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BOUNDARY VARIATIONAL INEQUALITY APPROACH IN THE ANISOTROPIC ELASTICITY FOR THE SIGNORINI PROBLEM

A. GACHECHILADZE AND D. NATROSHVILI

Abstract. The purpose of the paper is reducing the three-dimensional Signorini problem to a variational inequality which occurs on the two-dimensional boundary of a domain occupied by an elastic anisotropic body. The uniqueness and existence theorems for the solution of the boundary variational inequality are proved and a boundary element procedure together with an abstract error estimate is described for the Galerkin numerical approximation.

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1. Introduction

Signorini type problems in the elasticity theory are well studied (see [11], [12], [21], [7], [19], and the references therein). The main tool to investigate these problems is the theory of spatial variational inequalities. By this method the uniqueness and existence theorems are proved and the regularity properties of solutions are established in various functional spaces.

The purpose of the present paper is reducing of the three-dimensional Signorini problem to a variational inequality which occurs on the two-dimensional boundary of a domain occupied by the elastic *anisotropic* body under consideration. We will show that the *spatial variational inequality* (SVI) is equivalent to the *boundary variational inequality* (BVI) obtained. The uniqueness and existence theorems for the solution of BVI are proved, and a boundary element procedure together with an abstract error estimate is described for the Galerkin numerical approximation.

A similar approach in the *isotropic* case is considered in [16], [17], [15], where the Signorini problem is reduced to a system consisting of a BVI and a boundary singular integral equation (for related problems see also [9], [8] and the references therein).

2. Classical and Spatial Variational Inequality Formulations of the Signorini Problem

2.1. Let an elastic homogeneous anisotropic body in the natural configuration

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occupies a bounded region $\overline{\Omega^+}$ of the three-dimensional space $\mathbb{R}^3$: $\overline{\Omega^+} = \Omega^+ \cup S$, $S = \partial\Omega^+$. For simplicity, we assume that the boundary $S = \partial\Omega^+$ is C^∞-smooth. Further, let $\Omega^- = \mathbb{R}^3 \backslash \overline{\Omega^+}$. The elastic coefficients C_{kjpq} satisfy the symmetry conditions

$$C_{kjpq} = C_{pqkj} = C_{jkpq}.$$

The stress tensor $\{\sigma_{kj}\}$ and the strain tensor $\{\varepsilon_{kj}\}$ are related by Hooke's law

$$\sigma_{kj}(u) = C_{kjpq}\varepsilon_{pq}(u), \ \varepsilon_{kj}(u) = \frac{1}{2}\left(\partial_k u_j + \partial_j u_k\right),$$
$$\partial = \left(\partial_1, \partial_2, \partial_3\right), \ \partial_k := \frac{\partial}{\partial x_k},$$

where $u = (u_1, u_2, u_3)^\top$ is the displacement vector; here and in what follows the summation over the repeated indices is meant from 1 to 3 unless stated otherwise; the subscript $\top$ denotes transposition.

Define the symmetric bilinear forms:

$$E(u,v) = \sigma_{kj}(u)\varepsilon_{kj}(v) = C_{kjpq}\varepsilon_{pq}(u)\varepsilon_{kj}(v) = E(v,u), \tag{2.1}$$

$$a(u,v) = \int\limits_{\Omega^+} E(u,v)dx = a(v,u) = \int\limits_{\Omega^+} C_{kjpq}\, \partial_k u_j\, \partial_p v_q\, dx. \tag{2.2}$$

As usual, the quadratic form corresponding to the potential energy is supposed to be positive definite in the symmetric variables $\varepsilon_{kj} = \varepsilon_{jk}$ (see, e.g., [12])

$$\begin{aligned} E(u,u) = \sigma_{kj}(u)\varepsilon_{kj}(u) = C_{kjpq}\varepsilon_{kj}(u)\varepsilon_{pq}(u) \\ \geq \delta_1\, \varepsilon_{kj}(u)\, \varepsilon_{kj}(u), \quad \delta_1 = \text{const} > 0. \end{aligned} \tag{2.3}$$

Let the boundary S be divided into three disjoint open subsurfaces S_1, S_2, and S_3, where $S_k \cap S_j = \varnothing$ for $k \neq j$, $S_2 \neq \varnothing, S_3 \neq \varnothing$, $\overline{S}_1 \cup \overline{S}_2 \cup \overline{S}_3 = S$, $\overline{S}_k = S_k \cup \partial S_k$; for simplicity, we assume ∂S_k, $k = 1,2,3$, to be C^∞-smooth curves.

By $T(\partial, n)u$ we denote the stress vector acting on a surface element with the unit normal vector $n = (n_1, n_2, n_3)$:

$$[\,T(\partial_x, n(x))u(x)\,]_k = \sigma_{kj}(u)n_j(x) = C_{kjpq}n_j(x)\partial_q u_p(x), \quad k = 1,2,3. \tag{2.4}$$

We note that throughout the paper we will use the following notation (when it causes no confusion):

(a) if all elements of a vector $v = (v_1, \ldots, v_m)^\top$ (a matrix $N = [N_{kj}]_{m\times n}$) belong to one and the same space X, we will write $v \in X$ $(N \in X)$ instead of $v \in X^m$ $(N \in X_{m\times n})$;

(b) if $K:\ X_1 \times X_2 \times \cdots \times X_m \to Y_1 \times Y_2 \times \cdots \times Y_n$ and $X_1 = X_2 = \cdots = X_m$, $Y_1 = Y_2 = \cdots = Y_n$, we will write $K:\ X \to Y$ instead of $K:\ X^m \to Y^n$;

(c) if $a, b \in \mathbb{R}^m$, then $a \cdot b := \sum_{k=1}^m a_k b_k$ denotes the usual scalar product in $\mathbb{R}^m$;

(d) by $\|u; X\|$ we denote the norm of the element u in the space X.

As usual, $H^{\alpha}(\Omega^+)$, $H^{\alpha}_{loc}(\Omega^-)$ and $H^{\alpha}(S)$ denote the Sobolev-Slobodetski (Bessel potential) spaces; here α is a real number (see, e.g., [22], [31], [32]). By $\widetilde{H}^{\alpha}(S_k)$ we denote the subspace of $H^{\alpha}(S)$:

$$\widetilde{H}^{\alpha}(S_k) = \{\omega : \ \omega \in H^{\alpha}(S), \ \operatorname{supp} \omega \subset \overline{S}_k\},$$

while $H^{\alpha}(S_k)$ denotes the space of restriction on S_k of functionals from $H^{\alpha}(S)$

$$H^{\alpha}(S_k) = \{r_{S_k} f : \ f \in H^{\alpha}(S)\},$$

where r_{S_k} denotes the restriction operator on S_k.

