak. Suppose that g is an infinitely differentiable function on the curve $S\backslash\{O\}$ and let g have the following representations on the arcs $S_\pm$:*

$$g_\pm(z) = \sum_{k=0}^{n+1} Q_\pm^{(k+1)}(\log x)x^{k+\nu} + O\Big(x^{n+2+\nu-\varepsilon}\Big), \quad z = x+iy, \quad \nu > -1,$$

where $Q_\pm^{(j)}$ are polynomials of degree j and ε is a small positive number.

Suppose the above representations can be differentiated $n+2$ times. Then the problem $\mathcal{D}^+$ has a solution u of the form

$$u(z) = \frac{1}{2\mu}\Big[\kappa\varphi_n(z) - z\overline{\varphi_n'(z)} - \overline{\psi_n(z)}\Big] + u_0(z), \quad z \in \Omega, \tag{7}$$

where $\nabla^k u_0(z) = O(|z|^{n-2k})$ for $k = 0,\dots,n$ and

$$\varphi_n(z) = \sum_{k=0}^{n+1} P_\varphi^{(k+2)}(\log z)z^{\nu+k-1},$$
$$\psi_n(z) = \sum_{k=0}^{n+1} P_\psi^{(k+2)}(\log z)z^{\nu+k-1}\,.$$

Here $P_\varphi^{(j)}$ and $P_\psi^{(j)}$ are polynomials of degree j.

Proof. (i) We are looking for a displacement vector u_n such that the vector-valued function $g_n = g - u_n$ belong to $C^\infty(S\backslash\{O\})$ and $(g_n)_\pm(x) = x^\nu (q_n)_\pm(x)$, where $(q_n)_\pm$ are infinitely differentiable on $[0,\delta]$ and satisfy $\nabla^k (q_n)_\pm(x) = O(x^{n+1-k-\varepsilon})$, $k = 0,\dots,n+2$, on the arcs $S_\pm$ with ε being a small positive number.

To this end, we use the method of complex stress functions (see [16], Ch. II). The displacement vector u is related to complex potentials φ and ψ as follows:

$$2\mu u(z) = \kappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)},$$

where functions φ and ψ are to be defined by the boundary data of the problem $\mathcal{D}^+$.

It suffices to consider a function $g(z)$ coinciding with $A_\pm x^\nu(\log x)^m$ on $S_\pm$. We shall seek the functions φ and ψ in the form

$$\varphi(z) = z^{\nu-1}\sum_{k=0}^{m}\beta_k(\log z)^{m-k} + \varepsilon_0 z^\nu(\log z)^m,$$
$$\psi(z) = z^{\nu-1}\sum_{k=0}^{m}\gamma_k(\log z)^{m-k} + \delta_0 z^\nu(\log z)^m$$

for $\nu \neq 1$. There exist β_k, γ_k, ε_0 and δ_0 such that $\kappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}$ restricted to $S_\pm$ is equal to $2\mu A_\pm x^\nu(\log x)^m$ plus terms of the form $c_\pm x^i(\log x)^j$, admitting the estimate $O(x^\nu(\log x)^{m-1})$.

We substitute expansions of φ and ψ in powers of x along $S_\pm$ into the equation

$$\frac{1}{2\mu}(\kappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}) = g(z), \quad z = x + iy \in S\,.$$

Comparing the coefficients in $x^\nu(\log x)^m$ and $x^{\nu-1}(\log x)^m$ we obtain the system

$$\begin{cases} i(\kappa''_+(0) - \kappa''_-(0))(\nu-1)(\kappa\beta_0 + (\nu-3)\overline{\beta_0} + \overline{\gamma_0} = 4\mu(A_+ - A_-) \\ \kappa\beta_0 - (\nu-1)\overline{\beta_0} - \overline{\gamma_0} = 0 \end{cases}$$

with respect to β_0 and γ_0. Let us choose ε_0 arbitrarily. Then δ_0 is defined by the equation

$$\begin{aligned}\kappa\varepsilon_0 - \nu\overline{\varepsilon_0} - \overline{\delta_0} &= \mu(A_+ + A_-) \\ &\quad - \frac{i}{4}(\kappa''_+(0) + \kappa''_-(0))(\nu-1)(\kappa\beta_0 + (\nu-3)\overline{\beta_0} + \overline{\gamma_0})\,.\end{aligned}$$

If β_k are given, then γ_k $(k \geq 1)$ are found from the chain of equations

$$\kappa\beta_k - (\nu-1)\overline{\beta_k} - \overline{\gamma_k} - (m-k+1)\overline{\beta_{k-1}} = 0\,.$$

In the case $\nu = 1$ we seek the functions φ and ψ in the form

$$\varphi(z) = \sum_{k=0}^{m+1} \beta_k(\log z)^{m+1-k} + \varepsilon_0 z(\log z)^m,$$

$$\psi(z) = \sum_{k=0}^{m+1} \gamma_k(\log z)^{m+1-k} + \delta_0 z(\log z)^m\,.$$

The coefficients β_0 and γ_0 are found from the system

$$\begin{cases} \kappa\beta_0 - \overline{\gamma_0} = 0\,, \\ i(m+1)(\kappa''_+(0) - \kappa''_-(0))(\kappa\beta_0 - \overline{\beta_0} + \overline{\gamma_0}) = 2\mu(A_+ - A_-)\,. \end{cases}$$

Further, we choose ε_0 arbitrarily and find δ_0 from the equation

$$\kappa\varepsilon_0 - \overline{\varepsilon_0} - \overline{\delta_0} = (m+1)\overline{\beta_0} + \mu(A_+ + A_-) - \frac{i}{4}(\kappa''_+(0) + \kappa''_-(0))(\kappa\beta_0 - \overline{\beta_0} + \overline{\gamma_0})\,.$$

Given β_k, we find γ_k $(k \geq 1)$ from the chain of equations

$$\kappa\beta_k - \overline{\gamma_k} = (m+1-k)\overline{\beta_{k-1}}\,.$$

(ii) By $u^{(1)}$ we denote a vector-valued function equal to g_n on $S\backslash\{O\}$ and satisfying the estimates

$$u^{(1)}(z) = O(|z|^{n+1+\nu-\varepsilon}), \quad \nabla^k u^{(1)}(z) = O(|z|^{n+\nu-k-\varepsilon}), \quad k = 1, \dots, n+2\,.$$

Let the vector-valued function $u^{(2)}$ be the unique solution of the boundary value problem

$$\triangle^* u^{(2)} = -\triangle^* u^{(1)} \text{ in } \Omega, \quad u^{(2)} \in \mathring{W}{}^1_2(\Omega). \tag{8}$$

After the change of the variable $z = \zeta^{-1}$ ($\zeta = \xi + i\eta$), equation (8) with respect to $U^{(2)}(\xi,\eta) = u^{(2)}(\frac{\xi}{|\zeta|^2}, -\frac{\eta}{|\zeta|^2})$ takes the form

$$\mathcal{L}(\partial_\xi,\partial_\eta)\,U^{(2)} = \Delta^*\,U^{(2)} + L(\partial_\xi,\partial_\eta)\,U^{(2)} = F^{(1)} \text{ in } \Lambda,$$

where a curvilinear semi-infinite strip Λ is the image of Ω, $L(\partial_\xi,\partial_\eta)$ is the second order differential operator with coefficients having the estimate $O(1/\xi)$ as $\xi \to +\infty$, and $\nabla^k F^{(1)}(\zeta) = O(|\zeta|^{-n-\nu-2-k+\varepsilon})$, $k = 0,\dots,n$.

Let ρ be a function from the class $C_0^\infty(\mathbf{R})$ vanishing for $\xi < 1$ and equal to 1 for $\xi > 2$, and let $\rho_r(\xi) = \rho(\xi/r)$. Clearly,

$$\xi^n\mathcal{L}(\partial_\xi,\partial_\eta)\xi^{-n}\widetilde{U}^{(2)} = \Delta^*\widetilde{U}^{(2)} + R(\partial_\xi,\partial_\eta)\widetilde{U}^{(2)},$$

where $R(\partial_\xi,\partial_\eta)$ is the second order differential operator with coefficients admitting the estimate $O(1/\xi)$ as $\xi \to +\infty$. Therefore the boundary value problem

$$\Delta^*\widetilde{U}^{(2)} + \rho_r\widetilde{U}^{(2)} = F^{(2)} \text{ in } \Lambda, \quad \widetilde{U}^{(2)} = 0 \text{ on } \partial\Lambda,$$

where

$$F^{(2)}(\xi,\eta) = \xi^n F^{(1)}(\xi,\eta) \text{ and } \nabla^k F^{(2)}(\xi,\eta) = O(\xi^{-2-\nu-k+\varepsilon}), \quad k = 0,\dots,n\,,$$

is uniquely solvable in $\mathring{W}_2^1(\Lambda)$ for large r. From the local estimate

$$\|\widetilde{U}^{(2)}\|_{W_2^{n+2}(\Lambda\cap\{\ell-1<\xi<\ell+1\})} \le \text{const}\Big(\|\chi F^{(2)}\|_{W_2^n(\Lambda)} + \|\chi\widetilde{U}^{(2)}\|_{L_2(\Lambda)}\Big), \tag{9}$$

where χ belongs to $C_0^\infty(\ell-2,\ell+2)$ and equals to one in $(\ell-1,\ell+1)$, and from the Sobolev embedding theorem it follows that the vector-valued function $\widetilde{U}^{(2)}$ and its derivatives up to order n are bounded as $\xi \to \infty$. We set

$$U^{(3)}(\xi,\eta) = \xi^{-n}\widetilde{U}^{(2)}(\xi,\eta) \text{ and } \nabla^k U^{(3)}(\xi,\eta) = O(\xi^{-n}), \quad k = 0,\dots,n.$$

Clearly, $U^{(3)}$ belongs to the space $\mathring{W}_2^1(\Lambda)$ and satisfies

$$\mathcal{L}(\partial_\xi,\partial_\eta)U^{(3)} = F^{(1)}$$

for $\xi > 2r$. Using a partition of unity and the same local estimate we obtain that $U^{(2)} - U^{(3)} \in \mathring{W}_2^1 \cap W_2^{n+2}(\Lambda_{2r})$, where $\Lambda_{2r} = \Lambda \cap \{\xi > 2r\}$.

Let $\mathcal{D}(\partial_\xi,\partial_\eta)$ denote the differential operator $\triangle^*$ continuously mapping $\mathring{W}_{2,\beta}^1 \cap W_{2,\beta}^{n+2}(\Pi)$ into $W_{2,\beta}^n(\Pi)$, where $\Pi = \Big\{(\xi,\eta) : -\kappa_+''(x)/2 < \eta < -\kappa_-''(x)/2\Big\}$. Eigenvalues of the operator pencil $\mathcal{D}(ik,\partial_\eta)$ are nonzero roots of the equation

$$\alpha^2k^2 = \kappa(\sinh\,\alpha k)^2,$$

where $\alpha = (\kappa_+''(0) - \kappa_-''(0))/2$ and $\kappa = (\lambda+3\mu)/(\lambda+\mu)$. Since the operator $\mathcal{D}(\partial_\xi,\partial_\eta)$ is the "limit" operator for $\mathcal{L}(\partial_\xi,\partial_\eta)$ and since the real axis has no eigenvalues of $\mathcal{D}(ik,\partial_\eta)$, there exists $\beta > 0$ such that

$$U^{(2)} - U^{(3)} \in W_{2,\beta}^{n+2} \cap \mathring{W}_{2,\beta}^1(\Lambda)$$

(cf. [5], [8]). Now, since $U^{(2)} = U^{(3)} + (U^{(2)} - U^{(3)})$, it follows from the Sobolev embedding theorem that

$$\nabla^k U^{(2)}(\xi,\eta) = O(|\xi|^{-n}) \text{ for } k = 0,\dots,n\,.$$

Therefore from (i) and (ii) we find that the function $u = u_n + u^{(1)} + u^{(2)}$ is a solution of the problem $\mathcal{D}^+$ and has the required representation (7) with $u_0 = u^{(1)} + u^{(2)}$. $\square$

Corollary 1.1. *Let g have the following representations on the arcs $S_\pm$:*

$$g_\pm(x) = \sum_{k=0}^{n+1} q_\pm^{(k)} x^{k+\nu} + O(x^{n+2+\nu}), \quad \nu > -1,$$

with real coefficients $q_\pm^{(k)}$. Then the functions φ_n and ψ_n in (7) have the form

$$\varphi_n(z) = \beta_0 z^{-1} + (\beta_{1,0} + \beta_{1,1}\log z) + \sum_{k=2}^{n+1} \beta_k z^{k-1},$$
$$\psi_n(z) = \gamma_0 z^{-1} + (\gamma_{1,0} + \gamma_{1,1}\log z) + \sum_{k=2}^{n+1} \gamma_k z^{k-1}$$

for $\nu = 0$,

$$\varphi_n(z) = (\beta_{0,0} + \beta_{0,1}\log z) + \sum_{k=1}^{n+1} \beta_k z^k, \quad \psi_n(z) = (\gamma_{0,0} + \gamma_{0,1}\log z) + \sum_{k=1}^{n+1} \gamma_k z^k$$

for $\nu = 1$, and

$$\varphi_n(z) = \sum_{k=0}^{n+1} \beta_k z^{k+\nu-1}, \quad \psi_n(z) = \sum_{k=0}^{n+1} \gamma_k z^{k+\nu-1}$$

otherwise.

Theorem 2. *Let Ω have an inward peak. Suppose g is an infinitely differentiable function on the curve $S\backslash\{O\}$ and its restrictions to the arcs $S_\pm$ have the representations*

$$g_\pm(z) = \sum_{k=0}^{n+1} Q_\pm^{(k+1)}(\log x) x^{k+\nu} + O(x^{n+\nu+2-\varepsilon}), \quad \nu > -1,$$

where $Q_\pm^{(j)}$ are polynomials of degree j and ε is a small positive number. Suppose that these representations can be differentiated $n+2$ times. Then the problem $\mathcal{D}^+$ has a solution of the form

$$u(z) = \frac{1}{2\mu}\Big[\kappa(\varphi_n(z) + \varphi_*(z)) - z\overline{(\varphi_n'(z) + \varphi_*'(z))}$$
$$- \overline{(\psi_n(z) + \psi_*(z))}\Big] + u_0(z)\,, \tag{10}$$

where $\nabla^\ell u_0(z) = O(|z|^{n+[\nu]+1-\ell-\varepsilon})$, $\ell = 1,\ldots,n$. *The complex potentials* φ_n, ψ_n, φ_* *and* ψ_* *are represented as follows:*

$$\varphi_n(z) = \sum_{k=0}^{n} P_\varphi^{(k+2)}(\log z) z^{k+\nu}, \quad \psi_n(z) = \sum_{k=0}^{n} P_\psi^{(k+2)}(\log z) z^{k+\nu},$$
$$\varphi_*(z) = \sum_{m=1}^{p} R_{\varphi,m}(\log z) z^{m/2}, \quad \psi_*(z) = \sum_{m=1}^{p} R_{\psi,m}(\log z) z^{m/2}.$$

Here $P_\varphi^{(j)}$, $P_\psi^{(j)}$ *are polynomials of degree* j, $R_{\varphi,m}$, $R_{\psi,m}$ *are polynomials of degree* $[(m-1)/2]$, *and* $p = 2(n+[\nu]+1)$.