2.2. The mathematical formulation of the typical Signorini problem reads as follows: Find the displacement vector $u = (u_1, u_2, u_3)^{\top} \in H^1(\Omega^+)$ by the following conditions:

$$A(\partial)u(x) = 0 \qquad \text{in} \quad \Omega^+, \tag{2.5}$$

$$[\,T(\partial_x, n(x))u(x)\,]^+ = g \quad \text{on} \quad S_1, \tag{2.6}$$

$$[\,u(x)\,]^+ = 0 \quad \text{on} \quad S_2, \tag{2.7}$$

$$\left.\begin{aligned} &[\,T(\partial, n)u(x)\,]^+ - n\,[\,T(\partial, n)u(x)\cdot n\,]^+ = 0,\\ &-[\,u(x)\cdot n\,]^+ \geq 0,\\ &-[\,T(\partial, n)u(x)\cdot n\,]^+ \geq 0,\\ &[\,T(\partial, n)u(x)\cdot n\,]^+\,[\,u(x)\cdot n\,]^+ = 0, \end{aligned}\right\} \quad \text{on} \ S_3, \tag{2.8}$$

where $A(\partial)$ is a matrix differential operator of elastostatics

$$A(\partial) = \,[\,A_{kp}(\partial)\,]_{3\times 3}, \quad A_{kp}(\partial) = C_{kjpq}\partial_j\partial_q,$$

the symbols $[\,\cdot\,]^{\pm}([\,\cdot\,]^{\pm}_S)$ denote limits (traces) on S from $\Omega^{\pm}$, $n = n(x)$ is the unit outward normal vector to S at the point $x \in S$.

Equation (2.5) corresponds to the equilibrium state of the elastic body in question (with bulk forces equal to zero). Condition (2.6) describes that the body is subjected to assigned surface forces on S_1, while (2.7) shows that the body is fixed along the subsurface S_2. The unilateral Signorini conditions ("ambiguous boundary conditions" – due to the original terminology of Signorini [11]) mean that the body remains on or "above" the portion S_3 of the boundary $\partial\Omega^+ = S$ (the "upper" direction on S is defined by the outward normal n).

The first equality in (2.8) means that tangent stresses vanish on S_3 (i.e., we have contact without friction with a rigid support along the S_3). The mechanical meaning of the last three conditions in (2.8) are described in detail, e.g., in [12], Part 2, Section 10.

We assume that the vector-function g in condition (2.6) belongs to the space $L_2(S_1) = H^0(S_1)$.

Note that if $u \in H^1(\Omega^+)$ and $A(\partial)u \in L_2(\Omega^+)$, then, in general, the limit $[\,T(\partial,n)u\,]^+_S$ is defined as a functional of the class $H^{-\frac{1}{2}}(S)$ defined by the duality relation (see, e.g., [22]),

$$\langle\,[\,T(\partial,n)u(x)\,]^+_S, [v]^+_S\rangle_S := \int\limits_{\Omega^+} A(\partial)u\cdot v\,dx + \int\limits_{\Omega^+} E(u,v)\,dx, \tag{2.9}$$

where $v=(v_1,v_2,v_3)^\top \in H^1(\Omega^+)$; here $\langle\cdot,\cdot\rangle_S$ is the duality between the spaces $H^{\frac{1}{2}}(S)$ and $H^{-\frac{1}{2}}(S)$, which coincides with the usual $[\,L_2(S)\,]^3$ scalar product for regular (in general, complex valued) vector-functions, i.e., if $f,h \in [\,L_2(S)\,]^3$, then

$$\langle f,h\rangle_S = \int\limits_S f_k h_k\,dS =: (f,\overline{h})_{L_2(S)},$$

where the over-bar denotes complex conjugation.

Equation (2.9) can be interpreted as a Green formula for the operator $A(\partial)$.

Due to the regularity results obtained in [19] if, in addition, $g=[G]^+_{S_1}$, where $G \in H^1(\Omega^+)$, then all conditions in (2.8) can be understood in the usual classical sense (see also [12], Part 2, Section 10).

2.3. The above-formulated Signorini problem is equivalent to the following spatial variational inequality (see [12], [19]): Find $u \in \mathbf{K}$ such that

$$a(u,v-u) \geq P(v-u), \qquad \forall v \in \mathbf{K}, \tag{2.10}$$

where the bilinear form $a(u,v)$ is given by (2.2),

$$\mathbf{K} = \Big\{u=(u_1,u_2,u_3)^\top \in H^1(\Omega^+)\ :\ [n(x)\cdot u(x)]^+ \leq 0 \ \text{ on } \ S_3 \\ \text{and } \ [u(x)]^+ = 0 \ \text{ on } \ S_2\Big\}, \tag{2.11}$$

and the linear functional P is defined by the equation

$$P(v) = \langle g,[v]^+\rangle_{S_1} = \int\limits_{S_1} g\cdot[v]^+ dS \tag{2.12}$$

with $g \in L_2(S_1)$.

In turn, the variational inequality (2.10) is equivalent to the minimization problem for the energy functional (see [12])

$$\mathcal{E}(v) = 2^{-1}\,a(u,v) - \langle g,[v]^+\rangle_{S_1}, \qquad \forall v \in \mathbf{K}. \tag{2.13}$$

Observe that the bilinear form $a(\cdot,\cdot)$ is coercive on the space

$$H^1(\Omega^+;S_2) := \Big\{v \in H^1(\Omega):\ [v]^+ = 0 \ \text{ on } \ S_2\Big\} \tag{2.14}$$

since the measure of the subsurface S_2 is positive (see, e.g., [29], [7]). Thus there exists a positive constant c_0 such that

$$a(u,u) \geq c_0\,\|u;\,H^1(\Omega^+)\|^2, \quad \forall u \in H^1(\Omega^+;S_2).$$

Therefore the variational inequality (2.10) together with problem (2.5)–(2.8) and the minimization problem for functional (2.13) is uniquely solvable (see, e.g., [12], [7], [19], [13]).

In the subsequent sections, on the basis of the potential theory, we will equivalently reduce the spatial variational inequality (2.10) to a boundary variational inequality.

First we expose the mapping and coercive properties of the integral (pseudodifferential) operators.

3. Properties of Boundary Integral (Pseudodifferential) Operators

3.1. Single- and double-layer potentials and their properties. Let $\Gamma(\cdot)$ be the fundamental matrix of the operator $A(\partial)$

$$A(\partial)\Gamma(x) = \delta(x)I,$$

where $\delta(\cdot)$ is the Dirac distribution and $I = [\delta_{kj}]_{3\times 3}$ is the unit matrix (δ_{kj} is the Kronecker symbol). This matrix reads [Na1]

$$\Gamma(x) = \mathcal{F}^{-1}_{\xi\to x}\,[\,A^{-1}(-i\xi)\,] = -\frac{1}{8\pi|x|}\int\limits_0^{2\pi} A^{-1}(a\eta)d\varphi, \tag{3.1}$$

where $A^{-1}(-i\xi)$ is the matrix inverse to $A(-i\xi)$, $\xi \in \mathbb{R}^3\backslash\{0\}$, $x \in \mathbb{R}^3\backslash\{0\}$, $a = [a_{kj}]_{3\times 3}$ is an orthogonal matrix with the property $a^\top x = (0,0,|x|)^\top$, $\eta = (\cos\varphi, \sin\varphi, 0)^\top$; $\mathcal{F}^{-1}_{\xi\to x}$ denotes the generalized inverse Fourier transform.

Further, let us introduce the single- and double-layer potentials

$$V(g)(x) = \int\limits_S \Gamma(x-y)\,g(y)\,dS_y, \tag{3.2}$$

$$W(g)(x) = \int\limits_S [T(\partial_y, n(y))\Gamma(x-y)]^\top\, g(y)\,dS_y, \tag{3.3}$$

where $g = (g_1, g_2, g_3)^\top$ is a density vector and $x \in \mathbb{R}^3\backslash\, S$.

The properties of these potentials and the corresponding boundary integral (pseudodifferential) operators in the Hölder ($C^{k+\alpha}$), Bessel potential (H^s_p) and Besov ($B^s_{p,q}$) spaces are studied in [20], [25], [2], [26], [27], [5], [6], [28] (see also [18], [3], [24], where the coerciveness of boundary operators and Lipschitz domains are considered).

In the sequel we need some results obtained in the above-cited papers and we recall them here for convenience.

Theorem 3.1 ([25], [2], [26]). *Let $k \geq 0$ be an integer and $0 < \gamma < 1$. Then the operators*

$$\begin{aligned} V &: C^{k+\gamma}(S) \to C^{k+1+\gamma}(\overline{\Omega^\pm}), \\ W &: C^{k+\gamma}(S) \to C^{k+\gamma}(\overline{\Omega^\pm}), \end{aligned} \tag{3.4}$$

are bounded.

For any $g \in C^{k+\gamma}(S)$ *and any* $x \in S$

$$
\begin{aligned}
[V(g)(x)]^{\pm} &= V(g)(x) = \mathcal{H}\, g(x), \\
[T(\partial_x, n(x))V(g)(x)]^{\pm} &= [\,\mp 2^{-1} I + \mathcal{K}\,]\, g(x) \\
[W(g)(x)]^{\pm} &= [\pm 2^{-1} I + \mathcal{K}^*]\, g(x), \\
[T(\partial_x, n(x))W(g)(x)]^{+} &= [T(\partial_x, n(x))W(g)(x)]^{-} = \mathcal{L}\, g(x), \quad k \geq 1,
\end{aligned}
\tag{3.5}
$$

where

$$\mathcal{H}\, g(x) := \int_S \Gamma(x-y)\, g(y)\, dS_y, \tag{3.6}$$

$$\mathcal{K}\, g(x) := \int_S T(\partial_x, n(x))\Gamma(y-x)\, g(y)\, dS_y, \tag{3.7}$$

$$\mathcal{K}^*\, g(x) := \int_S [T(\partial_y, n(y))\Gamma(x-y)]^{\top}\, g(y)\, dS_y, \tag{3.8}$$

$$\mathcal{L}\, g(x) := \lim_{\Omega^{\pm} \ni z \to x \in S} T(\partial_z, n(x)) \int_S [T(\partial_y, n(y))\Gamma(y-z)]^{\top}\, g(y)\, dS_y. \tag{3.9}$$

Theorem 3.2 ([5])**.** *Operators* (3.4) *can be extended by continuity to the bounded mappings*

$$
\begin{aligned}
V &: H^s(S) \to H^{s+1+\frac{1}{2}}(\overline{\Omega^+}) \quad [H^s(S) \to H^{s+1+\frac{1}{2}}_{loc}(\Omega^-)], \\
W &: H^s(S) \to H^{s+\frac{1}{2}}(\overline{\Omega^+}) \quad [H^s(S) \to H^{s+\frac{1}{2}}_{loc}(\Omega^-)],
\end{aligned}
$$

with $s \in \mathbb{R}$. *The jump relations* (3.5) *on* S *remain valid for the extended operators in the corresponding functional spaces.*

Theorem 3.3 ([25], [5])**.** *Let* $k \geq 0$ *be an integer,* $0 < \gamma < 1$, *and* $s \in \mathbb{R}$. *Then the operators*

$$
\begin{aligned}
\mathcal{H} &: C^{k+\gamma}(S) \to C^{k+1+\gamma}(S), \\
&: H^s(S) \to H^{s+1}(S),
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
\pm 2^{-1} I + \mathcal{K},\ \pm 2^{-1} I + \mathcal{K}^* &: C^{k+\gamma}(S) \to C^{k+\gamma}(S), \\
&: H^s(S) \to H^s(S),
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
\mathcal{L} &: C^{k+1+\gamma}(S) \to C^{k+\gamma}(S), \\
&: H^{s+1}(S) \to H^s(S)
\end{aligned}
\tag{3.12}
$$

are bounded.

Moreover,

(i) *the operators* $\pm 2^{-1} I + \mathcal{K}$ *and* $\pm 2^{-1} I + \mathcal{K}^*$ *are mutually adjoint singular integral operators of normal type with the index equal to zero. The operators* $\mathcal{H}$,

$2^{-1}I+\mathcal{K}$ *and* $2^{-1}I+\mathcal{K}^*$ *are invertible. The inverse of* $\mathcal{H}$

$$\mathcal{H}^{-1}: C^{k+1+\gamma}(S) \to C^{k+\gamma}(S) \quad [H^{s+1}(S) \to H^{s}(S)]$$

is a singular integro-differential operator;

(ii) *the* $\mathcal{L}$ *is a singular integro-differential operator and the following equalities hold in appropriate functional spaces:*

$$\mathcal{K}^*\mathcal{H} = \mathcal{H}\mathcal{K},\ \mathcal{L}\mathcal{K}^* = \mathcal{K}\mathcal{L},\ \mathcal{H}\mathcal{L} = -4^{-1}I + (\mathcal{K}^*)^2,\ \mathcal{L}\mathcal{H} = -4^{-1}I + \mathcal{K}^2; \tag{3.13}$$

(iii) *The operators* $-\mathcal{H}$ *and* $\mathcal{L}$ *are self-adjoint and non-negative elliptic pseudodifferential operators with the index equal to zero:*

$$\langle -\mathcal{H}h, h\rangle_S \geq 0, \qquad \langle \mathcal{L}g, g\rangle_S \geq 0, \tag{3.14}$$

$$\forall h \in C^{\gamma}(S),\ \forall g \in C^{1+\gamma}(S),\ \left[\forall h \in H^{-\frac{1}{2}}(S),\ \forall g \in H^{\frac{1}{2}}(S)\right],$$

with equality only for $h=0$ *and for*

$$g = [a \times x] + b, \tag{3.15}$$

respectively; here $a, b \in \mathbb{R}^3$ *are arbitrary constant vectors and* $[\cdot \times \cdot]$ *denotes the cross product of two vectors;*

(iv) *a general solution of the homogeneous equations* $[-2^{-1}I+\mathcal{K}^*]g=0$ *and* $\mathcal{L}g=0$ *is given by* (3.15)*, i.e.,*

$$\begin{aligned}
&\ker \mathcal{L} = \ker\left(-2^{-1}I + \mathcal{K}^*\right) \quad \textit{and}\\
&\dim\ker \mathcal{L} = \dim\operatorname{coker}\mathcal{L} = \dim\ker\left(-2^{-1}I + \mathcal{K}^*\right)\\
&\qquad = \dim\operatorname{coker}\left(-2^{-1}I + \mathcal{K}^*\right) = 6.
\end{aligned}$$

Theorem 3.4 ([5]**, citeNa1).** *Let* u *be a solution of the homogeneous equation* $A(D)u = 0$ *in* $\Omega^{\pm}$*. Then*

$$W\left([\,u\,]^{\pm}\right) - V\left([\,Tu\,]^{\pm}\right) = \begin{cases} \pm u(x), & x \in \Omega^{\pm},\\ 0, & x \in \Omega^{\mp}, \end{cases}$$

where either $u \in C^{k+\gamma}(\overline{\Omega^{\pm}})$ *with* $k \geq 1$, $0<\gamma<1$, *or* $u \in H^1(\Omega^+)$ $[\,H^1_{loc}(\Omega^-)$, $u(x) \to 0$ *as* $|x| \to +\infty\,]$.