Proof. We are looking for a displacement vector u_n such that the vector-valued function $g_n = g - u_n$ on $S\backslash\{O\}$ belong to $C^\infty(S\backslash\{O\})$ and $\nabla^k (g_n)_\pm(z) = O(x^{n+\nu+3-k})$ for $k = 1,\ldots,n+2$. We use the method of complex stress functions. It suffices to take $g(z)$ equal to $A_\pm x^\nu (\log x)^m$ on $S_\pm$. As in Theorem 1, we introduce the potentials

$$\varphi(z) = \beta_m z^\nu (\log z)^m \quad\text{and}\quad \psi(z) = \gamma_m z^\nu (\log z)^m$$

for $\nu \neq m/2$, $m \in \mathbf{Z}$ such that $\kappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}$ on $S_\pm$ is the sum of $2\mu A_\pm x^\nu (\log x)^m$ and terms of the form $c_\pm x^i (\log x)^j$, admitting the estimate $O(x^\nu (\log x)^{m-1})$. We substitute the expansions of φ and ψ in powers of x along $S_\pm$ into the equation

$$\frac{1}{2\mu}(\kappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}) = g(z), \quad z = x+iy \in S\,.$$

The coefficients β_m and γ_m are found from the system

$$\begin{cases} \kappa\beta_m - \nu\overline{\beta_m} - \overline{\gamma_m} = 2\mu A_+ \\ e^{4i\pi\nu}\kappa\beta_m - \nu\overline{\beta_m} - \overline{\gamma_m} = 2\mu e^{2i\pi\nu} A_-\,. \end{cases}$$

If $\nu = m/2$ we seek the functions φ and ψ in the form

$$\varphi(z) = \Big(\beta_{m,1}(\log z)^{m+1} + \beta_{m,0}(\log z)^m\Big) z^\nu,$$
$$\psi(z) = \Big(\gamma_{m,1}(\log z)^{m+1} + \gamma_{m,0}(\log z)^m\Big) z^\nu\,.$$

In this case $\beta_{m,1}$ and $\gamma_{m,1}$ are found from the system

$$\begin{cases} \kappa\beta_{m,1} - \nu\overline{\beta_{m,1}} - \overline{\gamma_{m,1}} = 0\,, \\ \kappa\beta_{m,1} + \nu\overline{\beta_{m,1}} + \overline{\gamma_{m,1}} = i\mu\dfrac{A_+ - (-1)^m A_-}{\pi(m+1)}\,. \end{cases}$$

Finally, we choose $\beta_{m,0}$ arbitrarily. Then $\gamma_{m,0}$ is defined by the equation

$$\kappa\beta_{m,0} - \nu\overline{\beta_{m,0}} - \overline{\gamma_{m,0}} = 2\mu A_+ + (m+1)\overline{\beta_{m,1}}\,.$$

(ii) Let $u^{(1)}$ be a vector-valued function equal to g_n on $S\backslash\{O\}$ and admitting the estimates

$$u^{(1)}(z) = O(|z|^{n+\nu+3}) \text{ and } \nabla^k u^{(1)}(z) = O(|z|^{n+\nu+2-k}), \quad k = 1, \ldots, n+2,$$

in a neighborhood of the peak. By $u^{(2)} = (u_1^{(2)}, u_2^{(2)})$ we denote the solution of the Dirichlet problem

$$\triangle^* u^{(2)} = -\triangle^* u^{(1)} \text{ in } \Omega, \quad u^{(2)} \in \mathring{W}_2^1(\Omega)\,.$$

Let Λ be the image of Ω under the mapping $(r, \theta) \to (t, \theta)$, where r, θ are polar coordinates of (x, y) and $t = \log(1/r)$. The vector-valued function $U^{(2)}(t, \theta)$ with the components

$$u_1^{(2)}(e^{-t}, \theta) \cos\theta + u_2^{(2)}(e^{-t}, \theta) \sin\theta \text{ and } u_2^{(2)}(e^{-t}, \theta) \cos\theta - u_1^{(2)}(e^{-t}, \theta) \sin\theta,$$

is a solution of the equation

$$\triangle^* U^{(2)} + K U^{(2)} = F^{(1)} \text{ in } \mathring{W}_2^1(\Lambda)\,,$$

where $F^{(1)}(t, \theta) = O(e^{-(n+\nu+2)t})$. Here K is the first order differential operator

$$K = \begin{pmatrix} -\lambda + 2\mu & -(\lambda + 3\mu)(\partial/\partial\theta) \\ (\lambda + 3\mu)(\partial/\partial\theta) & -\mu \end{pmatrix}.$$

From the local estimate

$$\|U^{(2)}\|_{W_2^{n+2}(\Lambda \cap \{\ell-1<\xi<\ell+1\})} \le \text{const}\Big(\|\chi F^{(1)}\|_{W_2^n(\Lambda)} + \|\chi U^{(2)}\|_{L_2(\Lambda)}\Big)\,, \tag{11}$$

where χ belongs to $C_0^\infty(\ell-2, \ell+2)$ and equals to 1 in $(\ell-1, \ell+1)$, it follows that $U^{(2)} \in W_2^{n+2} \cap \mathring{W}_2^1(\Lambda)$.

By $\mathcal{D}(\partial_t, \partial_\theta)$ we denote the operator $\triangle^* + K$ continuously mapping $\mathring{W}_{2,\beta}^1 \cap W_{2,\beta}^{n+2}(\Pi)$ into $W_{2,\beta}^n(\Pi)$, where $\Pi = \{(t, \theta) : 0 < \theta < 2\pi, t \in \mathbf{R}\}$. Eigenvalues of the operator pencil $\mathcal{D}(ik, \partial_\theta)$ are the numbers $k = i\ell/2$, where $\ell \in \mathbf{Z}$, $\ell \neq 0$. The multiplicity of each eigenvalue is equal to 2 and the maximum length of the Jordan chain for each eigenvector (multiplicity of eigenvector) is equal to 1. Therefore, the strip $0 < \operatorname{Im} z < \beta$, where $\beta \in (n + [\nu] + 1, n + [\nu] + 3/2)$, contains $p = 2(n + [\nu] + 1)$ eigenvalues of $\mathcal{D}(ik, \partial_\theta)$.

Since $F^{(1)} \in W_{2,\beta}^n(\Lambda)$, $U^{(2)}$ admits the representation (cf. [5], [8])

$$U^{(2)} = \sum_{k=1}^{p} c_k V^{(k)} + W^{(1)},$$

where $V^{(k)} = (V_1^{(k)}, V_2^{(k)})$ are linear independent vector-valued functions satisfying $(\triangle^* + K)V^{(k)} = 0$ in $\Lambda_R = \Lambda \cap \{t > R\}$ and vanishing on $\partial\Lambda \cap \{t > R\}$, $V^{(k)} \notin W_{2,\beta}^{n+2}(\Lambda_R)$ and $W^{(1)} \in W_{2,\beta}^{n+2}(\Lambda_R)$. Making the inverse change $t = -\log r$ we obtain

$$u^{(2)}(r, \theta) = \sum_{k=1}^{p} c_k v^{(k)}(r, \theta) + w^{(1)}(r, \theta)\,,$$

where $v^{(k)}(r,\theta) = V^{(k)}(\log(1/r),\theta)\cdot e^{i\theta}$ and $\nabla^\ell w^{(1)}(r,\theta) = O(r^{n+[\nu]+1-\ell})$ for $\ell = 1,\dots,n$. Using the method of complex stress functions and repeating the above-mentioned arguments we find

$$\sum_{k=1}^{p} c_k v^{(k)}(z) = \frac{1}{2\mu}\left[\kappa\varphi_*(z) - z\overline{\varphi_*'(z)} - \overline{\psi_*(z)}\right] + w^{(2)}(z)\,,$$

where $\nabla^\ell w^{(2)}(z) = O(|z|^{n+[\nu]+1-\ell-\varepsilon})$ for $\ell = 1,\dots,n$ and

$$\varphi_*(z) = \varepsilon_1 z^{1/2} + \varepsilon_2 z + (\varepsilon_{3,0} + \varepsilon_{3,1}\log z)z^{3/2} + \cdots + R_{\varphi,\,p-1}(\log z)z^{n+[\nu]+1/2},$$
$$\psi_*(z) = \delta_1 z^{1/2} + \delta_2 z + (\delta_{3,0} + \delta_{3,1}\log z)z^{3/2} + \cdots + R_{\psi,\,p-1}(\log z)z^{n+[\nu]+1/2}.$$

It follows from (i) and (ii) that the vector-valued function $u = u_n + u^{(1)} + u^{(2)}$ is the required solution of the problem $\mathcal{D}^+$ with $u_0 = u^{(1)} + w^{(1)} + w^{(2)}$. $\square$

Corollary 2.1. *Let g have the following representations on the arcs $S_\pm$:*

$$g_\pm(z) = \sum_{k=0}^{n+1}(\alpha_\pm^{(k,1)}\log x + \alpha_\pm^{(k,0)})x^{k+\nu} + O(x^{n+\nu+2})$$

for $\nu \neq m/2$, $m \in \mathbf{Z}$, where $\alpha_\pm^{(k,i)}$ are real numbers. Then the functions φ_n and ψ_n in (10) *have the form*

$$\varphi_n(z) = \sum_{k=0}^{n}(\beta^{(k,1)}\log z + \beta^{(k,0)})z^{k+\nu},$$
$$\psi_n(z) = \sum_{k=0}^{n}(\gamma^{(k,1)}\log z + \gamma^{(k,0)})z^{k+\nu}$$

with $\beta^{(k,i)}$, $\gamma^{(k,i)} \in \mathbf{C}$.

2.2. Asymptotic behaviour of solutions to the problem $\mathcal{N}^-$. We introduce the weighted space $W^{k,\rho}(\Omega^c)$ with the inner product

$$(f_1,f_2)_{k,\rho} := \sum_{|\alpha|\le k}\int_{\Omega^c}\rho^{-2k+2|\alpha|}D^\alpha f_1\overline{D^\alpha f_2}\,dxdy,$$

where $\rho(z) = (1+|z|^2)$. By $\mathring{W}^{k,\rho}(\Omega^c)$ we denote the completion of $C_0^\infty(\Omega^c)$ in $W^{k,\rho}(\Omega^c)$.

Theorem 3. *Let Ω have an inward peak. Suppose that h is an infinitely differentiable vector-valued function on $S\backslash\{O\}$, $\int_S h\,ds = 0$ and let the restriction of h to $S_\pm$ admit the representation*

$$h_\pm(z) = \sum_{k=0}^{n-1} H_\pm^{(k+1)}(\log x)x^{k+\nu} + O(x^{n+\nu-\varepsilon}),\quad \nu > -1,$$

where $H_\pm^{(j)}$ are polynomials of degree j and ε is a small positive number. Let this representation be differentiable n times. Then the problem $\mathcal{N}^-$ with the boundary data h has a solution v bounded at infinity, satisfying the condition

$$V.P.\int_S Tv\,ds = \lim_{\varepsilon\to 0} \int_{\{q\in S,\,|q|\geq\varepsilon\}} Tv\,ds = 0\,, \tag{12}$$

and, up to a linear function $\alpha + icz$ with real coefficient c, represented in the form

$$v(z) = \frac{1}{2\mu}\left[\kappa\varphi_n(z) - z\overline{\varphi_n'(z)} - \overline{\psi_n(z)}\right] + v_0(z)\,. \tag{13}$$

Here $\nabla^k v_0(z) = O(|z|^{n-2k-1})$ for $k = 1, \dots, n-1$,

$$\begin{aligned}
\varphi_n(z) = {}& i\left(\sum_{m=0}^{p}\beta_{0,m}\left(\log\frac{zz_0}{z_0-z}\right)^m\right)\left(\frac{zz_0}{z_0-z}\right)^{\nu-2} \\
&+ \sum_{k=1}^{n+1} P_\varphi^{(k+2)}\left(\log\frac{zz_0}{z_0-z}\right)\left(\frac{zz_0}{z_0-z}\right)^{k+\nu-2}, \\
\psi_n(z) = {}& i\left(\sum_{m=0}^{p}\gamma_{0,m}\left(\log\frac{zz_0}{z_0-z}\right)^m\right)\left(\frac{zz_0}{z_0-z}\right)^{\nu-2} \\
&+ \sum_{k=1}^{n+1} P_\psi^{(k+2)}\left(\log\frac{zz_0}{z_0-z}\right)\left(\frac{zz_0}{z_0-z}\right)^{k+\nu-2},
\end{aligned}$$

where $\beta_{0,m}$ and $\gamma_{0,m}$ are real numbers, $p = 1$ if $\nu \neq 0, 1, 2, 3$, and $p = 2$ otherwise, $P_\varphi^{(j)}$ and $P_\psi^{(j)}$ are polynomials of degree j.

Proof. (i) We are looking for a displacement vector v_n such that the traction $h_n = h - Tv_n$ belong to $C^\infty(S\backslash\{0\})$ and admit the estimates

$$\nabla^k(h_n)_\pm(z) = O(x^{n+\nu-k-\varepsilon}), \quad z = x + iy,$$

on $S_\pm$ for $k = 0, \dots, n$. To this end we represent the boundary condition of the problem $\mathcal{N}^-$ with the boundary data h in the Muskhelishvili form (see [16], Ch. II, Sect. 30)

$$\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} = f(z), \quad z \in S\backslash\{0\}\,. \tag{14}$$

Here φ and ψ are complex stress functions and f has the form

$$f(z) = -i\int_{(0z)^\smile} h\,ds + \text{const}, \quad z \in S\,,$$

where by $(0z)^\smile$ we denote the arc of S connecting 0 and z. As f in (14), it suffices to consider the function $\pm ih_\pm x^{\nu+1}(\log x)^m$ on $S_\pm$. In a small neighborhood of the peak, we are looking for complex potentials φ and ψ in the form

$$\varphi(z) = \sum_{r=0}^{3}\left(\sum_{k=0}^{p}\frac{m!}{(m-k)!}{\beta'}_{r,k}(\log z)^{p-k}\right)z^{\nu+r-2}$$

and

$$\psi(z)=\sum_{r=0}^{3}\Big(\sum_{k=0}^{p}\frac{m!}{(m-k)!}\gamma'_{r,k}(\log z)^{p-k}\Big)z^{\nu+r-2}\,,$$

where $p=m$ if $\nu\neq 0,1,2,3$ and $p=m+1$ otherwise. As in Theorem 1, we find coefficients of φ and ψ so that the restriction of $\varphi(z)+z\overline{\varphi'(z)}+\overline{\psi(z)}$ to $S_\pm$ is the sum of $\pm ih_\pm x^{\nu+1}(\log x)^m$ and terms of the form $c_\pm x^i(\log x)^j$, admitting the estimate $O(x^{\nu+1}(\log x)^{m-1})$.