We also need some additional properties of the above-introduced operators, which will be proved below.

Theorem 3.5. *Let* $f \in H^{\frac{1}{2}}(S)$ *and* $h \in H^{-\frac{1}{2}}(S)$ *satisfy the condition*

$$\left(-2^{-1}I + \mathcal{K}^*\right) f = \mathcal{H}h, \quad \textit{i.e.,} \quad h = \mathcal{H}^{-1}\left(-2^{-1}I + \mathcal{K}^*\right) f. \tag{3.16}$$

Then there exists a unique vector-function $u \in H^1(\Omega^+)$ *such that*

$$\begin{aligned}
A(\partial)\,u(x) &= 0 \quad \textit{in} \quad \Omega^+,\\
[\,u\,]^+ = f \quad \textit{and} \quad [\,Tu\,]^+ &= h \quad \textit{on} \quad S = \partial\Omega^+.
\end{aligned}$$

Moreover,

$$\mathcal{L}f = \left(2^{-1}I + \mathcal{K}\right)h, \quad i.e., \quad h = \left(2^{-1}I + \mathcal{K}\right)^{-1}\mathcal{L}f. \tag{3.17}$$

Proof. We put

$$u\,(x) = W(f)(x) - V\,(h)\,(x)\,, \quad x \in \Omega^{\pm}.$$

Clearly, $u \in H^1(\Omega^+) \cap H^1_{loc}(\Omega^-)$ and $A\,(\partial)\,u(x) = 0$ in Ω^+. Due to Theorem 3.2 and equality (3.16) we have

$$[\,u\,]^- = \left(-2^{-1}I + \mathcal{K}^*\right)f - \mathcal{H}h = 0.$$

Therefore, with the help of the uniqueness theorem for the exterior Dirichlet boundary value problem (in Ω^-) for the operator $A(\partial)$, we conclude that $u(x) = 0$ in Ω^-. Thus

$$W(g)(x) - V(h)(x) = \begin{cases} u(x) & \text{for} \quad x \in \Omega^+, \\ 0 & \text{for} \quad x \in \Omega^-. \end{cases}$$

Whence, applying again Theorem 3.2, we get

$$[\,u\,]^+ - [\,u\,]^- = f \quad \text{and} \quad [\,Tu\,]^+ - [\,Tu\,]^- = h \quad \text{on} \quad S,$$

i.e.,

$$[\,u\,]^+ = f \quad \text{and} \quad [\,Tu\,]^+ = h.$$

The uniqueness of u can be easily shown by the uniqueness theorem for the Dirichlet problem.

To complete the proof we note that (3.16) and Theorem 3.3, (ii) imply

$$\mathcal{H}\mathcal{L}f = \left(2^{-1}I + \mathcal{K}^*\right)\left(-2^{-1}I + \mathcal{K}^*\right)f = \left(2^{-1}I + \mathcal{K}^*\right)\mathcal{H}h = \mathcal{H}\left(2^{-1}I + \mathcal{K}\right)h,$$

from which (3.17) follows since $\mathcal{H}$ is an invertible operator. □

Further, we discuss in detail the coercive properties of the bilinear forms in (3.14) and the operators related to them. Note that similar problems for the isotropic case and the Laplace equation are studied in [18], [3], [24], Ch.10, [4], Ch. XI, § 4 with the help of Korn's inequalities. Here we treat the general anisotropic case and prove the coerciveness of the corresponding operators by means of different arguments.

3.2. Coercive properties of the boundary bilinear forms. We start by the following simple lemma.

Lemma 3.6. *Let $\langle\cdot,\cdot\rangle_S$ be the bracket of duality (bilinear form) between the dual pair $H^r(S)$ and $H^{-r}(S)$, $\forall r \in \mathbb{R}$, and $(\cdot,\cdot)_{H^{-r}(S)}$ denote a scalar product in $H^{-r}(S)$.*

There exists a linear bounded bijective operator

$$P_{2r}\,:\,H^r(S) \to H^{-r}(S) \tag{3.18}$$

and positive constants C_1 and C_2, such that

$$\langle f, \overline{g}\rangle_S = (P_{2r}f\,,\,g)_{H^{-r}(S)}\,, \quad \forall f \in H^r(S),\ \forall g \in H^{-r}(S), \tag{3.19}$$

and

$$C_1 \, \| f; H^r(S) \| \le \| P_{2r} f; H^{-r}(S) \| \le C_2 \, \| f;\, H^r(S) \|, \ \forall f \in H^r(S). \quad (3.20)$$

Moreover,

$$\langle f, P_{2r} \overline{f} \rangle_S \ge 0, \quad \forall f \in H^r(S), \tag{3.21}$$

with the equality only for $f = 0$.

Proof. First we consider the case of real functions. Let $f \in H^r(S)$. Then

$$|\langle f, g \rangle_S| \le C \| f; H^r(S) \| \, \| g; H^{-r}(S) \|, \quad \forall g \in H^{-r}(S),$$

where C is independent of f and g, i.e., $\langle f, \cdot \rangle_S$ is the linear bounded functional on $H^{-r}(S)$ and due to the Riesz theorem there exists a unique element $F \in H^{-r}(S)$ such that

$$\langle f, \overline{g} \rangle_S = (F, g)_{H^{-r}(S)}\,, \quad \forall g \in H^{-r}(S). \tag{3.22}$$

We put $F = P_{2r} f$. It is evident that the mapping $P_{2r} \, : \, H^r(S) \to H^{-r}(S)$ is linear and injective. Now we show that it is surjective, i.e., for arbitrary $F \in H^{-r}(S)$ there exists unique $f \in H^{-r}(S)$ such that equality (3.22) holds.

Let Λ_r be an equivalent lifting (i.e., order reducing) pseudodifferential operator of order $-r$ (see, e.g., [22], [10], [14], [1]):

$$\Lambda_r \, : \, H^0(S) \to H^r(S).$$

This mapping is an isomorphism. Then the adjoint operator (with respect to the duality bracket) Λ_r^* is also an equivalent lifting operator

$$\Lambda_r^* \, : \, [\, H^r(S) \,]^* \to [\, H^0(S) \,]^*, \text{ i.e., } \ \Lambda_r^* \, : \, H^{-r}(S) \to H^0(S).$$

Obviously, there exist positive constants a_1, a_2, b_1, and b_2 such that

$$a_1 \, \| f; H^0(S) \| \le \| \Lambda_r f; H^r(S) \| \le a_2 \, \| f; H^0(S) \|, \quad \forall f \in H^0(S),$$

$$b_1 \, \| g; H^{-r}(S) \| \le \| \Lambda_r^* g; H^0(S) \| \le b_2 \, \| g; H^{-r}(S) \|, \quad \forall g \in H^{-r}(S).$$

Here a_j and b_j do not depend on f and g, respectively. Note that

$$B(\tilde{f}, g) := \langle \overline{\Lambda_r} \Lambda_r^* \tilde{f}, g \rangle_S$$

is a bilinear form on $H^{-r}(S) \times H^{-r}(S)$.

Consider the equation

$$B(\tilde{f}, g) = (F, g)_{H^{-r}(S)}\,, \quad \forall g \in H^{-r}(S), \tag{3.23}$$

where $F \in H^{-r}(S)$ is some given element and $\tilde{f} \in H^{-r}(S)$ is the sought for element. Obviously, the linear functional in the right-hand side of (3.23) is bounded on $H^{-r}(S)$.