The coefficients $\beta'_{0,k}$, $\gamma'_{0,k}$, $k=0,\dots,m$, are defined by the system

$$\begin{cases}\beta'_{0,k}+(\nu-2)\overline{\beta'_{0,k}}+\overline{\gamma'_{0,k}}=A_{0,k}\,,\\ \beta'_{0,k}-(\nu-4)\overline{\beta'_{0,k}}-\overline{\gamma'_{0,k}}=B_{0,k}\,,\end{cases}$$

where $A_{0,k}=B_{0,k}=0$ for $k=0$ and $A_{0,k}=-\overline{\beta'_{0,k-1}}$, $B_{0,k}=\overline{\beta'_{0,k-1}}$ if $k=1,\dots,m$. We find

$$\operatorname{Re}\beta'_{0,k}=\operatorname{Re}\gamma'_{0,k}=0,\quad k=0,\dots,m,$$

and

$$\begin{aligned}&(3-\nu)\operatorname{Im}\beta'_{0,0}=\operatorname{Im}\gamma'_{0,0}\,,\\ &(3-\nu)\operatorname{Im}\beta'_{0,k}=\operatorname{Im}\gamma'_{0,k}+\operatorname{Im}\beta'_{0,k-1},\quad k=1,\dots,m\,.\end{aligned}$$

The coefficients $\beta'_{1,0}$ and $\gamma'_{1,0}$ satisfy

$$\begin{cases}\beta'_{1,0}+(\nu-1)\overline{\beta'_{1,0}}+\overline{\gamma'_{1,0}}=0\,,\\ \beta'_{1,0}-(\nu-3)\overline{\beta'_{1,0}}-\overline{\gamma'_{1,0}}=\dfrac{(\nu-3)(\nu-2)}{\nu-1}(\kappa''_+(0)+\kappa''_-(0))\operatorname{Im}\beta'_{0,0}\,.\end{cases}$$

Hence it follows that

$$\operatorname{Re}\beta'_{1,0}=\frac{(\nu-3)(\nu-2)}{4(\nu-1)}(\kappa''_+(0)+\kappa''_-(0))\operatorname{Im}\beta'_{0,0},$$

$$\operatorname{Re}\gamma'_{1,0}=-\nu\operatorname{Re}\beta'_{1,0}\quad\text{and}\quad\operatorname{Im}\gamma'_{1,0}=-(\nu-2)\operatorname{Im}\beta'_{1,0}.$$

We set

$$\operatorname{Im}\gamma'_{1,0}=\operatorname{Im}\beta'_{1,0}=0.$$

The coefficients $\beta'_{1,1}$ and $\gamma'_{1,1}$ are found from the system

$$\begin{cases}\beta'_{1,1}+(\nu-1)\overline{\beta'_{1,1}}+\overline{\gamma'_{1,1}}=-\overline{\beta'_{1,0}}\,,\\ \beta'_{1,1}-(\nu-3)\overline{\beta'_{1,1}}-\overline{\gamma'_{1,1}}=\overline{\beta'_{1,0}}\\ \qquad\qquad+\dfrac{(\nu-3)(\nu-2)}{\nu-1}(\kappa''_+(0)+\kappa''_-(0))\operatorname{Im}\beta'_{0,1}\\ \qquad\qquad+\dfrac{\nu^2-2\nu-1}{(\nu-1)^2}(\kappa''_+(0)+\kappa''_-(0))\operatorname{Im}\beta'_{0,0}\,.\end{cases}\tag{15}$$

Summing up these equations we obtain

$$\operatorname{Re}\beta'_{1,1} = \frac{(\nu-3)(\nu-2)}{4(\nu-1)}(\kappa''_+(0)+\kappa''_-(0))\operatorname{Im}\beta'_{0,1} + \frac{\nu^2-2\nu-1}{4(\nu-1)^2}(\kappa''_+(0)+\kappa''_-(0))\operatorname{Im}\beta'_{0,0},$$

$$\operatorname{Re}\gamma'_{1,1} = -\nu\operatorname{Re}\beta'_{1,1} - \operatorname{Re}\beta'_{1,0}.$$

From the first equation of system (15) we find

$$\operatorname{Im}\gamma'_{1,1} = -(\nu-2)\operatorname{Im}\beta'_{1,1} - \operatorname{Im}\beta'_{1,0}\,.$$

We set

$$\operatorname{Im}\beta'_{1,1} = \operatorname{Im}\gamma'_{1,1} = 0\,.$$

The real parts of $\beta'_{1,k}$ and $\gamma'_{1,k}$, $k=2,\dots,m$, are found from the systems

$$\begin{cases} \beta'_{1,k} + (\nu-1)\overline{\beta'_{1,k}} + \overline{\gamma'_{1,k}} = A_{1,k}\,, \\ \beta'_{1,k} - (\nu-3)\overline{\beta'_{1,k}} - \overline{\gamma'_{1,k}} = B_{1,k}\,, \end{cases}$$

where

$$A_{1,k} = -\overline{\beta'_{1,k-1}},$$

$$B_{1,k} = -\frac{1}{\nu-1}\left(\beta'_{1,k-1} - 2(\nu-2)\overline{\beta'_{1,k-1}} - \overline{\gamma'_{1,k-1}} - \beta'_{1,k-2}\right) + \frac{(\kappa''_+(0)+\kappa''_-(0))}{\nu-1}\Big(\operatorname{Im}\beta'_{0,k-2} + (2\nu-5)\operatorname{Im}\beta'_{0,k-1} + (\nu-3)(\nu-2)\operatorname{Im}\beta'_{0,k}\Big)\,.$$

Let $\operatorname{Im}\beta'_{1,k}=0$, $k=2,\dots,m$. The coefficients $\operatorname{Im}\beta'_{2,0}$ and $\operatorname{Im}\gamma'_{2,0}$ are calculated by the system

$$\begin{cases} \beta'_{2,0} + \nu\overline{\beta'_{2,0}} + \overline{\gamma'_{2,0}} = A_{2,0}\,, \\ \beta'_{2,0} - (\nu-2)\overline{\beta'_{2,0}} - \overline{\gamma'_{2,0}} = B_{2,0}\,, \end{cases} \tag{16}$$

where

$$A_{2,0} = -\frac{1}{2}i\kappa''_+(0)\kappa''_-(0)(\nu-3)(\nu-2)\operatorname{Im}\beta'_{0,0},$$

$$B_{2,0} = 2\frac{h_+ + h_-}{\nu(\kappa''_+(0)-\kappa''_-(0))} + \frac{(\nu-2)(\nu-3)}{3\nu}(\kappa'''_+(0)+\kappa'''_-(0))\operatorname{Im}\beta'_{0,0} - i\frac{(\nu-2)(\nu-3)(\nu-4)}{6\nu}((\kappa''_+(0))^2 + \kappa''_+(0)\kappa''_-(0) + (\kappa''_-(0))^2)\operatorname{Im}\beta'_{0,0} + \frac{(\nu-1)(\nu-2)}{\nu}(\kappa''_+(0)+\kappa''_-(0))\operatorname{Re}\beta'_{1,0}\,.$$

System (16) is solvable if

$$\frac{(\nu-2)(\nu-3)(\nu+2)}{3}(\kappa''_+(0)-\kappa''_-(0))^3\operatorname{Im}\beta'_{0,0} = -8\operatorname{Im}(h_+ + h_-). \tag{17}$$

We express $\operatorname{Im}\beta'_{0,0}$ from (17). Summing up the equations of system (16) we find $\operatorname{Re}\beta'_{2,0}$. Then $\operatorname{Re}\gamma'_{2,0}$ is defined from the first equation in (16)

$$\operatorname{Re}\gamma'_{2,0} = -(\nu+1)\operatorname{Re}\beta'_{2,0}.$$

We choose $\operatorname{Im}\beta'_{2,0}$ arbitrarily and find $\operatorname{Im}\gamma'_{2,0}$ from the equation

$$(\nu-1)\operatorname{Im}\beta'_{2,0} + \operatorname{Im}\gamma'_{2,0} = \frac{1}{2}\kappa''_{+}(0)\kappa''_{-}(0)(\nu-3)(\nu-2)\operatorname{Im}\beta'_{0,0}\,.$$

The coefficients $\beta'_{2,1}$ and $\gamma'_{2,1}$ are found by the equation

$$\beta'_{2,1} + \nu\overline{\beta'_{2,1}} + \overline{\gamma'_{2,1}} = -\frac{1}{2}i\kappa''_{+}(0)\kappa''_{-}(0)\Big((2\nu-5)\operatorname{Im}\beta'_{0,0} + (\nu-3)(\nu-2)\operatorname{Im}\beta'_{0,1}\Big)\,.$$

We set

$$\operatorname{Im}\beta'_{0,1} = \operatorname{Im}\gamma'_{0,1} = 0 \ \text{ and } \ \operatorname{Re}\beta'_{2,1} = \operatorname{Re}\gamma'_{2,1} = 0.$$

Let $\beta'_{2,1}$ be chosen. Then $\gamma'_{2,1}$ is subject to

$$-(\nu-1)\operatorname{Im}\beta'_{2,1} - \operatorname{Im}\gamma'_{2,1} = \frac{1}{2}\kappa''_{+}(0)\kappa''_{-}(0)(2\nu-5)\operatorname{Im}\beta'_{0,0}\,.$$

Finally, given $\beta'_{2,k}$ arbitrarily, we define $\gamma'_{2,k}$, $k \geq 2$, recursively

$$\beta'_{2,k} + \nu\overline{\beta'_{2,k}} + \overline{\gamma'_{2,k}} = \frac{1}{2}i\kappa''_{+}(0)\kappa''_{-}(0)\,\Big(\operatorname{Im}\beta'_{0,k-2} + (2\nu-5)\operatorname{Im}\beta'_{0,k-1} + (\nu-3)(\nu-2)\operatorname{Im}\beta'_{0,k}\Big)\,.$$

We set

$$\operatorname{Im}\beta'_{0,k} = \operatorname{Im}\gamma'_{0,k} = 0 \ \text{ and } \ \operatorname{Re}\beta'_{2,k} = \operatorname{Re}\gamma'_{2,k} = 0, \quad k \geq 2\,,$$

and choose $\operatorname{Im}\beta'_{2,k}$ and $\operatorname{Im}\gamma'_{2,k}$ to satisfy

$$-(\nu-1)\operatorname{Im}\beta'_{2,k} - \operatorname{Im}\gamma'_{2,k} = \frac{1}{2}\kappa''_{+}(0)\kappa''_{-}(0)(\operatorname{Im}\beta'_{0,k-2} + (2\nu-5)\operatorname{Im}\beta'_{0,k-1})\,.$$

The exceptional cases $\nu = 0, 1, 2, 3$ can be treated similarly. We set

$$\varphi_n(z) = \sum_{r=0}^{3}\left(\sum_{k=0}^{p}\beta_{r,k}\left(\log\frac{zz_0}{z_0-z}\right)^{p-k}\right)\left(\frac{zz_0}{z_0-z}\right)^{r+\nu-2},$$
$$\psi_n(z) = \sum_{r=0}^{3}\left(\sum_{k=0}^{p}\gamma_{r,k}\left(\log\frac{zz_0}{z_0-z}\right)^{p-k}\right)\left(\frac{zz_0}{z_0-z}\right)^{r+\nu-2}, \quad z \in \Omega^c\,,$$

where z_0 is a fixed point of Ω. We choose $\beta_{r,k}$ and $\gamma_{r,k}$ so that in the decompositions of φ_n and ψ_n along $S_\pm$ the coefficients in $x^{\nu-k}(\log x)^m$, $k = 0, 1, 2,\ m = 1, \ldots, p$, coincide with the corresponding coefficients in the decompositions of φ and ψ.

The displacement vector v_n is defined by

$$v_n(z) = (2\mu)^{-1}\left(\kappa\varphi_n(z) - z\overline{\varphi'_n(z)} - \overline{\psi_n(z)}\right)\,.$$

Then $\nabla v_n = O(|z|^{-2})$ as $|z| \to \infty$ and satisfies $V.P.\int_S Tv_n\,ds = 0$.

(ii) Let $f_{n,1}$ and $f_{n,2}$ denote the components of the vector-valued function f_n on $S\backslash\{O\}$ defined by

$$f_n(z) = -i \int\limits_{\stackrel{\smile}{(0z)}} h_n\, ds + \text{const.}$$

We set

$$A_n = -\frac{i}{2\pi} \int\limits_S f_{n,1}dx + f_{n,2}dy$$

and consider the function

$$\chi_n(z) = A_n \left(\log\left(1 - \frac{z}{z_0}\right) + \sum_{k=1}^{n+[\nu]+1} c_k \left(\frac{zz_0}{z_0 - z}\right)^k \right), \quad z \in \Omega^c,$$

where z_0 is a fixed point in Ω and the coefficients c_k are chosen so that $\chi_n(z) = O(|z|^{n+[\nu]+2})$ as z tends to zero. If by $v_n^{(1)}$ we denote the displacement vector $v_n^{(1)}(z) = (1/2\mu)\overline{\chi'(z)}$, then $Tv_n^{(1)} = i(\partial/\partial s)\overline{\chi_n'}$. For the traction $h_n^{(1)} = h_n - Tv_n^{(1)}$ on $S\backslash\{O\}$ with the components $h_{n1}^{(1)}$ and $h_{n2}^{(1)}$ the principal vector $\int_S h_n^{(1)}(z)ds$ and the principal moment $\int_S \Big[(x-x_0)h_{n2}^{(1)}(x,y) - (y-y_0)h_{n1}^{(1)}(x,y)\Big]ds$ with respect to $z_0 = (x_0, y_0) \in \Omega$ are equal to zero.

We need to construct a displacement vector $v_n^{(2)}$ with the given stress $h_n^{(1)}$ on S. To this end, we represent the boundary conditions of this problem via the Airy function $F(z)$ biharmonic in Ω^c (see [16], Ch. II, Sect. 30). We have $F = b$ and $\partial F/\partial n = d$ on $S\backslash\{O\}$, where the functions b and d belong to $C^\infty(S\backslash\{O\})$ and admit the estimates

$$b(p) = O(|p|^{n+2+\nu-\varepsilon}), \quad d(p) = O(|p|^{n+1+\nu-\varepsilon}), \quad p \in S\backslash\{O\}\,,$$

because of their relation with $h_n^{(1)}$:

$$\begin{gathered} F(z) = \int\limits_{\stackrel{\smile}{(0z)}} [(x_s - x)h_{n1}^{(1)}(x_s, y_s) - (y_s - y)h_{n2}^{(1)}(x_s, y_s)]ds, \quad z = (x, y) \in S\backslash\{O\}, \\ \frac{\partial F}{\partial x}(z) = -\int\limits_{\stackrel{\smile}{(0z)}} h_{n2}^{(1)} ds + C_1, \quad \frac{\partial F}{\partial y}(z) = \int\limits_{\stackrel{\smile}{(0z)}} h_{n1}^{(1)} ds + C_2. \end{gathered} \tag{16}$$

Let $F_1(z)$ in Ω^c satisfy

$$F_1(p) = b(p), \quad (\partial F_1/\partial n)(p) = d(p) \text{ on } S\backslash\{O\},$$

and admit the estimate

$$\nabla^k F_1(z) = O(|z|^{n+1+\nu-k-\varepsilon}), \quad k = 0, \ldots, n+2, \quad \text{as } z \to 0\,.$$

We can assume also that $\rho^{-2}\Delta^2 F_1 \in L_2(\Omega^c)$. Then the boundary value problem

$$\triangle^2 F_2 = -\Delta^2 F_1 \text{ in } \Omega^c, \quad F_2 = \partial F_2/\partial n = 0 \text{ on } S,$$

is uniquely solvable in $\mathring{W}^{2,\rho}(\Omega^c)$ (cf. [4]).

We use the Kelvin transform setting

$$U(\zeta) = |\zeta|^2 F_2(1/\zeta), \ \ \zeta = \xi + i\eta \in \Lambda,$$

where Λ is the image of Ω^c under the inversion $\zeta = 1/z$. We have $\triangle^2 U = H$ in Λ, where $H(\zeta) = O(|\zeta|^{-n-\nu-3+\varepsilon})$. As in Theorem 1, we can construct a vector-valued function $U_n^{(1)} \in \mathring{W}_2^2(\Lambda \cap \{\xi > 2R\})$ such that $\nabla^k U_n^{(1)}(\xi,\eta) = O(\xi^{-n+1})$, $k = 0,\dots,n$, $\triangle^2 U_n^{(1)} = H$ in $\Lambda \cap \{\xi > 2R\}$ for large R and $U - U_n^{(1)}$ belongs to $(\mathring{W}_2^2 \cap W_2^{n+2})(\Lambda \cap \{\xi > 2R\})$.

By $\mathcal{U}(\partial_\xi, \partial_\eta)$ we denote the operator $\triangle^2$ continuously mapping $\mathring{W}_{2,\beta}^2 \cap W_{2,\beta}^{n+2}(\Pi)$ into $W_{2,\beta}^{n-2}(\Pi)$, where $\Pi = \Big\{(\xi,\eta): \ -\kappa_+''(x)/2 < \eta < -\kappa_-''(x)/2\Big\}$. The eigenvalues of the operator pencil $\mathcal{U}(ik, \partial_\eta)$ are nonzero roots of the equation

$$(\alpha k)^2 - (\sinh \alpha k)^2 = 0,$$

where $\alpha = (\kappa_+''(0) - \kappa_-''(0))/2$. Since the real axis contains no eigenvalues of $\mathcal{U}(ik, \partial_\eta)$, there exists a positive β such that

$$U - U_n^{(1)} \in W_{2,\beta}^{n+2} \cap \mathring{W}_{2,\beta}^2 \, (\Lambda \cap \{\xi > 2R\})$$

(cf. [5], [8]). Hence and from the Sobolev embedding theorem it follows that $U - U_n^{(1)}$ and its derivatives up to order n have the estimate $O\Big(\exp(-\beta\xi)\Big)$ as $\xi \to +\infty$. Thus $U = U_n^{(1)} + (U - U_n^{(1)})$ admits the estimate $\nabla^k U(\xi,\eta) = O(|\xi|^{-n+1})$ for $k = 1,\dots,n$. Therefore the Airy function $F(z)$, equal to $F_1(z) + F_2(z)$, has the representation

$$F(z) = O(|z|^{n-1}) \ \text{ and } \ \nabla^k F(z) = O(|z|^{n-1-2k}), \ \ k = 1,\dots,n\,.$$

A displacement vector $v_n^{(2)}$ corresponding to $F(z)$ has the form

$$v_n^{(2)}(z) = \alpha + icz + v_{n0}^{(2)}(z),$$

where c is a real coefficient and $\nabla^k v_{n0}^{(2)}(z) = O(|z|^{n-2k-2})$. Since the gradient $\nabla v_n^{(2)}$ of the displacement vector $v_n^{(2)}$ is square summable in a neighborhood of infinity, we have

$$\int\limits_{|z|=R_n} T v_n^{(2)} ds \to 0$$

for a certain sequence $\{R_n\}$, $R_n \to \infty$. This and the condition $\int_S h\, ds = 0$ imply

$$\int\limits_S T v_n^{(2)}\, ds = 0.$$

From the first Kolosov formula

$$\Delta F(z) = 4\,\mathrm{Re}(\varphi_n^{(2)})'(z)$$

(see [3], Sect. 8.4) it follows that $\mathrm{Re}(\varphi_n^{(2)})'(z)$, where $\varphi_n^{(2)}$ is the complex potential corresponding to $v_n^{(2)}$, is square summable in a neighborhood of infinity. Therefore

$$\varphi_n^{(2)}(z) = icz + \sum_{k=-\infty}^{0} b_k z^{-k}, \quad c \in \mathbf{R}.$$

The second Kolosov formula

$$\frac{\partial^2 F}{\partial x^2}(z) - \frac{\partial^2 F}{\partial y^2}(z) - 2i\frac{\partial^2 F}{\partial x \partial y}(z) = 2\Big[\overline{z}(\varphi_n^{(2)})''(z) + (\psi_n^{(2)})'(z)\Big]$$

(see [3], Sect. 8.4) implies that the derivative of the second complex potential $\psi_n^{(2)}(z)$ is also square summable in a neighborhood of infinity. Therefore we have

$$\psi_n^{(2)}(z) = \sum_{k=-\infty}^{0} a_k z^{-k}.$$

Thus the displacement vector $v_n^{(2)}$ admits the representation

$$v_n^{(2)}(z) = \frac{1}{2\mu}\,(\kappa+1)icz + O(1).$$

Hence, up to a rigid displacement equal to $(2\mu)^{-1}(\kappa+1)icz$, the constructed displacement vector $v_n^{(2)}$ is bounded at infinity. We remove this rigid displacement term since it is a solution of the homogeneous problem $\mathcal{N}^-$. Thus the displacement vector $v = v_n + v_n^{(1)} + v_n^{(2)}$ is a solution of the problem $\mathcal{N}^-$. It has the required representation (13) and satisfies condition (12). $\square$

Corollary 3.1. *Let the polynomials $H_\pm^{(1)}$ satisfy*

$$\mathrm{Im}\,\Big(H_+^{(1)}(0) + H_-^{(1)}(0)\Big) = \mathrm{Im}\,\Big((\partial H_+^{(1)}/\partial t)(0) + (\partial H_-^{(1)}/\partial t)(0)\Big) = 0\,.$$

Then the solution of the problem $\mathcal{N}^-$ has a finite energy integral and can be represented in form (13) *with*

$$\varphi_n(z) = i\left(\sum_{m=0}^{p+1} \beta_{1,m}\left(\log\frac{zz_0}{z_0 - z}\right)^m\right)\left(\frac{zz_0}{z_0-z}\right)^{\nu-1} + \sum_{k=2}^{n+1} P_\varphi^{(k+2)}\left(\log\frac{zz_0}{z_0-z}\right)\left(\frac{zz_0}{z_0-z}\right)^{k+\nu-2},$$

$$\psi_n(z) = i\left(\sum_{m=0}^{p+1} \gamma_{1,m}\left(\log\frac{zz_0}{z_0 - z}\right)^m\right)\left(\frac{zz_0}{z_0-z}\right)^{\nu-1} + \sum_{k=2}^{n+1} P_\psi^{(k+2)}\left(\log\frac{zz_0}{z_0-z}\right)\left(\frac{zz_0}{z_0-z}\right)^{k+\nu-2},$$

where $\beta_{1,m}$, $\gamma_{1,m}$ are real numbers, $p = 1$ if $\nu = -1, 0, 1, 2$, and $p = 2$ otherwise, and $P_\varphi^{(j)}$, $P_\psi^{(j)}$ are polynomials of degree j.

Corollary 3.2. *Let h have the representation*

$$h_\pm(z) = h_\pm^{(1)} x^{-1/2} + h_\pm^{(2)} x^{1/2} \log x + h_\pm^{(3)} x^{1/2} + \dots$$

on $S_\pm$. Then the problem $\mathcal{N}^-$ with the boundary data h has a solution in the class $\mathfrak{M}$ if and only if

$$\mathrm{Im}\,(h_+^{(1)} + h_-^{(1)}) = 0, \quad \mathrm{Im}\,(h_+^{(2)} + h_-^{(2)}) = 0,$$

$$Re\, h_+^{(1)} \alpha_- + \ Re\, h_-^{(1)} \alpha_+ - \frac{1}{2}\ \mathrm{Im}\,(h_+^{(3)} + h_-^{(3)}) = 0\,.$$

Theorem 4. *Let Ω have an outward peak. Suppose that h is a C^∞-function on $S \backslash \{O\}$ and its restrictions to the arcs $S_\pm$ have the representations*

$$h_\pm(z) = \sum_{k=0}^{n-1} H_\pm^{(k)}(\log x) x^{\nu+k} + O(x^{\nu+n}), \ \ \nu > -2.$$

Here $H_\pm^{(j)}$ are polynomials of degree at most j. Suppose the above representations can be differentiated n times and

$$V.P. \int_S h\, ds = 0\,.$$

Then the problem $\mathcal{N}^-$ with the boundary data h on S has a solution v bounded at infinity, admitting the representation

$$v(z) = \frac{1}{2\mu}\Big[\kappa\,(\varphi_n(z) + \varphi_*(z)) - z\overline{(\varphi_n'(z) + \varphi_*'(z))} \\ - \Big(\overline{\psi_n(z) + \psi_*(z)}\Big)\Big] + v_0(z) \qquad (18)$$

up to a linear function $\alpha + icz$ with real coefficient c, and satisfying the condition

$$\int_S Tv\, ds = 0\,.$$

Here $\nabla^k v_0(z) = O(|z|^{n+\nu-k})$, $k = 0, \dots, n-1$, the complex potentials φ_n, ψ_n have the form

$$\varphi_n(z) = \sum_{k=1}^{n} P_\varphi^{(k+1)}\left(\log \frac{zz_0}{z_0 - z}\right)\left(\frac{zz_0}{z_0 - z}\right)^{k+\nu},$$

$$\psi_n(z) = \sum_{k=1}^{n} P_\psi^{(k+1)}\left(\log \frac{zz_0}{z_0 - z}\right)\left(\frac{zz_0}{z_0 - z}\right)^{k+\nu},$$

where $P_\varphi^{(j)}$ and $P_\psi^{(j)}$ are polynomials of degree j, and

$$\varphi_*(z) = \sum_{k=1}^{m} R_{\varphi,k}\left(\log \frac{zz_0}{z_0 - z}\right)\left(\frac{zz_0}{z_0 - z}\right)^{k/2},$$

$$\psi_*(z) = \sum_{k=1}^{m} R_{\psi,k}\left(\log \frac{zz_0}{z_0 - z}\right)\left(\frac{zz_0}{z_0 - z}\right)^{k/2},$$

where $R_{\varphi,k}$ and $R_{\psi,k}$ are polynomials of degree at most $[(k-1)/2]$ and m is the largest integer not exceeding $2(n+\nu)+1$.

Proof. We choose a displacement vector v_n such that $h_n = h - Tv_n$ belongs to $C^\infty(S\backslash\{O\})$ and $(h_n)_\pm(z) = O(x^{n+\nu})$. To this end, as in Theorem 3, we use the method of complex stress functions. It is convenient to write the boundary conditions of the problem $\mathcal{N}^-$ in the Muskhelishvili form

$$\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} = f(z), \quad z \in S\,.$$

Here

$$f(z) = -i \int\limits_{(0z)^\smile} h\, ds + \text{ const}\,,$$

where by $(0z)^\smile$ we denote the arc of S connecting 0 and z. It suffices to consider a function $f(z)$ of the form $\pm ih_\pm x^{\nu+1}(\log x)^m$ on arcs $S_\pm$. We set

$$\varphi_n(z) = \beta_m \left(\frac{zz_0}{z-z_0}\right)^{\nu+1} \left(\log\frac{zz_0}{z-z_0}\right)^m\,,$$
$$\psi_n(z) = \gamma_m \left(\frac{zz_0}{z-z_0}\right)^{\nu+1} \left(\log\frac{zz_0}{z-z_0}\right)^m$$

for $\nu \neq n/2,\ \ n \in \mathbf{Z}$, where z_0 is a fixed point in Ω.

We show that there exist β_m and γ_m such that the function $\varphi_n(z) + z\overline{\varphi_n{}'(z)} + \overline{\psi_n(z)}$ on $S_\pm$ is equal to the sum of $\pm ih_\pm x^{\nu+1}(\log x)^m$ and a linear combination of the functions $c_\pm x^i(\log x)^j$, admitting the estimate $O(x^{\nu+1}(\log x)^{m-1})$.

As in Theorem 2, we decompose φ_n and ψ_n in powers of x along $S_\pm$. Coefficients β_m and γ_m in $x^{\nu+1}(\log x)^m$ are found from the system

$$\begin{cases} \beta_m + (\nu+1)\overline{\beta_m} + \overline{\gamma_m} = ih_+ \\ e^{4i\pi\nu}\beta_m + (\nu+1)\overline{\beta_m} + \overline{\gamma_m} = -ih_- e^{2i\pi\nu}\,. \end{cases}$$

In the case $\nu = n/2,\ \ n \in \mathbf{Z}$ the functions φ_n and ψ_n are defined by

$$\varphi_n(z) = \left(\beta_{m,1}\left(\log\frac{zz_0}{z-z_0}\right)^{m+1} + \beta_{m,0}\left(\log\frac{zz_0}{z-z_0}\right)^m\right)\left(\frac{zz_0}{z-z_0}\right)^{\nu+1}\,,$$
$$\psi_n(z) = \left(\gamma_{m,1}\left(\log\frac{zz_0}{z-z_0}\right)^{m+1} + \gamma_{m,0}\left(\log\frac{zz_0}{z-z_0}\right)^m\right)\left(\frac{zz_0}{z-z_0}\right)^{\nu+1}\,,$$

where z_0 is a fixed point in Ω. The above coefficients $\beta_{m,1}$ and $\gamma_{m,1}$ are found from the system

$$\begin{cases} \beta_{m,1} + (\nu+1)\overline{\beta_{m,1}} + \overline{\gamma_{m,1}} = 0\,, \\ \beta_{m,1} - (\nu+1)\overline{\beta_{m,1}} - \overline{\gamma_{m,1}} = -\dfrac{h_+ + (-1)^m h_-}{2\pi(m+1)}\,. \end{cases}$$

Given $\beta_{m,0}$, we find $\gamma_{m,0}$ by the equation

$$\beta_{m,0} + (\nu+1)\overline{\beta_{m,0}} + \overline{\gamma_{m,0}} = ih_+ + (m+1)\overline{\beta_{m,1}}\,.$$

After the complex stress functions φ_n and ψ_n have been found, the displacement vector v_n is defined by means of

$$v_n(z) = (2\mu)^{-1}\left(\kappa\varphi_n(z) - z\overline{\varphi'_n(z)} - \overline{\psi_n(z)}\right).$$

Since $V.P. \int_S h\, ds = 0$, we have $H_+^{(0)} = -H_-^{(0)}$ for $-2 < \nu \le -1$. Therefore

$$\lim_{\varepsilon\to 0} \int_{\{q\in\Omega^c,\ |q|=\varepsilon\}} T v_n ds = 0\,.$$

Hence we arrive at $V.P. \int_S T v_n ds = 0$.

(ii) As in Theorem 3, we can construct the displacement vector $v_n^{(1)}$ so that

$$v_n^{(1)}(z) = O(|z|^{n+[\nu]+1}) \text{ as } z \to 0$$

and $v_n^{(1)}$ vanishes at infinity. Furthermore, the stress function $h_n^{(1)} = h_n - T v_n^{(1)}$ on S has the zero principal vector and the zero principal moment with respect to any point $z_0 \in \Omega$.

We find a displacement vector $v_n^{(2)}$ with given $h_n^{(1)}$ on S. To this end, we express the boundary condition of this problem via the Airy function F in Ω^c. Let $F = b$ and $\partial F/\partial n = d$ on S. Taking into account the relation of b and d with $h_n^{(1)}$ we have $d(z) = O(|z|^{n+1+\nu})$, $b(z) = O(|z|^{n+2+\nu})$ for $z \in S\backslash\{O\}$.

We represent the Airy function $F(z)$ as the sum $F_1(z) + F_2(z)$, where $F_1(z)$ is chosen so that it vanishes at infinity, satisfies the conditions

$$F_1(z) = b(z),\ \ \partial F_1(z)/\partial n = d(z) \text{ on } S$$

and admits the estimate $\nabla^k F_1(z) = O(|z|^{n+2+\nu-k})$, $k = 0, \dots, n+2$, in a neighborhood of the origin. Then $F_2(z)$ is a unique solution of problem

$$\triangle^2 F_2 = -\triangle^2 F_1 \text{ in } \Omega^c,\ \ F_2 = \partial F_2/\partial n = 0 \text{ on } S$$

in $\mathring{W}^{2,\rho}(\Omega^c)$ with $\rho(z) = (1+|z|)^2$ (see [4]).