It is also evident that $B(\cdot, \cdot)$ is continuous:

$$\begin{aligned} |B(\tilde{f}, g)| = |\langle \overline{\Lambda_r} \Lambda_r^* \tilde{f}, g \rangle_S| &\le C \, \| \overline{\Lambda_r} \Lambda_r^* \tilde{f}; H^r(S) \| \, \| g; H^{-r}(S) \| \\ &\le c \, a_2 \, b_2 \, \| \tilde{f}; H^{-r}(S) \| \, \| g; H^{-r}(S) \|. \end{aligned}$$

Moreover, $B(\cdot,\cdot)$ is coercive:

$$B(\tilde{f},\tilde{f}) = \langle \overline{\Lambda_r} \Lambda_r^* \tilde{f}, \tilde{f}\rangle_S = \langle \Lambda_r^* \tilde{f}, \overline{\Lambda_r^* \tilde{f}} \rangle_S = \left(\Lambda_r^* \tilde{f}, \Lambda_r^* \tilde{f} \right)_{H^0(S)}$$

$$= \|\Lambda_r^* f; H^0(S)\|^2 \geq b_1^2 \, \|\tilde{f}; H^{-r}(S)\|^2.$$

Due to the Lax–Milgram theorem then there exists a unique solution $\tilde{f} \in H^{-r}(S)$ of equation (3.23), i.e.,

$$\langle \overline{\Lambda_r} \Lambda_r^* \tilde{f}, g\rangle_S = (F, g)_{H^{-r}(S)}\,, \quad g \in H^{-r}(S),$$

which proves the surjectivity of the operator P_{2r} since $\overline{\Lambda_r} \Lambda_r^* \tilde{f} \in H^{-r}(S)$. Thus mapping (3.18) is bijective. Inequality (3.20) then follows from the Banach theorem on inverse operators.

Now let $f = f_1 + i f_2$ and $g = g_1 + i g_2$, where $f_j \in H^r(S)$ and $g_j \in H^{-r}(S)$ are real functions.

Due to the results just proven we have

$$\langle f, \overline{g}\rangle_S = \langle f_1 + i f_2, g_1 - i g_2 \rangle_S = \langle f_1, g_1\rangle_S + i\langle f_2, g_1\rangle_S - i\langle f_1, g_2\rangle_S$$

$$-i^2 \langle f_2, g_2\rangle_S = (P_{2r} f_1, g_1)_{H^{-r}(S)} + i\,(P_{2r} f_2, g_1)_{H^{-r}(S)}$$

$$-i\,(P_{2r} f_1, g_2)_{H^{-r}(S)} - i^2\,(P_{2r} f_2, g_2)_{H^{-r}(S)} = (P_{2r} f, g)_{H^{-r}(S)}\,,$$

whence (3.20) and (3.21) follow. □

Lemma 3.7. *There exists a positive constant C_3 such that*

$$\langle -\mathcal{H}\varphi, \overline{\varphi}\rangle_S \geq C_3\, \|\varphi; H^{-\frac{1}{2}}(S)\|^2, \quad \forall \varphi \in H^{-\frac{1}{2}}(S).$$

Proof. Due to Lemma 3.6

$$\langle -\mathcal{H}\varphi, \overline{\varphi}\rangle_S = (-P_1 \mathcal{H}\varphi, \varphi)_{H^{-\frac{1}{2}}(S)}\,, \quad \forall \varphi \in H^{-\frac{1}{2}}(S).$$

Moreover, the mapping

$$-P_1\mathcal{H} \,:\, H^{-\frac{1}{2}}(S) \to H^{-\frac{1}{2}}(S)$$

is an isomorphism and

$$(-P_1 \mathcal{H}\varphi, \varphi)_{H^{-\frac{1}{2}}(S)} = \langle -\mathcal{H}\varphi, \overline{\varphi}\rangle_S \geq 0$$

for an arbitrary (complex-valued) function $\varphi \in H^{-\frac{1}{2}}(S)$. Evidently, $-P_1\mathcal{H}$ is a self-adjoint positive operator (see, e.g., [30], Theorem 12.32).

By the square root theorem (see, e.g., [30], Theorem 12.33, [23], Ch.7, §3, Theorem 2) we conclude that there exists a unique positive linear bounded bijective operator

$$Q \,:\, H^{-\frac{1}{2}}(S) \to H^{-\frac{1}{2}}(S)$$

such that $Q^2 = -P_1\mathcal{H}$. Consequently, Q is self-adjoint and invertible, and

$$e_1\, \|\varphi; H^{-\frac{1}{2}}(S)\| \leq \|Q\varphi; H^{-\frac{1}{2}}(S)\| \leq e_2\, \|\varphi; H^{-\frac{1}{2}}(S)\|$$

with some positive constants e_1 and e_2. Therefore

$$\begin{aligned}\langle -\mathcal{H}\varphi, \overline{\varphi}\rangle_S &= (-P_1\mathcal{H}\varphi, \varphi)_{H^{-\frac{1}{2}}(S)} \\ &= \left(Q^2\varphi, \varphi\right)_{H^{-\frac{1}{2}}(S)} = \|Q\varphi; H^{-\frac{1}{2}}(S)\|^2 \geq e_1^2\, \|\varphi; H^{-\frac{1}{2}}(S)\|^2,\end{aligned}$$

which completes the proof. □

We denote by $X(\Omega^+)$ the space of rigid displacements in Ω^+, and by $X(S) = X(\partial\Omega^+)$ the space of their restrictions on $S = \partial\Omega^+$ (see (3.15)). Note that $\dim X(\Omega^+) = \dim X(S) = 6$. Let $\Psi(S) = \{\chi^j\}_{j=1}^6$ be the orthonormal (in the $H^0(S)$-sense) basis in $X(S)$:

$$\left(\chi^j, \chi^k\right)_{H^0(S)} = \langle \chi^j, \chi^k\rangle_S = \delta_{kj},$$

where δ_{kj} is Kronecker's symbol. Clearly, the system $\{\chi^j\}_{j=1}^6$ can be obtained by the orthonormalization (in the $H^0(S)$-sense) procedure of the following basis of $X(S)$:

$$\begin{aligned}&\nu^{(1)} = (1,0,0), \quad \nu^{(2)} = (0,1,0), \quad \nu^{(3)} = (0,0,1), \\ &\nu^{(4)} = (x_3, 0, -x_1), \quad \nu^{(5)} = (-x_2, x_3, 0), \quad \nu^{(6)} = (0, -x_3, x_2).\end{aligned}$$

Further, let $r \in \mathbb{R}$ and

$$H^r_*(S) := \{\varphi : \varphi \in H^r(S),\ \langle \varphi, \chi^j\rangle_S = 0,\ j = \overline{1,6}\}. \tag{3.24}$$

It is evident that $H^r_*(S)$ is a Hilbert space with the scalar product induced by $(\cdot,\cdot)_{H^r(S)}$.

Lemma 3.8. $\mathcal{L}(H^{\frac{1}{2}}(S)) = H_*^{-\frac{1}{2}}(S)$, *where the operator* $\mathcal{L}$ *is given by* (3.9).