We make the change of variable $t = \log(1/r)$, $r = |z|$. Let Λ be the image of the domain Ω^c under the mapping $(r,\theta) \to (t,\theta)$, where r, θ are polar coordinates of (x,y). The function $U(t,\theta) = F_2(e^{-t},\theta)$ solves the equation

$$\begin{aligned}\mathcal{L}(\partial_t,\partial_\theta)U(t,\theta) &= \left[\left(\frac{\partial}{\partial t}+2\right)^2 + \frac{\partial^2}{\partial\theta^2}\right]\left[\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial\theta^2}\right]U(t,\theta)\\ &= H(t,\theta) \text{ in } \Lambda\cap\{t>R\}\end{aligned}$$

and satisfies $U = \partial U/\partial n = 0$ on $\partial\Lambda\cap\{t > R\}$, where $\nabla^k H(t,\theta) = O(e^{-(n+2+\nu)t})$, $k = 0, \dots, n-2$. From the local estimate

$$\|U\|_{W_2^{n+2}(\Lambda\cap\{\ell-1<\xi<\ell+1\})} \le \text{const}\Big(\|\chi H\|_{W_2^{n-2}(\Lambda)} + \|\chi U\|_{L_2(\Lambda)}\Big),$$

where χ belongs to $C_0^\infty(\ell-2,\ell+2)$ and equals 1 in $(\ell-1,\ell+1)$, it follows that U belongs to $\mathring{W}^2_{2,1}\cap W^{n+2}_{2,1}(\Lambda\cap\{t>R\})$. By $\mathcal{U}(\partial_t,\partial_\theta)$ we denote the operator of the boundary value problem

$$\mathcal{U}(\partial_t,\partial_\theta)U=F \text{ in } \Pi=\{(t,\theta):\, 0<\theta<2\pi\},\;\; U=(\partial/\partial n)U=0 \text{ on } \partial\Pi$$

continuously mapping $W_2^{n+2}\cap\mathring{W}_2^2(\Pi)$ into $W_2^{n-2}(\Pi)$. Let β satisfy $n+\nu+1<\beta<n+\nu+3/2$. The operator pencil $\mathcal{U}(ik,\partial_\theta):\, W_2^{n+2}\cap\mathring{W}_2^2(0,2\pi)\to W_2^{n-2}(0,2\pi)$ has $p=2n+[2\nu]+2$ eigenvalues of the form $k=i\ell/2$, $\ell\in\mathbf{N}$, in the strip $\{k:0<\text{Im } k<\beta\}$. The eigenvalues have multiplicity 2 and multiplicity of each eigenvector equals 1. Therefore the solution U admits the representation (cf. [5], [8])

$$U=\sum_{k=1}^{p}c_kU_k+W.$$

Here U_k are linearly independent, each U_k satisfies the equation $\mathcal{L}U_k=0$ in the domain $\Lambda\cap\{t>R\}$, vanishes on $\partial\Lambda\cap\{t>R\}$ and $U_k\notin W^{n+2}_{2,\beta}(\Lambda\cap\{t>R\})$, in addition, $W\in W^{n+2}_{2,\beta}(\Lambda\cap\{t>R\})$.

Making the inverse change of variable we obtain

$$F(r,\theta)=\sum_{k=1}^{p}c_kU_k(-\log r,\theta)+O(r^\beta)$$

and this equality can be differentiated n times. By the method of complex stress functions the displacement vector $v_n^{(2)}$ corresponding to F is given by

$$v_n^{(2)}(z)=\frac{1}{2\mu}\left[\kappa\varphi_*(z)-z\overline{\varphi_*'(z)}-\overline{\psi_*(z)}\right]+w^{(1)}(z)\,,$$

where $w^{(1)}(z)=\alpha+icz+O(|z|^{n+\nu})$ with real α and this equality can be differentiated $n-1$ times,

$$\varphi_*(z)=\sum_{k=1}^{m}R_{\varphi,k}\left(\log\frac{zz_0}{z_0-z}\right)\left(\frac{zz_0}{z_0-z}\right)^{k/2}$$

and

$$\psi_*(z)=\sum_{k=1}^{m}R_{\psi,k}\left(\log\frac{zz_0}{z_0-z}\right)\left(\frac{zz_0}{z_0-z}\right)^{k/2}.$$

Here m is the largest integer not exceeding $2(n+\nu)+1$. As in Theorem 3, we can prove that

$$\int_S Tv_n^{(2)}ds=0.$$

Hence the displacement vector $v=v_n+v_n^{(1)}+v_n^{(2)}$ is a solution of the problem $\mathcal{N}^-$ with the boundary data h and admits the required representation. $\square$

Corollary 4.1. *Let h have representations*

$$h_\pm(z) = \sum_{k=0}^{n-1} h_\pm^{(k)} x^{k+\nu} + O(x^{n+\nu}), \quad x \in S_\pm,$$

for $\nu > -1$, $\nu \neq \ell/2$, $\ell \in \mathbf{Z}$. Then the functions φ_n and ψ_n in (18) *have the form*

$$\varphi_n(z) = \sum_{k=1}^{n} \beta_k \left(\frac{zz_0}{z-z_0}\right)^{k+\nu}, \quad \psi_n(z) = \sum_{k=1}^{n} \gamma_k \left(\frac{zz_0}{z-z_0}\right)^{k+\nu}.$$

2.3. Properties of solutions of the problem $\mathcal{D}^-$.

Proposition 1. *Let Ω have an outward peak and let a vector-valued function g satisfy on $S_\pm$ the conditions*

$$\frac{\partial^k g}{\partial s^k}(z) = O(|z|^{\beta-k}), \quad k = 0, 1, 2.$$

Then the problem $\mathcal{D}^-$ with the boundary data g has a solution u bounded at infinity and subject to

$$\int_S |q|\Big|\nabla u(q)\Big| ds_q < +\infty, \quad u(z) = O(|z|^\gamma) \tag{19}$$

if $-1 < \gamma < \beta \leq 0$, and to

$$\int_S |\nabla u(q)| ds_q < +\infty, \quad u(z) = O(|z|^\gamma) \tag{20}$$

if $0 < \gamma < \min\{\beta, 1/2\}$.

Proof. As in Theorem 2, we can construct a solution u satisfying either (19) or (20) and such that u belongs to $W^{1,p}(|z| > R)$ and is square summable in a neighborhood of infinity for sufficiently large R. We check now that u is bounded at infinity. By the first Kolosov formula $\mathrm{Re}\{\varphi'(z)\}$ is a linear combination of the components of ∇u. It follows that $\mathrm{Re}\,\varphi'(z)$ is square summable in a neighborhood of infinity. So, $\varphi(z)$ has the decomposition

$$\varphi(z) = icz + \sum_{k=-\infty}^{0} b_k z^{-k}, \quad c \in \mathbf{R}.$$

From the equality

$$u(z) = (2\mu)^{-1}(\kappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)})$$

it follows that $\psi(z)$ is also square summable in a neighborhood of infinity. Therefore $\psi(z)$ can be presented in the form

$$\psi(z) = \sum_{k=-\infty}^{0} d_k z^{-k}.$$

Finally we have

$$u(z) = i(2\mu)^{-1}(\kappa+1)cz + O(1).$$

Since the function $\rho^{-1}u^-$ is square summable in a neighborhood of infinity, $c = 0$. Consequently, u is bounded at infinity. □

Proposition 2. *Let Ω have an inward peak and let g on $S_\pm$ satisfy the conditions*

$$\frac{\partial^k g}{\partial s^k}(z) = O(|z|^{\beta-k}), \quad k = 0, 1, 2.$$

Then the problem $\mathcal{D}^-$ with the boundary data g has a solution u bounded at infinity and satisfying

$$\int_S |u(q)| ds_q < +\infty, \quad \int_S |q||\nabla u(q)| ds_q < +\infty. \tag{21}$$

Proof. Let $u^{(1)}$ be an extension of g onto Ω^c equal to zero outside a certain disk and satisfying $\Delta^* u^{(1)} = O(|z|^{\beta-3})$ as $z \to 0$. We look for a function $u^{(2)}$ such that

$$\triangle^* u^{(2)} = -\triangle^* u^{(1)} \;\text{ in }\; \Omega^c, \quad u^{(2)} = 0 \;\text{ on }\; S\backslash\{O\}\,. \tag{22}$$

After the change of variables $z = 1/\zeta$ ($\zeta = \xi + i\eta$) the equation in (22) takes the form

$$\Delta^* U + L(\partial_\xi, \partial_\eta)U = F \;\text{ in }\; \Lambda, \tag{23}$$

where $U(\xi, \eta) = u^{(2)}(\xi/|\zeta|^2, \eta/|\zeta|^2)$, Λ is the image of Ω and $L(\partial_\xi, \partial_\eta)$ is the second order differential operator with coefficients admitting the estimate $O(\xi^{-\beta-1})$. Since $F \in L_2(\Lambda)$, problem (23) has a unique solution in the space $W_2^2(\Lambda)$. Hence $u = u^{(1)} + u^{(2)}$ satisfies conditions (21). □

3. Integral Equation on the Contour with an Inward Peak

3.1. Integral equation for the problem $\mathcal{D}^+$. We shall use the following lemma.

Lemma 1. *Let Ω have an inward peak and let $\alpha = \max(|\kappa_+''(0)|,\ |\kappa_-''(0)|)$. Then, for any $\sigma = \left(\sigma^{(1)}, \sigma^{(2)}\right)$ from $\mathfrak{M}$,*

$$\left(W\sigma^{(1)}\right)(z) = \mp\frac{1}{2}\left(\sigma_+^{(1)} - \sigma_-^{(1)}\right) \pm \frac{\mu}{2\pi(\lambda+2\mu)} \int_0^\delta \left(\sigma_+^{(2)} - \sigma_-^{(2)}\right)(\tau)\frac{d\tau}{x-\tau} + O(1),$$

$$\left(W\sigma^{(2)}\right)(z) = \mp\frac{1}{2}\left(\sigma_+^{(2)} - \sigma_-^{(2)}\right) \mp \frac{\mu}{2\pi(\lambda+2\mu)} \int_0^\delta \left(\sigma_+^{(1)} - \sigma_-^{(1)}\right)(\tau)\frac{d\tau}{x-\tau} + O(1)$$

with the upper sign if

$$z \in \{(x, y) :\ \kappa_+(x) < y < \alpha x,\ x \in (0, \delta)\}\,,$$

and with the lower sign if

$$z \in \left\{(x, y) :\ -\alpha x^2 < y < \kappa_-(x),\ x \in (0, \delta)\right\}\,.$$

For the proof see [9]. Now we state a uniqueness theorem.

Theorem 5. *Let Ω have an inward peak. The boundary integral equation*

$$2^{-1}\sigma - W\sigma + c_1\mathcal{A}_1 + c_2\mathcal{A}_2 + c_3\mathcal{A}_3 = 0 \tag{24}$$

has only the trivial solution in the Cartesian $\mathfrak{M} \times \mathbf{R}^3$.

Proof. Let a pair (σ, c) with $\sigma \in \mathfrak{M}$, $c \in \mathbf{R}^3$, be a solution of equation (24). Consider the displacement vector

$$W\sigma(z) + c_1\mathcal{A}_1(z) + c_2\mathcal{A}_2(z) + c_3\mathcal{A}_3(z), \quad z \in \Omega, \tag{25}$$

with zero boundary value on $S\backslash\{O\}$. We show that this vector-valued function is equal to zero in Ω.

We use the change of the variable $t = \log 1/r$ and denote by $W_1(z)$, $W_2(z)$ the components of (25). Then $U(t,\theta)$ with the components

$$W_1(e^{-t},\theta)\cos\theta + W_2(e^{-t},\theta)\sin\theta, \quad -W_1(e^{-t},\theta)\sin\theta + W_2(e^{-t},\theta)\cos\theta$$

is a solution of the problem

$$\mathcal{L}(\partial_t, \partial_\theta)U = (\triangle^* + K)U = 0 \text{ in } \Lambda, \quad U = 0 \text{ on } \partial\Lambda,$$

where r, θ are polar coordinates of (x,y) and Λ is the image of Ω. Here by K we denote the first order differential operator with constant coefficients

$$K = \begin{pmatrix} -\lambda + 2\mu, & -(\lambda+3\mu)\partial/\partial\theta \\ (\lambda+3\mu)\partial/\partial\theta & -\mu \end{pmatrix}.$$

Since the potential $W\sigma(z)$ grows not faster than a power function as $z \to 0$, there exists $\beta < 0$ such that $U \in L_2(\Lambda)$. By the local estimate (11) we obtain that U belongs to $W^2_{2,\beta} \cap \mathring{W}^1_{2,\beta}(\Lambda)$. The eigenvalues of the operator pencil $\mathcal{D}(ik, \partial_\theta) : W^2_2 \cap \mathring{W}^1_2(0, 2\pi) \to L_2(0, 2\pi)$ are the numbers $i\ell/2$, where ℓ is a nonvanishing integer. According to [5], [8], the operator $\mathcal{L}$ is invertible if $\beta \neq i\ell/2$, $\ell = \pm1, \pm2, \ldots$. Since $U \in \ker \mathcal{L}$, from its asymptotic representation of U (cf. [5]) it follows that either $U(t,\theta) = O(e^{-t/2})$ or $U(t,\theta)$ grows faster than $e^{t/(2+\varepsilon)}$ $(\varepsilon > 0)$ as $t \to +\infty$. In the first case the energy integral of (W_1, W_2) is finite and therefore it is equal to zero in Ω. Now we shall show that the second case is impossible.

Let χ be a C^∞-function equal to zero outside of a small neighborhood of the origin and $\chi = 1$ near the origin. The components of $\chi\sigma$ will be denoted by $\sigma^{(1)}, \sigma^{(2)}$. Since the boundary values of $W\sigma$ are bounded on $S\backslash\{O\}$, by Lemma 1 we have

$$-\frac{1}{2}\left(\sigma^{(1)}_+ - \sigma^{(1)}_-\right)(x) + \frac{\mu}{2\pi(\lambda+2\mu)}\int_0^\delta \left(\sigma^{(2)}_+ - \sigma^{(2)}_-\right)(\tau)\,\frac{d\tau}{x-\tau} = O(1),$$

$$-\frac{1}{2}\left(\sigma^{(2)}_+ - \sigma^{(2)}_-\right)(x) - \frac{\mu}{2\pi(\lambda+2\mu)}\int_0^\delta \left(\sigma^{(1)}_+ - \sigma^{(1)}_-\right)(\tau)\,\frac{d\tau}{x-\tau} = O(1).$$

These equalities imply that the functions $\left(\sigma_+^{(1)} - \sigma_-^{(1)}\right)$ and $\left(\sigma_+^{(2)} - \sigma_-^{(2)}\right)$ are p-summable for any $p > 1$ in a neighborhood of the origin. Therefore $W\sigma(z)$ is also p-summable for any $p > 1$ along any ray

$$\{(x, y) : x = \alpha t,\ y = \beta t,\ t > 0\}$$

near the origin with $\alpha < 0$. Since $\mathcal{A}_j = O(1),\ j = 1, 2, 3$, our goal is achieved. Hence we have

$$W\sigma(z) + c_1\mathcal{A}_1(z) + c_2\mathcal{A}_2(z) + c_3\mathcal{A}_3(z) = 0 \quad \text{in} \ \ \Omega.$$

Let u^- be a solution of the problem $\mathcal{D}^-$ with the boundary data σ. We notice that

$$VTu^- - Wu^- = u^-(\infty) \quad \text{in} \ \ \Omega\,.$$

Here by $V\tau(z)$ we denote the simple-layer potential with a density τ and with the kernel

$$\Gamma(z, q) - \Gamma(z, 0), \quad q \in S\backslash\{O\}.$$

Since

$$Wu^- = -\sum_{k=1}^{3} c_k\mathcal{A}_k\,,$$

by substituting Wu^- into the previous equality, we find

$$VTu^- + \sum_{k=1}^{3} c_k\mathcal{A}_k = u^-(\infty) \quad \text{in} \ \ \Omega.$$

The limit relations for the simple-layer potential imply

$$T\left(VTu^- + \sum_{k=1}^{3} c_k\mathcal{A}_k + \frac{1}{2}u^-\right) = 0 \quad \text{on} \ \ S\backslash\{O\}.$$

Let w_k^- be the solution of the problem

$$\triangle^* w_k^- = 0 \ \text{ in } \ \Omega^c, \quad Tw_k^- = T\mathcal{A}_k \ \text{ on } \ S\backslash\{O\}$$

constructed in Theorem 3. Since the boundary function $T\mathcal{A}_k$ does not satisfy the conditions of Corollary 3.2, it follows that the solution w_k^- does not belong to $\mathfrak{M}$. Then we have

$$VTu^-(z) = -u^-(z) - \sum_{k=1}^{3} c_k w_k^-(z) + u_0(z) \ \text{ in } \ \Omega^c, \tag{26}$$

where u_0 is the displacement vector satisfying $Tu_0 = 0$ on $S\backslash\{O\}$. We substitute (26) into the identity

$$Wu^- - VTu^- = u^- - u^-(\infty) \ \text{ in } \ \Omega^c$$

and obtain

$$W\sigma(z) = u_0(z) - \sum_{k=1}^{3} c_k w_k^- - u^-(\infty),\ z \in \Omega^c\,. \tag{27}$$

From the jump formula for $W\sigma$ it follows

$$\sigma(p) = -u_0(z) + \sum_{k=1}^{3} c_k w_k^- + u^-(\infty) - \sum_{k=1}^{3} c_k \mathcal{A}_k.$$

Since the density σ belongs to $\mathfrak{M}$, according to (27) the function w_0, defined by $w_0(\zeta) = u_0(\zeta^{-1})$, and ∇w_0 grow not faster than a power function in the image Λ of Ω^c. Therefore $w_0 \in W^2_{2,\beta}(\Lambda)$ with a small negative β. Since $\nabla u_0(z) = O(|z|^{-2})$ as $z \to \infty$ and $Tu_0 = 0$ on $S\backslash\{O\}$, it follows from Betti's formula that

$$\int_{\{\kappa_-(x)<y<\kappa_+(x)\}} Tu_0\, ds = 0 \quad \text{for} \quad 0 < x < \delta\,.$$

We represent the displacement vector u_0 via the Airy function U in the domain $\Omega_\delta = \Omega^c \cap \{|z| < \delta\}$. Since

$$\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y} = -i \int_{(0z)^\smile} Tu_0\, ds + \text{ const}\,,$$

where by $(0z)^\smile$ we denote the arc of S connecting 0 and z, and since $(\partial U/\partial x) + i(\partial U/\partial y)$ is defined up to a constant term, we assume that

$$\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y} = 0 \ \text{ on } \ S_\pm.$$

Therefore U is equal to the constants $c_\pm$ on $S_\pm$. We use the Kelvin transform

$$W(\zeta) = |\zeta|^2 U(1/\zeta), \ \ \zeta \in \Lambda_\delta,$$

where Λ_δ is the image Ω_δ under the mapping $\zeta = 1/z$. The expressions $W(\zeta)$ and $\nabla^k W(\zeta)$, $k = 1, 2$, grow not faster than a power function. Therefore, W belongs to $W^2_{2,\beta}(\Lambda_\delta)$ with $\beta < 0$, W is biharmonic in Λ_δ and satisfies the conditions

$$W(\zeta) = c_\pm|\zeta|^2, \quad (\partial W/\partial n) = c_\pm(\partial|\zeta|^2/\partial n) \ \text{ on } \ \partial\Lambda_\delta \cap \{|\zeta| > \delta^{-1}\}.$$

From the local estimate

$$\begin{aligned} &\|W\|_{W^3_2(\Lambda_1\cap\{\ell-1<Re\zeta<\ell+1\})} \\ &\quad \le \text{const}\Big[\|\chi W\|_{W_2^{5/2}(\partial\Lambda_\delta)} + \|\chi(\partial W/\partial n)\|_{W_2^{3/2}(\partial\Lambda_\delta)} + \|\chi W\|_{W^2_2(\Lambda_\delta)}\Big], \end{aligned} \tag{28}$$

where a cut-off C^∞ function χ equals 1 on $(\ell-1, \ell+1)$ and vanishes outside $(\ell-2, \ell+2)$, it follows that W belongs to $W^3_{2,\beta}(\Lambda_\delta)$.

Since $\nabla^2 w_0$ is represented as a linear combination of derivatives up to the third order of W with coefficients growing not faster than a power function, we have $w_0 \in W^3_{2,\beta}(\Lambda_\delta)$.

Let $\mathcal{N}(\partial_\xi, \partial_\eta)$ denote the operator of the boundary value problem

$$\begin{aligned} &\triangle^* u = f \ \text{ in } \ \Pi = \Big\{(\xi,\eta) : \xi \in \mathbf{R}^1, -\kappa''_+(0)/2 < \eta < -\kappa''_-(0)/2\Big\}\,, \\ &Tu = h \ \text{ on} \Big\{(\xi,\eta) : \eta = -\kappa''_\pm(0)/2\Big\}\,, \end{aligned}$$

continuously mapping $W^2_{2,\beta}(\Pi)$ into $L_{2,\beta}(\Pi) \times W^{1/2}_{2,\beta}(\partial\Pi)$. It is well known that the eigenvalues of the operator pencil $\mathcal{N}(ik, \partial_\eta)$: $W^2_2(-\kappa''_+(0)/2, -\kappa''_-(0)/2) \to L_2(-\kappa''_+(0)/2, -\kappa''_-(0)/2) \times \mathbf{R}^2$ are found from the equation

$$k^2\alpha^2 = (\sinh k\,\alpha)^2\,,$$

where $\alpha = \Big(\kappa''_+(0) - \kappa''_-(0)\Big)/2$. One can check that the dimension of the space $\operatorname{Ker}\mathcal{N}(0, D_\eta)$ equals two. The vector-valued function $u_1^{(0)}(\eta) = (0,1)$ is the basis element of multiplicity 4. The second linear independent eigenvector $u_2^{(0)}(\eta) = (1,0)$ has multiplicity 2. The generalized eigenvectors corresponding to $u_1^{(0)}, u_2^{(0)}$ are

$$\begin{aligned}
u_1^{(1)}(\eta) &= \Big(-i\eta - \frac{1}{2}\, i\,(\alpha_+ + \alpha_-), 0\Big)\,,\\
u_1^{(2)}(\eta) &= \left(0, -\frac{1}{2}\frac{\lambda}{\lambda+2\mu}\,\eta^2 - \frac{1}{2}\frac{\lambda}{\lambda+2\mu}\,(\alpha_+ + \alpha_-)\eta\right),\\
u_1^{(3)}(\eta) &= \Big(-\frac{i}{6}\frac{3\lambda+4\mu}{\lambda+2\mu}\,\eta^3 - \frac{i}{4}(\alpha_+ + \alpha_-)\frac{3\lambda+4\mu}{\lambda+2\mu}\,\eta^2 -\\
&\quad - 2i\,\frac{\lambda+\mu}{\lambda+2\mu}\,\alpha_+\alpha_-\eta, 0\Big),\\
u_2^{(1)}(\eta) &= \left(0, -i\,\frac{\lambda}{\lambda+2\mu}\,\eta\right).
\end{aligned}$$

The operator pencil $\mathcal{N}(ik, \partial_\eta)$ has no other eigenvalues on $\mathbf{R}$.

Therefore $u_0(z)$ is represented near the origin in the form

$$u_0(z) = \sum_{k=0}^{6} d_k Z_k(z) + u_1(z)$$

(cf. [8]). Here Z_k are linearly independent solutions of the equation $\Delta^* u = 0$ in a vicinity of the pick in Ω^c satisfying the boundary condition $Tu = 0$ near O on S. The last term u_1 and its gradient exponentially vanish as $z \to 0$. Three of the vector-valued functions Z_k, $1 \le k \le 6$, say Z_1, Z_2, Z_3, form rigid displacements. The others are represented in the form

$$\begin{aligned}
&Z_4(z) = x^{-1} + O(1),\ Z_5(z) = ix^{-3} + O(x^{-2}),\\
&Z_6(z) = ix^{-4} + O(x^{-3}).
\end{aligned}$$

Since the functions $w_1^-, w_2^-, w_3^-, Z_4, Z_5, Z_6$ have different orders of singularities and since $\sigma \in \mathfrak{M}$, we have

$$c_1 = c_2 = c_3 = d_4 = d_5 = d_6 = 0.$$

In particular,

$$u_0(z) = d_1 Z_1(z) + d_2 Z_2(z) + d_3 Z_3(z) + u_1(z).$$

Hence it follows that u_0 and its gradient ∇u_0 are bounded as $z \to 0$. On the other hand, according to (25) we have

$$u_0(z) - u_0(\infty) = O(|z|^{-1}), \quad \nabla u_0(z) = O(|z|^{-2}) \ \text{as} \ z \to \infty.$$

By the classical uniqueness theorem (cf. [16], Ch. I, Sect. 20) it follows that $u_0(z) = u_0(\infty)$, $z \in \Omega^c$. Thus

$$\sigma(p) = -u_0(p) + u^-(\infty) = \text{ const} \quad \text{on} \ S\backslash\{O\}.$$

Since the nonzero constant does not satisfy the homogeneous equation (24), it follows that $\sigma = 0$. $\square$

Consider the equation

$$2^{-1}\sigma - W\sigma - c_1\mathcal{A}_1 - c_2\mathcal{A}_2 - c_3\mathcal{A}_3 = -g \tag{29}$$

on S with respect to a density σ and a vector (c_1, c_2, c_3). The functions $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ have been defined in the Introduction.

Theorem 6. *Let Ω have an inward peak and let g belong to the class $\mathfrak{N}_\nu$, $\nu > 3$. Then equation (29) has a unique solution $\{\sigma, c_1, c_2, c_3\}$ in $\mathfrak{M} \times \mathbf{R}^3$, and the density σ can be represented in the form*

$$\sigma(z) = \Big(\alpha(\log x)^2 + \beta \log x + \gamma\Big)\, x^{-1/2} + O(x^{-\varepsilon}) \ \ \textit{on} \ \ S_\pm\,,$$

with small positive ε.

Proof. Let $U_{m,k}$, $m = 1, \ldots, 4$, and $k = 1, 2$, denote the solutions of the problem

$$\triangle^* u \,=\, 0 \ \text{ in } \ \Omega \cap B_r, \quad u = 0 \ \text{ on } \ B_r \cap S\backslash\{O\}$$

corresponding to the eigenvalues $k = i\ell/2$, $\ell \in \mathbf{Z}$, $\ell \neq 0$, of the operator pencil introduced in Theorem 2. Here $B_r = \{|z| < r\}$ with small positive r. We normalize the functions $U_{m,k}$ in the following way:

$$\begin{aligned}
U_{11}(z) = \ & \frac{i}{2\mu}\Big[2\kappa \operatorname{Im}\, z^{1/2} - \overline{z^{-1/2}} \operatorname{Im}\, z\Big] + \\
& + i\,\frac{(\kappa-1)(\alpha_+ - \alpha_-)}{8\pi\kappa\mu}\Big[2\kappa \operatorname{Im}\,(z^{3/2}\log z) - 3\overline{z^{1/2}\log z} \operatorname{Im}\, z - \\
& - 2\overline{z^{1/2}} \operatorname{Im}\, z\Big] - i\,\frac{(\kappa-1)\alpha_+}{2\mu}\overline{z^{3/2}} + O(z^{5/2}(\log z)^2),
\end{aligned}$$

$$\begin{aligned}
U_{12}(z) = \ & -\frac{1}{2\mu}\Big[2\kappa \operatorname{Im}\, z^{1/2} + \overline{z^{-1/2}} \operatorname{Im}\, z\Big] - \\
& - \frac{(\kappa+1)(\alpha_+ - \alpha_-)}{8\pi\kappa\mu}\Big[2\kappa \operatorname{Im}\,(z^{3/2}\log z) + 3\overline{z^{1/2}\log z} \operatorname{Im}\, z + \\
& + 2\overline{z^{1/2}} \operatorname{Im}\, z\Big] + \frac{(\kappa+1)\alpha_+}{2\mu}\,\overline{z^{3/2}} + O(z^{5/2}(\log z)^2),
\end{aligned}$$

$$U_{21}(z) = i\,\frac{\kappa-1}{\mu}\,\mathrm{Im}\ z + i\,\frac{(\kappa-1)(\alpha_+-\alpha_-)}{4\pi\mu\kappa}\,\Big[2\kappa\,\mathrm{Im}\,(z^2\log z) -$$
$$- 4\overline{z\log z}\,\mathrm{Im}\ z - 2\overline{z}\,\mathrm{Im}\ z\Big] - i\,\frac{(\kappa-1)\alpha_+}{\mu}\,\overline{z^2} + O(z^3\log^2 z),$$

$$U_{22}(z) = -\frac{\kappa+1}{\mu}\ \mathrm{Im}\ z - \frac{(\alpha_+-\alpha_-)(\kappa+1)}{4\pi\mu\kappa}\Big[2\kappa\,\mathrm{Im}\,(z^2\log z) +$$
$$+ 4\overline{z\log z}\ \mathrm{Im}\ z + 2\overline{z}\,\mathrm{Im}\ z\Big] + \frac{(\kappa+1)\alpha_+}{\mu}\,\overline{z^2} + O(z^3(\log z)^2),$$

$$U_{31}(z) = \frac{i}{2\mu}\Big[2\kappa\,\mathrm{Im}\ z^{3/2} - 3\overline{z^{1/2}}\,\mathrm{Im}\ z\Big] + O(z^{5/2}\log z),$$
$$U_{32}(z) = -\frac{1}{2\mu}\Big[2\kappa\,\mathrm{Im}\ z^{3/2} + 3\overline{z^{1/2}}\,\mathrm{Im}\ z\Big] + O(z^{5/2}\log z),$$
$$U_{41}(z) = \frac{i}{\mu}\Big[\kappa\,\mathrm{Im}\ z^2 - 2\overline{z}\,\mathrm{Im}\ z\Big] + O(z^3\log z),$$
$$U_{42}(z) = -\frac{1}{\mu}\Big[\kappa\,\mathrm{Im}\ z^2 + 2\overline{z}\,\mathrm{Im}\ z\Big] + O(z^3\log z),$$

where $\alpha_\pm = \kappa''_\pm(0)/2$. According to Theorem 2, the solution of the problem $\mathcal{D}^+$ with the boundary data g admits the representation

$$u^+(z) = \sum_{m=1}^{4}\ \sum_{k=1}^{2} d_{mk}U_{mk}(z) + W(z)$$

(cf. [5], [8]), where coefficients $d_{m,k}$ are defined uniquely and

$$W(z) = O(|z|^{2+\varepsilon}), \quad \varepsilon > 0.$$

We set

$$c_1 = d_{11},\ c_2 = (Qd_{12} + d_{31})/(Q^2+1),\ \ c_3 = d_{22}.$$

Then the displacement vector

$$u^+(z) - \sum_{k=1}^{3} c_k\mathcal{A}_k(z)$$

is the sum of a linear combination of the functions

$$U_{12}(z) - QU_{31}(z), U_{21}(z), U_{32}(z), U_{41}(z), U_{42}(z)$$

and the function $W_1(z)$ admitting the estimate $O(|z|^{2+\varepsilon})$. According to Theorem 3 (see also Corollaries 3.1 and 3.2), the problem $\mathcal{N}^-$ with the boundary data

$$T\left(u^+ - \sum_{k=1}^{3} c_k\mathcal{A}_k\right)$$

has a solution whose trace on $S\backslash\{O\}$ is in the class $\mathfrak{M}$.