Proof. The Green identity

$$\int\limits_{\Omega^+} [\, Au \cdot v - u \cdot Av\,]\, dx = \int\limits_S \Big\{ [\,Tu\,]^+ \cdot [\,v\,]^+ - [\,u\,]^+ \cdot [\,Tv\,]^+ \Big\}\, dS$$

with $u = W(\varphi)$ and $\forall v \in X(\Omega^+)$ yields

$$\int\limits_S \mathcal{L}\varphi \cdot \chi^j dS = \langle \mathcal{L}\varphi, \chi^j\rangle_S = 0, \quad \forall \varphi \in H^{\frac{1}{2}}(S),\ j = \overline{1,6}.$$

Therefore

$$\mathcal{L} \,:\, H^{\frac{1}{2}}(S) \to H_*^{-\frac{1}{2}}(S), \quad \text{i.e.,} \quad \mathcal{L}(H^{\frac{1}{2}}(S)) \subset H_*^{-\frac{1}{2}}(S).$$

On the other hand, the equation

$$\mathcal{L}\varphi = f, \quad f \in H_*^{-\frac{1}{2}}(S), \tag{3.25}$$

is solvable, and a solution $\varphi \in H^{\frac{1}{2}}(S)$ can be represented in the form ([25], [5])

$$\varphi = \varphi_0 + \sum_{j=1}^6 c_j \chi^j, \quad \varphi_0 \in H^{\frac{1}{2}}(S),$$

where φ_0 is some particular solution of the above non-homogeneous equation and c_j are arbitrary constants. In addition, if we require that $\varphi \in H_*^{\frac{1}{2}}(S)$, then (3.25) is uniquely solvable and the solution reads

$$\varphi = \varphi_0 - \sum_{j=1}^{6} \langle \varphi_0, \chi^j \rangle_S \, \chi^j. \tag{3.26}$$

It can be easily shown that the right-hand side in (3.26) does not depend on the choice of the particular solution φ_0 since the homogeneous version of equation (3.25) (with $f = 0$) possesses only the trivial solution in the space $H_*^{\frac{1}{2}}(S)$. □

Corollary 3.9. *The operator*

$$\mathcal{L} \,:\, H_*^{\frac{1}{2}}(S) \to H_*^{-\frac{1}{2}}(S)$$

is an isomorphism, and the inequality

$$c_1^* \, \|\varphi; H^{\frac{1}{2}}(S)\| \le \|\mathcal{L}\varphi; H^{-\frac{1}{2}}(S)\| \le c_2^* \, \|\varphi; H^{\frac{1}{2}}(S)\|, \tag{3.27}$$

holds for all $\varphi \in H_*^{\frac{1}{2}}(S)$ with some positive constants c_1^* and c_2^* independent of φ.

Lemma 3.10. *There exists a positive constant c_3^* such that*

$$\langle \mathcal{L}\varphi, \overline{\varphi} \rangle_S \ge c_3^* \, \|\varphi; H^{\frac{1}{2}}(S)\|^2, \quad \forall \varphi \in H_*^{\frac{1}{2}}(S).$$

Proof. Let

$$\widetilde{\mathcal{L}}\varphi := \mathcal{L}\varphi + \sum_{j=1}^{6} \langle \varphi, \chi^j \rangle_S \chi^j.$$

Note that $\widetilde{\mathcal{L}}\varphi = \mathcal{L}\varphi$ if and only if $\varphi \in H_*^{\frac{1}{2}}(S)$.

Evidently,

$$\langle \widetilde{\mathcal{L}}\varphi, \overline{\varphi} \rangle_S = \langle \mathcal{L}\varphi, \overline{\varphi} \rangle_S + \sum_{j=1}^{6} |\langle \varphi, \chi^j \rangle_S|^2, \;\; \forall \varphi \in H^{\frac{1}{2}}(S),$$

and $\langle \widetilde{\mathcal{L}}\varphi, \overline{\varphi} \rangle_S = 0$ implies $\varphi = 0$, due to Theorem 3.3, (iii) and (iv). This also shows that the homogeneous equation

$$\widetilde{\mathcal{L}}\varphi = 0, \quad \varphi \in H^{\frac{1}{2}}(S),$$

possesses only the trivial solution, and since $\operatorname{Ind} \widetilde{\mathcal{L}} = \operatorname{Ind} \mathcal{L} = 0$, we conclude that the mapping

$$\widetilde{\mathcal{L}} \,:\, H^{\frac{1}{2}}(S) \to H^{-\frac{1}{2}}(S)$$

is an isomorphism.

Let $\Lambda_{\frac{1}{2}}$ be an equivalent lifting operator

$$\Lambda_{\frac{1}{2}} \,:\, H^0(S) \to H^{\frac{1}{2}}(S),$$

and construct the operator $\overline{\Lambda^*_{\frac{1}{2}}}\widetilde{\mathcal{L}}\Lambda_{\frac{1}{2}}$, where $\Lambda^*_{\frac{1}{2}}$ is the operator adjoint to $\Lambda_{\frac{1}{2}}$ with respect to the duality bracket. Therefore the isomorphism $\Lambda^*_{\frac{1}{2}} \,:\, H^{-\frac{1}{2}}(S) \to H^0(S)$ is also an equivalent lifting operator.

It is obvious that the mapping

$$\overline{\Lambda^*_{\frac{1}{2}}}\widetilde{\mathcal{L}}\Lambda_{\frac{1}{2}} \,:\, H^0(S) \to H^0(S) \tag{3.28}$$

is an isomorphism. Moreover,

$$\langle\overline{\Lambda^*_{\frac{1}{2}}}\widetilde{\mathcal{L}}\Lambda_{\frac{1}{2}}\varphi,\overline{\varphi}\rangle_S = \langle\widetilde{\mathcal{L}}\Lambda_{\frac{1}{2}}\varphi,\overline{\Lambda_{\frac{1}{2}}\varphi}\rangle_S \geq 0, \quad \forall\varphi \in L_2(S) = H^0(S),$$

with equality only for $\varphi = 0$. Thus (3.28) defines a positive invertible operator. Further, due to the square root theorem, there exists a positive (self-adjoint) invertible operator $\widetilde{\mathcal{Q}} \,:\, H^0(S) \to H^0(S)$ such that

$$\overline{\Lambda^*_{\frac{1}{2}}}\widetilde{\mathcal{L}}\Lambda_{\frac{1}{2}} = \widetilde{\mathcal{Q}}^2.$$

Therefore, with the help of the self-adjointness of the operator $\widetilde{\mathcal{Q}}$ and invoking the Banach theorem on inverse operator, we get

$$\begin{aligned}\langle\overline{\Lambda^*_{\frac{1}{2}}}\widetilde{\mathcal{L}}\Lambda_{\frac{1}{2}}\varphi,\overline{\varphi}\rangle_S &= \langle\widetilde{\mathcal{Q}}^2\varphi,\overline{\varphi}\rangle_S = \Big(\widetilde{\mathcal{Q}}^2\varphi,\varphi\Big)_{H^0(S)}\\ &= \|\widetilde{\mathcal{Q}}\varphi;H^0(S)\|^2 \geq c_4\|\varphi;H^0(S)\|^2, \quad \forall\varphi\in H^0(S). \qquad (3.29)\end{aligned}$$

Note that for arbitrary $\psi \in H^{\frac{1}{2}}(S)$ there exists unique $\varphi \in H^0(S)$ such that $\Lambda_{\frac{1}{2}}\varphi = \psi$, i.e., $\psi = \Lambda^{-1}_{\frac{1}{2}}\varphi$ and $\|\varphi;H^0(S)\| \geq c_5\|\psi;H^{\frac{1}{2}}(S)\|$, where c_5 is a positive constant independent of φ and ψ. Consequently, by virtue of (3.29) we derive

$$\begin{aligned}\langle\widetilde{\mathcal{L}}\psi,\overline{\psi}\rangle_S &= \langle\widetilde{\mathcal{L}}\Lambda_{\frac{1}{2}}\varphi,\overline{\Lambda_{\frac{1}{2}}\varphi}\rangle_S = \langle\overline{\Lambda^*_{\frac{1}{2}}}\widetilde{\mathcal{L}}\Lambda_{\frac{1}{2}}\varphi,\overline{\varphi}\rangle_S\\ &\geq c_4\,\|\varphi;H^0(S)\|^2 \geq c_4c_5^2\,\|\psi;H^{\frac{1}{2}}(S)\|^2,\end{aligned}$$

which completes the proof since $\widetilde{\mathcal{L}}\psi = \mathcal{L}\psi, \;\; \forall\psi \in H^{\frac{1}{2}}_*(S)$. □

Below we need some mapping and coercive properties of the operator

$$\mathcal{M} := \mathcal{L} - \Big(-2^{-1}I + \mathcal{K}\Big)\,\mathcal{H}^{-1}\,\Big(-2^{-1}I + \mathcal{K}^*\Big). \tag{3.30}$$

Applying equalities (3.13) we easily transform (3.30) to obtain

$$\mathcal{M} = \mathcal{H}^{-1}\,\Big(-2^{-1}I + \mathcal{K}^*\Big) = \Big(-2^{-1}I + \mathcal{K}\Big)\,\mathcal{H}^{-1}. \tag{3.31}$$

Corollary 3.11. *The operator*

$$\mathcal{M} \,:\, H^{\frac{1}{2}}(S) \to H^{-\frac{1}{2}}(S)$$

is a bounded, positive, formally self-adjoint (*with respect to the duality bracket*), *elliptic pseudodifferential operator of order* 1, *and*

$$\ker\mathcal{M} = \ker\mathcal{L} = \ker\Big(-2^{-1}I + \mathcal{K}^*\Big), \quad \operatorname{Ind}\mathcal{M} = 0.$$

Moreover,

$$\mathcal{M}\left(H^{\frac{1}{2}}(S)\right) = H_*^{-\frac{1}{2}}(S).$$

The mapping

$$\mathcal{M}\,:\, H_*^{\frac{1}{2}}(S) \to H_*^{-\frac{1}{2}}(S)$$

is an isomorphism, and the inequalities

$$c_1'\,\|\varphi; H^{\frac{1}{2}}(S)\| \le \|\mathcal{M}\varphi; H^{-\frac{1}{2}}(S)\| \le c_2'\,\|\varphi; H^{\frac{1}{2}}(S)\|,\ \forall\varphi \in H_*^{\frac{1}{2}}(S),$$

$$\langle \mathcal{M}\varphi, \overline{\varphi}\rangle_S \ge c_3'\,\|\varphi; H^{\frac{1}{2}}(S)\|^2, \quad \forall\varphi \in H_*^{\frac{1}{2}}(S),$$

hold with some positive constants c_1', c_2', and c_3' independent of φ.

Proof. It is a ready consequence of (3.30), (3.31), Theorem 3.3, Corollary 3.9, and Lemma 3.10. □

4. Reduction to BVI. Existence and Uniqueness Results

Let $u \in \mathbf{K}$ be the unique solution of the SVI (2.10). Due to Theorems 3.4 and 3.1 we have the following Steklov-Poincaré relations connecting the Dirichlet and Neumann boundary data on $S = \partial\Omega^+$ of the vector u:

$$\mathcal{L}[u]^+ = \left(2^{-1}I + \mathcal{K}\right)[Tu]^+, \quad \left(-2^{-1}I + \mathcal{K}^*\right)[u]^+ = \mathcal{H}[Tu]^+. \tag{4.1}$$

These equalities imply

$$\begin{aligned}[Tu]^+ &= \mathcal{L}[u]^+ - \left(-2^{-1}I + \mathcal{K}\right)[Tu]^+ \\ &= \mathcal{L}[u]^+ - \left(-2^{-1}I + \mathcal{K}\right)\mathcal{H}^{-1}\left(-2^{-1}I + \mathcal{K}^*\right)[u]^+ = \mathcal{M}[u]^+,\end{aligned} \tag{4.2}$$

where $\mathcal{M}$ is defined by (3.30).

The Green formula (2.9) with the vector u and arbitrary $v \in H^1(\Omega^+)$ can be rewritten as

$$\int\limits_{\Omega^+} E(u,v)dx = \langle\,[\,T(\partial,n)u(x)\,]^+, [v]^+\rangle_S = \langle \mathcal{M}[u]^+, [v]^+\rangle_S, \tag{4.3}$$

whence by virtue of (2.2)

$$a(u,v) = \langle \mathcal{M}[u]^+, [v]^+\rangle_S. \tag{4.4}$$

Substituting (4.4) into (2.10) leads to the BVI: Find $\varphi \in \widetilde{\mathbf{K}}(S)$ such that

$$\langle \mathcal{M}\varphi, \psi - \varphi\rangle_S \ge \int\limits_{S_1} g\cdot(\psi-\varphi)dS, \quad \forall\psi \in \widetilde{\mathbf{K}}(S), \tag{4.5}$$

where, in our case, $g \in L_2(S_1)$ is a given function (see(2.6)),

$$\varphi = [u]^+, \quad \psi = [v]^+, \tag{4.6}$$

$$\widetilde{\mathbf{K}}(S) := \left\{f :\ f \in H^{\frac{1}{2}}(S),\ n\cdot f \le 0 \text{ on } S_3,\ f = 0 \text{ on } S_2\right\}. \tag{4.7}$$

Obviously, the closed convex cone $\widetilde{\mathbf{K}}(S)$ coincides with the space of traces (on S) of functions from $\mathbf{K}$ (see (2.11)).

In what follows we show that the BVI (4.5) is equivalent to the SVI (2.10) in the following sense. If u is a solution of the SVI (2.10), then $[u]^+ = \varphi$ solves the BVI (4.5) which has just been proved.

Vice versa, if φ is a solution of the BVI (4.5), then the vector-function

$$u(x) = W\left(\varphi\right)(x) - V\left(h\right)(x), \quad x \in \Omega^+, \tag{4.8}$$

where (see (3.31))

$$h = \mathcal{H}^{-1}\left(-2^{-1}I + \mathcal{K}^*\right)\varphi = \left(-2^{-1}I + \mathcal{K}\right)\mathcal{H}^{-1}\varphi = \mathcal{M}\varphi \tag{4.9}$$

solves the SVI (2.10). To prove this, we show that the vector-function (4.8) meets conditions (2.5)–(2.8). Since $\varphi \in H^{\frac{1}{2}}(S)$ and $h \in H^{-\frac{1}{2}}(S)$, it is evident that $u \in H^1(\Omega^+)$ by Theorem 3.2 and $A(\partial)u = 0$ in Ω^+. In accord with Theorem 3.5 (see the proof of Theorem 3.5) we have $\varphi = [u]^+_S$ and $h = [Tu]^+_S = \mathcal{M}\varphi$.

Obviously, condition (2.7) holds since $\varphi \in \widetilde{\mathbf{K}}$.