Now, when the vector $c = (c_1, c_2, c_3)$ is chosen, by U^+ and U^- we denote solutions of the problems $\mathcal{D}^+$ and $\mathcal{D}^-$ with the data

$$g - \sum_{k=1}^{3} c_k \mathcal{A}_k \quad \text{on } S\backslash\{O\}.$$

Then TU^+ admits the estimate $O\left(x^{-1/2}\right)$ on $S\backslash\{O\}$. According to Proposition 2 the displacement vector $U^-(z)$ is bounded at infinity and

$$\int_S \left|TU^-(p)\right||p|ds_p < +\infty\,.$$

Hence (cf. [7], Ch. 5, Sect. 1) we obtain

$$g(p) - \sum_{k=1}^{3} c_k \mathcal{A}_k(p) = V\left(TU^+ - TU^-\right)(p) + U^e(\infty), \;\; p \in S\backslash\{O\}.$$

Let V^- be a solution of the problem $\mathcal{N}^-$:

$$V^- = 0 \text{ in } \Omega^c, \quad TV^- = TU^+ \text{ on } S\backslash\{O\}, \;\; V^-(\infty) = 0\,.$$

Applying Betti's formula to $w = V^- - U^- + U^-(\infty)$ in Ω^c and using the jump relation for Ww we obtain

$$\begin{aligned} w(p) - 2Ww(p) &= -\,2V\left(TU^+ - TU^-\right) = \\ &= -2g(p) + 2\sum_{k=1}^{3} c_k \mathcal{A}_k(p) + 2U^-(\infty), \;\; p \in S\backslash\{O\}. \end{aligned}$$

Hence the function

$$\sigma(p) = w(p) - U^-(\infty) = V^-(p) - g(p) + \sum_{k=1}^{3} c_k \mathcal{A}_k(p), \;\; p \in S\backslash\{O\}$$

with chosen c_1, c_2, c_3 is a solution of equation (29) in the class $\mathfrak{M}$. According to Theorems 2 and 3 σ has the required asymptotics and according to Theorem 5 the obtained solution (σ, c) is unique. $\square$

3.2. Integral equation for the problem $\mathcal{N}^-$. We say that a function $v(z)$, $z \in \Omega$, belongs to the space $\mathring{W}_2^1$ outside the peak if for any C_0^∞-function κ such that $O \notin \operatorname{supp}\kappa$ one has $\kappa v \in \mathring{W}_2^1(\Omega)$.

Lemma 2. *Let Ω have an outward peak. If a vector-valued function $v(z)$, $z \in \Omega$, belongs to $\mathring{W}_2^1$ outside the peak, satisfies $\triangle^* v = 0$ in a vicinity of the peak, and $v(z)$ and $\nabla v(z)$ grow not faster than a power function, then there exists $\beta < 0$ such that $v(z)$ and $\nabla v(z)$ admit the estimate $O\Big(\exp(\beta \operatorname{Re} \frac{1}{z})\Big)$ as $z \to 0$.*

Proof. The function w defined by $w(\zeta) = (\kappa v)(\zeta^{-1})$ satisfies $(\triangle^* + L)w = 0$ in $\Lambda = \{\zeta : \zeta^{-1} \in \Omega\}$ and $w = 0$ on $\partial\Lambda$. Here κ belongs to $C_0^\infty(\mathrm{R}^2)$ and is equal to 1 in a neighborhood of the peak and L is a second order differential operator with coefficients vanishing as $\zeta \to \infty$. For small negative β we have $w \in \mathring{W}^1_{2,\beta}(\Lambda)$. Hence and by the local estimate (9) it follows that $w \in W^3_{2,\beta} \cap W^1_{2,\beta}(\Lambda)$.

The operator $\mathcal{D}(\partial_\xi, \partial_\eta)$ of the boundary value problem

$$\triangle^* u = f \text{ in } \Pi = \Big\{(\xi,\eta) : -\kappa''_+(0)/2 < \eta < -\kappa''_-(0)/2\Big\}, \quad u = 0 \text{ on } \partial\Pi$$

continuously maps $W^2_{2,\beta} \cap \mathring{W}^1_{2,\beta}(\Pi)$ into $L_{2,\beta}(\Pi)$. The eigenvalues of the operator pencil

$$\mathcal{D}(ik, \partial_\eta) : W^2_2 \cap \mathring{W}^1_2(-\kappa''_+(0)/2, -\kappa''_-(0)/2) \to L_2(-\kappa''_+(0)/2, -\kappa''_-(0)/2)\,.$$

are nonzero solutions of the equation

$$\alpha^2 k^2 = \kappa\,(\sinh k\,\alpha)^2$$

where $\alpha = (\kappa''_+(0) - \kappa''_-(0))/2$ and $\kappa = (\lambda + 3\mu)/(\lambda + \mu)$. We assume that there are no eigenvalues in the strip $\{k : \beta < \operatorname{Im} k < -\beta\}$. Therefore w belongs to $W^3_{2,-\beta} \cap \mathring{W}^1_{2,-\beta}(\Lambda)$ (cf. [5], [8]). The Sobolev embedding theorem implies that w and ∇w admit the estimate $O\Big(\exp(\beta\xi)\Big)$ as $\xi \to \infty$. $\square$

Theorem 7. *Let Ω have an inward peak. Then the homogeneous integral equation*

$$-2^{-1}\tau + TV\tau = 0 \tag{30}$$

has only the trivial solution in the class $\mathfrak{M}$.

Proof. Let $\tau \in \mathfrak{M}$ satisfy (30). Integrating this equation over S we obtain

$$\int_S \tau\, ds = 0.$$

Hence the potential $v(z) = V\tau(z)$ is a solution of the problem $\mathcal{N}^+$ with the boundary data τ, that is

$$Tv(p) = \tau(p), \quad p \in S\backslash\{O\}.$$

By the Betti integral representation for v we obtain $Wv = 0$ in Ω. Therefore the density v of this double-layer potential is a solution of the homogeneous integral equation of the problem $\mathcal{D}^+$

$$(1 - 2W)\,v(p) = 0, \quad p \in S\backslash\{O\}.$$

Since the restriction of v to $S\backslash\{O\}$ belongs to $\mathfrak{M}$, it follows by Theorem 5 that

$$v(p) = 0 \text{ for } p \in S\backslash\{O\}.$$

Since $v(z)$ and $\nabla v(z)$ vanish at infinity and grow not faster than a power function as $z \to 0$, Lemma 2 implies that $v(z) = O\Big(\exp(\beta \operatorname{Re}\frac{1}{z})\Big)$ with negative β. According to the classical uniqueness theorem we have $v = 0$ in Ω^c (cf. [16]).

We make the change of the variable $t = \log 1/r$ and denote by $v^{(1)}(z)$, $v^{(2)}(z)$ the components of the displacement vector $v(z)$, $z \in \Omega$. Then the vector-valued function $U(t,\theta) = v(e^{-t},\theta)e^{-it}$, where r, θ are the polar coordinates of (x,y), is a solution of the equation $(\triangle^* + K)U = 0$ in the image Λ of Ω satisfying $U = 0$ at $\partial\Lambda$. Here K is the first order differential operator

$$K = \begin{pmatrix} -\lambda + 2\mu, & -(\lambda+3\mu)\partial/\partial\theta \\ (\lambda+3\mu)\partial/\partial\theta & -\mu \end{pmatrix}.$$

The vector-valued function v belongs to $\mathring{W}^1_{2,\beta}(\Lambda)$ with negative β. From the local estimate (11) it follows that v is an element of $W^2_{2,\beta}(\Lambda)$. Consider the operator $\mathcal{D}(\partial_t, \partial_\theta) : W^2_{2,\beta} \cap \mathring{W}^1_{2,\beta}(\Pi) \to L_{2,\beta}(\Pi)$ of the boundary value problem

$$(\triangle^* + K)U = F \ \text{ in } \Pi = \{(t,\theta) : \ 0 < \theta < 2\pi, t \in \mathbf{R}\}, \ \ U = 0 \ \text{ on } \partial\Pi,$$

and introduce the operator pencil $\mathcal{D}(ik, \partial_\theta) : k \in \mathbf{C}$, continuously mapping $W^2_2 \cap \mathring{W}^1_2(0, 2\pi)$ into $L_2(0,2\pi)$ and considered in Theorem 2. Then the displacement vector $v(z)$ is represented as a linear combination of linearly independent nonzero solutions of the homogeneous problem $\mathcal{D}^+$ (cf. [8]). Since the displacement vector $v(z)$, $z \in \Omega$, admits the estimate $O(|z|^\beta)$, $\beta > -1$, one has

$$v(z) = d_1\zeta_1(z) + d_2\zeta_2(z),$$

where ζ_1 and ζ_2 are solutions of the homogeneous problem $\mathcal{D}^+$ admitting the estimate $O(|z|^{-1/2})$.

From the limit relations for $V\tau$ it follows

$$\tau(p) = d_1 T\zeta_1(p) + d_2 T\zeta_2, \ \ p \in S\backslash\{O\}.$$

Since the stresses $T\zeta_k$, $k = 1,2$, do not belong to $\mathfrak{M}$ and are of different order and since $\tau \in \mathfrak{M}$, the coefficients d_1 and d_2 are zero. □

Theorem 8. *Let Ω have an inward peak and let h belong to $\mathfrak{N}_\nu$, $\nu > 3$, and satisfy*

$$\int_S h ds = 0.$$

Then the integral equation

$$-2^{-1}\tau + TV\tau = h$$

has a unique solution in the class $\mathfrak{M}$. The solution τ can be represented as

$$\tau(z) = \alpha_\pm x^{-1/2} + O(1).$$

Proof. Let v^- denote a solution of the problem $\mathcal{N}^-$ with the boundary data h on S, vanishing at infinity. According to Theorem 3, v^- has the representation

$$v^-(z) = (\alpha^{(0)} + \alpha^{(1)} \log x)x^{\nu-2} + O(x^{\nu-1-\varepsilon}) \ \text{ on } S.$$

By Theorem 2 the displacement vector u^+, equal to v^- on $S\backslash\{O\}$, is given by

$$u^+(z) = \frac{1}{2\mu}\left[\kappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}\right],$$

where $\varphi(z)$ and $\psi(z)$ admit the representations

$$\varphi(z) = \beta z^{1/2} + O(z), \quad \psi(z) = \gamma z^{1/2} + O(z).$$

Therefore we have

$$Tu^i(z) = \alpha_{\pm} x^{-1/2} + O(1) \ \text{ on } \ S\backslash\{O\}.$$

Hence

$$v^- = V(Tu^+ - h) \ \text{ on } \ S\backslash\{O\}.$$

Let us show that the same equality is valid in Ω^c. We put

$$v(z) = v^-(z) - V(Tu^+ - h)(z), \ \ z \in \Omega^c.$$

As in Theorem 2, we can prove that $\nabla v(z)$ admits the estimate $O(|z|^{-2})$ at infinity. Hence and by Lemma 2 it follows that v belongs to $\mathring{W}^{1,\rho}(\Omega^c)$ with $\rho(z) = (1+|z|)^2$. Since the problem $\mathcal{D}^-$ is uniquely solvable in $\mathring{W}^{1,\rho}(\Omega^c)$ (cf. [4]), v vanishes in Ω^c. So the density

$$\tau = Tu^+ - h$$

satisfies the integral equation of the problem $\mathcal{N}^-$. According to Theorem 7 the solution in the class $\mathfrak{M}$ is unique. □

4. Integral Equations on the Contour with an Outward Peak

4.1. Integral equation for the problem $\mathcal{D}^+$.

Theorem 9. *Let Ω have an outward peak. The homogeneous integral equation* (1) *has a two-dimensional space of solutions in $\mathfrak{M}$.*

Proof. By Theorem 4 the homogeneous problem $\mathcal{N}^-$ in the class of functions admitting the estimate $O\Big(|z|^{\beta}\Big)$, $\beta > -1$, has a nonzero solution vanishing at infinity and satisfying

$$\zeta(z) = cz^{-1/2} + O(1).$$

Furthermore, the function ζ span a two-dimensional real linear space. From the integral representation

$$\zeta(z) = (W\zeta)(z) - (VT\zeta)(z), \ \ z \in \Omega^c,$$

by the jump formula for $W\zeta$ we get

$$\zeta(p) - 2(W\zeta)(p) = -2(VT\zeta)(p) = 0, \ \ p \in S\backslash\{O\}.$$

Thus, $\zeta(p)$ is a solution of the homogeneous integral equation

$$(1 - 2W)\,\sigma = 0 \tag{31}$$

for the problem $\mathcal{D}^+$.

Now, let $\sigma \in \mathfrak{M}$ be a non-trivial solution of equation (31). The potential $W\sigma$ and $\nabla W\sigma$ in Ω have a power growth as $z \to 0$. According to Lemma 2, $W\sigma(z) = O\Big(\exp(\beta \operatorname{Re} \frac{1}{z})\Big)$ with negative β and therefore $W\sigma \in \mathring{W}^1_2(\Omega)$. Since the homogeneous problem $\mathcal{D}^+$ has only the trivial solution in $\overset{\circ}{W}{}_2^1(\Omega)$, we obtain that $W\sigma$ vanishes in Ω.

Let u^- denote the solution of the problem $\mathcal{D}^-$ with the boundary function σ. We have

$$VTu^- - Wu^- = u^-(\infty) \ \text{ in } \ \Omega.$$

Here, by $V\tau(z)$ we denote the simple-layer potential with the kernel

$$\Gamma(z,q) - \Gamma(z,0), \quad q \in S\backslash\{O\}.$$

Since Wu^- vanishes in Ω, we have $VTu^-(z) = u^-(\infty)$, $z \in \Omega$. The limit relations for the simple-layer potential imply

$$T\Big(VTu^- + \frac{1}{2}u^-\Big)(p) = 0, \ p \in S\backslash\{O\}.$$

So one has

$$VTu^-(z) = -u^-(z) + u_0(z) \ \text{ in } \ \Omega^c, \tag{32}$$

where u_0 is a displacement vector satisfying the boundary condition

$$Tu_0(p) = 0 \ \text{ on } \ S\backslash\{O\}.$$

From the integral representation of u^- and (32) it follows

$$Wu^-(z) = u_0(z) - u^-(\infty), \quad z \in \Omega^c.$$

Since the potential Wu^- vanishes at infinity, we have

$$u_0(\infty) = u^-(\infty).$$

The jump formula for Wu^- implies

$$\sigma(p) = u^-(p) = u_0(p) - u_0(\infty), \quad p \in S\backslash\{O\}.$$

Since σ belongs to the class $\mathfrak{M}$, from (32) it follows that u_0 and its gradient ∇u_0 grow not faster than a power function as $z \to 0$. For any function $\rho \in C_0^\infty(\mathbf{R}^2)$, satisfying $O \notin \operatorname{supp} \rho$, we have that ρu_0 belongs to $W_2^1(\Omega^c)$.

From the estimate $\nabla u_0(z) = O(|z|^{-2})$ $(z \to \infty)$ and Betti's formula we obtain

$$\int\limits_{\{z\in\Omega^c,\ |z|=r\}} Tu_0(z)ds = 0\,, \quad 0 < r < \delta\,.$$

We assume that the Airy function U generated by the displacement vector u_0 in the domain $\Omega_\delta = \Omega^c \cap \{|z| < \delta\}$ satisfies

$$\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y} = 0 \ \text{ on } \ S_\pm.$$

In particular, U is constant on S_+ and S_-.

We make the change of the variable $t = \log(1/r)$, $r = |z|$. Let Λ be the image of Ω_δ under the mapping $(r,\theta) \to (t,\theta)$, where r, θ are polar coordinates of

(x, y). By Γ_+, Γ_- we denote the images of S_+ and S_-. Then $W(t, \theta)$, equal to $U(e^{-t}, \theta)$, is a solution of the equation

$$\left[\left(\frac{\partial}{\partial t}+2\right)^2+\frac{\partial^2}{\partial \theta^2}\right]\left[\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial \theta^2}\right] W(t, \theta)=0 \quad \text{in } \Lambda$$

and satisfies $W = c_\pm$, $\partial W/\partial n = 0$ on $\Gamma_\pm$. Moreover, the function W belongs to $W^2_{2,\gamma}(\Lambda)$ with negative γ. From the local estimate (28) it follows that W belongs to $W^3_{2,\gamma}(\Lambda)$. Taking into account the relation between u_0 and U in Ω_δ we conclude that $W_0(t, \theta) = (\rho u_0)(e^{-t}, \theta) \cdot e^{-i\theta}$ belongs to $W^2_{2,\beta}(\Lambda)$ with negative β.

Let $\mathcal{D}(\partial_t, \partial_\theta)$ denote the operator of the boundary value problem

$$(\triangle^* + K)U = F \;\text{ in }\; \Pi = \{(t, \theta) : 0 < \theta < 2\pi, t \in \mathrm{R}\},\;\; TU = f \;\text{ on }\; \partial\Pi,$$

continuously mapping $W^2_{2,\beta}(\Pi)$ into $L_{2,\beta}(\Pi) \times W^{1/2}_{2,\beta}(\partial\Pi)$. Here K has the form

$$K = \begin{pmatrix} -\lambda + 2\mu & -(\lambda + 3\mu)(\partial/\partial\theta) \\ (\lambda + 3\mu)(\partial/\partial\theta) & -\mu \end{pmatrix}.$$

The eigenvalues of the operator pencil $\mathcal{D}(ik, \partial_\theta) : W^2_2(0, 2\pi) \to L_2(0, 2\pi) \times \mathbf{R}^2$ are numbers of the form $k = i\ell/2$, $\ell \in \mathbf{Z}$, and they have multiplicity two. Therefore W_0 admits the representation

$$W_0 = \sum_{k=1}^{p} c_k V_k + V_0$$

(cf. [8], Theorem 6.2), where p is the largest integer not exceeding 2β, V_k are linear independent vector-valued functions satisfying the equation $(\triangle^* + K)U = 0$ in $\Lambda_R = \Lambda \cap \{t > R\}$ and the boundary condition $TU = 0$ on $\partial\Lambda \cap \{t > R\}$. Moreover, we have $V_k \notin W^2_{2,\gamma}(\Lambda_R)$ and $V_0 \in W^2_{2,\gamma}(\Lambda_R)$ for $\gamma \in (-1/2, 0)$. Making the inverse change of the variable $r = e^{-t}$ we obtain

$$u_0(z) = (2\mu)^{-1}(\kappa\varphi_*(z) - z\overline{\varphi'_*(z)} - \overline{\psi_*(z)}) + O(|z|^{-\varepsilon}),$$

where $\varepsilon > -1/2$, and

$$\varphi_*(z) = \sum_{k=1}^{p} R^{(k)}_\varphi\left(\log\frac{zz_0}{z_0 - z}\right)\left(\frac{zz_0}{z_0 - z}\right)^{(k-p-1)/2},$$

$$\psi_*(z) = \sum_{k=1}^{p} R^{(k)}_\psi\left(\log\frac{zz_0}{z_0 - z}\right)\left(\frac{zz_0}{z_0 - z}\right)^{(k-p-1)/2}.$$

Here $R^{(k)}_\varphi$ and $R^{(k)}_\psi$ are polynomials of degree $[(k-1)/2]$, and z_0 is a fixed point in Ω. Since $\sigma \in \mathfrak{M}$, we have $u_0(z) = O\Big(|z|^{-1/2}\Big)$. Thus, σ is the restriction to $S\backslash\{O\}$ of a solution of the homogeneous problem $\mathcal{N}^-$. □

Theorem 10. *Let Ω have an outward peak and let g belong to $\mathfrak{N}_\nu$, $\nu > 0$. Then equation* (1) *is solvable in the class $\mathfrak{M}$ and the homogeneous equation has a two dimensional space of solutions. Among solutions of* (1) *there exists only one with the representation*

$$\sigma(x) = \beta_\pm x^{\nu-1} + O(1)$$

if $\nu \neq 1/2$, and

$$\sigma(x) = \beta_\pm x^{-1/2} \log x + O(1)$$

if $\nu = 1/2$.

Proof. Let $V\sigma$ be a simple-layer potential with the kernel $\Gamma(z,q)$. Let u^+, u^- be solutions of the problems $\mathcal{D}^+$ and $\mathcal{D}^-$. Theorem 1 implies that

$$Tu^+(z) = \pm\beta_0 x^{\nu-2} + O(x^{\nu-1}), \quad z = x + iy \in \Omega.$$

According to Theorem 2, Tu^- is integrable on S and the displacement vector $u^-(z)$ is bounded at infinity. Then

$$g(p) = V\Big(Tu^+ - Tu^-\Big)(p) + u^-(\infty), \quad p \in S\backslash\{O\}.$$

Here the integral is understood in the sense of the principal value.

Let v^- denote the solution of the problem $\mathcal{N}^-$ with the boundary data Tu^+. According to Theorem 4, we have

$$v^-(z) = O(x^{\nu-1}) \text{ for } \nu \neq 1/2,$$
$$v^-(z) = O(x^{-1/2}\log x) \text{ for } \nu = 1/2, \quad z \in S\backslash\{O\}.$$

Using the integral representation for $w = v^- - u^- + u^-(\infty)$ in Ω^c and the jump formula for Ww we obtain that

$$w - 2Ww = -2V\Big(Tu^+ - Tu^-\Big) = -2g + 2u^-(\infty) \text{ on } S\backslash\{O\}.$$

Thus the displacement vector $\sigma = w - u^-(\infty) = v^- - g$ is a solution of equation (1).

By Theorem 9 the restrictions of solutions of the homogeneous problem $\mathcal{N}^-$ to $S\backslash\{O\}$ (and only such functions) satisfy the homogeneous integral equation of the problem $\mathcal{D}^+$. The space of such solutions in $\mathfrak{M}$ is two-dimensional. We choose two linear independent solutions ζ_R and ζ_I such that

$$\zeta_R(z) = \mp\frac{\kappa+1}{\mu}\, x^{-1/2} + O(1),$$
$$\zeta_I(z) = \mp\frac{\kappa+1}{\mu}\, i\, x^{-1/2} + O(1), \quad z = x + iy \in S\backslash\{O\}.$$

Therefore solutions of equation (1) in $\mathfrak{M}$ have the form

$$\sigma = v^- - g - (\operatorname{Re} c)\zeta_R + (\operatorname{Im} c)\zeta_I\,.$$

The complex constant c can be chosen so that σ has the required asymptotics. $\square$

4.2. Integral equation for the problem $\mathcal{N}^-$.

Theorem 11. *Let Ω have an outward peak. Then the homogeneous equation*

$$-2^{-1}\tau + TV\tau + t_1 T\varrho_1 + t_2 T\varrho_2 = 0 \tag{33}$$

has only the trivial solution in the Cartesian product $\mathfrak{M} \times \mathbf{R}^2$.

Proof. Let τ be a solution of (33). We define the displacement vector $v(z)$ as follows:

$$v(z) = V\tau(z) + t_1\varrho_1(z) + t_2\varrho_2(z), \quad z \in \Omega^c.$$

By integrating (33) over the boundary S we find

$$\int_S \tau(p) ds_p = 0.$$

Therefore the potential $V\tau(z)$ takes the form

$$V\tau(z) = \int \{\Gamma(z,q) - \Gamma(z,0)\}\, \tau(q) ds_q\,.$$

Hence we have $v(z) = O(|z|^{-1})$, $\nabla v(z) = O(|z|^{-2})$, as $|z| \to \infty$. Therefore $v(z)$ is a solution of the problem $\mathcal{N}^-$ with zero boundary data and

$$\nabla v(z) = O(|z|^{\beta}), \quad \beta > -1, \quad \text{as } z \to 0.$$

The classical uniqueness theorem implies that

$$v(z) = \text{ const}, \quad z \in \Omega^c.$$

Since $v(z)$ vanishes at infinity, it follows that

$$V\tau(z) = -t_1\varrho_1(z) - t_2\varrho_2(z), \quad z \in \Omega^c.$$

We extend the functions $\varrho_k(z)$, $z \in S$, $k = 1, 2$, onto Ω as solutions of the homogeneous Lamé system. Let $\varrho_1^+(z)$ and $\varrho_2^+(z)$ denote these extensions. The vector-valued function

$$\varrho_0(z) = V\tau(z) + t_1\varrho_1^+(z) + t_2\varrho_2^+(z), \quad z \in S\backslash\{O\}$$

is bounded in Ω, vanishes on $S\backslash\{O\}$ and its gradient $\nabla\varrho_0$ grows slower than a power function. According to Lemma 2, $\rho_0(z)$ admits the estimate $O\Big(\exp(\beta \operatorname{Re} \frac{1}{z})\Big)$ with negative β as $z \to 0$. We have

$$V\tau(z) = -\sum_1^2 t_k\varrho_k^+(z) + \varrho_0(z).$$

The jump formula for the simple-layer potential entails

$$\tau(z) = -\sum_1^2 t_k T\varrho_k^+(z) + \sum_1^2 t_k T\varrho_k(z) + T\varrho_0(z), \quad z \in S.$$

We have $T\varrho_k(z) = O(|z|^{-1/2})$, $k = 0, 1$, and $T\varrho_1^+(z)$, $T\varrho_2^+(z)$ have order $-3/2$. Since the components of these functions have different singularity, t_1 and t_2 vanish. Thus, $V\tau(z) = 0$ in Ω^c. By Lemma 2 applied to $V\tau$ in Ω, and by the classical uniqueness theorem we can conclude that $V\tau(z) = 0,\ z \in \Omega$. So the density τ given by

$$\tau(z) = T(V\tau)^-(z) - T(V\tau)^+(z), \quad z \in S\backslash\{O\},$$

vanishes on S. $\square$

Let ζ_R and ζ_I be solutions of the homogeneous problem $\mathcal{N}^-$ introduced in Theorem 10.

Theorem 12. *Let Ω have an outward peak, let h belong to $\mathfrak{N}_\nu$, $\nu > 0$, and satisfy*

$$\int_S h ds = 0.$$

Then the integral equation (6) *has a unique solution in the direct product $\mathfrak{M}\times\mathbf{R}^2$ and this solution can be represented as follows:*

$$\begin{aligned}
\sigma(z) &= \beta_\pm x^{\nu-1} + O(x^{-1/2}) && \textit{for } 0 < \nu < 1/2,\\
\sigma(z) &= \Big(\gamma_\pm \log x + \beta_\pm\Big)x^{-1/2} + O(\log x) && \textit{for } \nu = 1/2,\\
\sigma(z) &= \gamma_\pm x^{-1/2} + \beta_\pm x^{\nu-1} + O(\log x) && \textit{for } 1/2 < \nu < 1,
\end{aligned}$$

where $z = x + iy \in S\backslash\{O\}$.

Proof. We consider only one of the statements of the theorem in detail. Let v^- denote a solution of the problem $\mathcal{N}^-$ with the boundary data h such that $v^-(\infty) = 0$.

We assume that $0 < \nu < 1/2$. From Theorem 4 it follows that v^- has the form

$$v^-(z) = \frac{1}{2\mu}\left[\kappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}\right] + O(z^{3/2}\log z),$$

where

$$\begin{aligned}
\psi(z) &= \delta_{0,0} + \delta_{1,0}\left(\frac{zz_0}{z-z_0}\right)^{1/2} + \delta_{2,0}\frac{zz_0}{z-z_0} + \beta_0\left(\frac{zz_0}{z-z_0}\right)^{1+\nu},\\
\psi(z) &= \varepsilon_{0,0} + \varepsilon_{1,0}\left(\frac{zz_0}{z-z_0}\right)^{1/2} + \varepsilon_{2,0}\frac{zz_0}{z-z_0} + \gamma_0\left(\frac{zz_0}{z-z_0}\right)^{1+\nu}, \quad z_0 \in \Omega.
\end{aligned}$$

Thus v^- admits the representation

$$v^e(z) = c_0 \pm c_1 x^{1/2} + c_2 x + d_\pm x^{1+\nu} + O(x^{3/2}\log x) \ \text{ on } S,$$

where

$$c_1 = (1+\kappa)\delta_{1,0}/2\mu, \quad c_2 = (1+\kappa)\delta_{2,0}/2\mu.$$

We apply Betti's formula to v^- and first to ζ_R and then to ζ_I in $\Omega_\varepsilon = \Omega^c\backslash\{|z| < \varepsilon\}$. Taking the limit as $\varepsilon \to 0$ we obtain

$$\operatorname{Re} c_1 = \frac{1}{4\pi}\int_S h\zeta_R\, ds, \quad \operatorname{Im} c_1 = \frac{1}{4\pi}\int_S h\zeta_I\, ds.$$

We set

$$t_1 = \frac{\mu}{1+\kappa}\,\frac{1}{2\pi}\int\limits_S h\,\zeta_R ds, \quad t_2 = \frac{\mu}{1+\kappa}\,\frac{1}{2\pi}\int\limits_S h\,\zeta_I ds.$$

Then the problem $\mathcal{N}^-$ with the boundary data $h - t_1 T\varrho_1 - t_2 T\varrho_2$ has a solution on $S_\pm$ of the form

$$v(z) = c_0 + c_2 x + d_\pm x^{1+\nu} + O(x^{3/2}\log x).$$

The solution of the problem $\mathcal{D}^+$ with the given displacement $c_2 z$ on S admits the estimate $O(z)$ and this estimate can be differentiated at least once. Hence and by Theorem 1 it follows that the solution u^+ of the the problem

$$\triangle u^+ = 0 \ \text{ in } \ \Omega, \quad u^+ = v^- \ \text{ on } \ S,$$

admits the representation

$$u^+(z) = \frac{1}{2\mu}\Big[\kappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}\Big] + O(z\log z).$$

differentiable at least once. Here

$$\varphi(z) = \beta_0 z^\nu + \beta_1 z^{1/2}, \ \ \psi(z) = \gamma_0 z^\nu + \gamma_1 z^{1/2}.$$

Therefore the stress Tu^+ has the following representation on $S_\pm$:

$$Tu^+(z) = \beta_\pm x^{\nu-1} + O(x^{-1/2}), \ \ z = x + iy.$$

On S, we have

$$v = V\Big(Tu^+ - h + t_1 T\varrho_1 + t_2 T\varrho_2\Big)\,. \tag{34}$$

The simple-layer potential VTu^+ is bounded in a neighborhood of the origin and its gradient admits the estimate $O(|z|^{\nu-1})$. Hence and by a classical uniqueness theorem we obtain that (34) remains valid in Ω^c. Consequently, the density

$$\tau = Tu^+ - h + t_1 T\varrho_1 + t_2 T\varrho_2$$

satisfies the integral equation of the problem $\mathcal{N}^-$ and has the required asymptotics. Theorem 11 implies that the constructed solution of equation (6) is unique.

The cases $\nu = 1/2$ and $1/2 < \nu < 1$ can be studied in a similar way. □

Applying Theorems 11 and 12, we obtain the following result.

Theorem 13. *Let Ω have an outward peak. Furthermore, let h belong to $\mathfrak{N}_\nu$, $\nu > 0$, and satisfy*

$$\int\limits_S h ds, \quad \int\limits_S h\zeta_R ds = 0 \quad \textit{and} \quad \int\limits_S h\zeta_I ds = 0.$$

Then equation (2) *is uniquely solvable in* $\mathfrak{M}$.

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