If in (4.5) we put $\psi = \varphi \pm \widetilde{\psi} \in \widetilde{\mathbf{K}}$, where $\widetilde{\psi} \in \widetilde{H}^{\frac{1}{2}}(S_1)$, we arrive at the equation

$$\langle \mathcal{M}\varphi, \widetilde{\psi}\rangle_S = \langle g, \widetilde{\psi}\rangle_{S_1} \quad \forall \widetilde{\psi} \in \widetilde{H}^{\frac{1}{2}}(S_1),$$

whence $\mathcal{M}\varphi = g$ on S_1 follows, i.e., condition (2.6) holds as well.

Further, if in (4.5) we put $\psi = \varphi \pm \omega \in \widetilde{\mathbf{K}}$, where $\omega \in \widetilde{H}^{\frac{1}{2}}(S_3)$ and $\omega \cdot n = 0$, we get

$$\langle \mathcal{M}\varphi, \omega\rangle_S = \langle g, \omega\rangle_{S_3}, \quad \forall \omega \in \widetilde{H}^{\frac{1}{2}}(S_3), \quad \omega \cdot n = 0,$$

which implies $\mathcal{M}\varphi - n(n \cdot \mathcal{M}\varphi) = 0$ on S_3. Thus, the first condition in (2.8) holds automatically due to the inclusion $\varphi \in \widetilde{\mathbf{K}}$.

Let us set $\psi = \varphi - n\,\nu$, where n is the outward normal vector and $\nu \in \widetilde{H}^{\frac{1}{2}}(S_3)$ is a non-negative scalar function .

From (4.5) then it follows

$$\langle \mathcal{M}\varphi, -n\nu\rangle_S \geq 0, \text{ i.e., } \langle -n \cdot \mathcal{M}\varphi, \nu\rangle_S \geq 0, \;\; \forall \nu \in \widetilde{H}^{\frac{1}{2}}(S_3),\; \nu \geq 0.$$

This shows that the third inequality in (2.10) holds.

Now, let h be a scalar function with the properties

$$0 \leq h(x) \leq 1, \quad h \in C^1(S), \quad \operatorname{supp} h \subset \overline{S_3},$$

and put $\psi = [1 + t\,h(x)]\varphi \in \widetilde{\mathbf{K}}$ with $t \in (-1, 1)$. From inequality (4.5) then we get

$$\langle \mathcal{M}\varphi, h(x)\varphi\rangle_S = 0,$$

which can be rewritten as

$$\langle n \cdot \mathcal{M}\varphi, h(n \cdot \varphi)\rangle_S = 0 \tag{4.10}$$

due to the equation $\mathcal{M}\varphi = n\,(n \cdot \mathcal{M}\varphi)$ on S_3.

Since h is an arbitrary function with the above-mentioned properties, from (4.10) we conclude that the fourth condition in (2.8) also holds. The above arguments prove that the vector u defined by (4.8) meets conditions (2.5)–(2.8) and therefore it solves the SVI (2.10).

Thus, there holds

Lemma 4.1. *The SVI* (2.10) *is equivalent* (*in the above-mentioned sense*) *to the following BVI: Find* $\varphi \in \widetilde{\mathbf{K}}$ *such that*

$$\langle \mathcal{M}\varphi, \psi - \varphi \rangle_S \geq \int\limits_{S_1} g \cdot (\psi - \varphi) dS, \quad \forall \psi \in \widetilde{\mathbf{K}}(S), \tag{4.11}$$

with given $g \in L_2(S_1)$.

From the existence and uniqueness theorems for the SVI (2.10) it follows that BVI (4.11) is also uniquely solvable. Observe that to develop the Galerkin method for approximation of solutions by means of the boundary element procedure and to obtain the corresponding abstract error estimate we need the coercive property of the pseudodifferential operator $\mathcal{M}$ on the cone $\widetilde{\mathbf{K}}(S)$. This property is also sine–qua–non to study the well–posedness of the BVI (4.11) independently (without invoking the mentioned SVI) on the basis of the theory of abstract variational inequalities in Hilbert spaces.

Note that Corollary 3.11 proves the coercivity of the operator $\mathcal{M}$ on the space $H_*^{\frac{1}{2}}(S)$. But, in general, the $\widetilde{\mathbf{K}}(S)$ is not a subset of $H_*^{\frac{1}{2}}(S)$. However, there holds

Lemma 4.2. *The bilinear form* $\langle \mathcal{M}\varphi, \psi \rangle_S$ *is bounded and coercive on the space* $\widetilde{H}_*^{\frac{1}{2}}(S \backslash \overline{S_2}) \times \widetilde{H}_*^{\frac{1}{2}}(S \backslash \overline{S_2})$:

$$\langle \mathcal{M}\varphi, \psi \rangle_S \leq \kappa_1 \, \| \varphi; H^{\frac{1}{2}}(S) \| \, \| \psi; H^{\frac{1}{2}}(S) \|,$$

$$\langle \mathcal{M}\varphi, \overline{\varphi} \rangle_S \geq \kappa_2 \, \| \varphi; H^{\frac{1}{2}}(S) \|^2$$

with positive κ_1 *and* κ_2 *independent of* φ *and* ψ.

Proof. Step 1. It is evident that for any vector-function $\varphi \in H^{\frac{1}{2}}(S)$ we have the unique representation

$$\varphi(x) = \varphi^{(1)}(x) + \varphi^{(0)}(x), \tag{4.12}$$

where

$$\varphi^{(1)}(x) = \varphi(x) - \sum_{j=1}^{6} c_j(\varphi) \chi^{(j)}(x), \quad c_j(\varphi) = \langle \varphi, \chi^{(j)} \rangle_S, \tag{4.13}$$

$$\varphi^{(0)}(x) = \sum_{j=1}^{6} c_j(\varphi) \chi^{(j)}(x); \tag{4.14}$$

here $\left\{ \chi^{(j)} \right\}_{j=1}^{6}$ is the above-introduced $H^0(S)$-orthogonal basis in the six-dimensional space $X(S)$ (traces on S of rigid displacements, i.e., vectors of type

(3.15)). Clearly,

$$\varphi^{(1)} \in H_*^{\frac{1}{2}}(S), \quad \varphi^{(0)} \in X(S) = \ker \mathcal{M} \subset H^{\frac{1}{2}}(S). \tag{4.15}$$

Note that

$$|c_j(\varphi)| = |\langle \varphi, \chi^{(j)} \rangle_S| \le \kappa_j \, \|\varphi; H^{\frac{1}{2}}(S)\|, \; j = \overline{1,6}, \; \varphi \in H^{\frac{1}{2}}(S), \tag{4.16}$$

with the constant κ_j independent of φ.

We put

$$l_j(\varphi) = \int\limits_{S_2} \varphi \cdot \chi^{(j)} dS. \tag{4.17}$$

It is easy to see that if $\varphi \in X(S)$, i.e., $\varphi(x) = \sum\limits_{j=1}^{6} c_j(\varphi)\chi^{(j)}(x)$, and $l_k(\varphi) = 0, \; k = \overline{1,6}$, then $\varphi(x) = 0$. Indeed, from these conditions we derive

$$0 = \sum_{k=1}^{6} c_k(\varphi) l_k(\varphi) = \int\limits_{S_2} \varphi \cdot \varphi \, dS = \int\limits_{S_2} |\varphi|^2 dS,$$

whence $\varphi = 0$ on S_2 and, consequently, $\varphi = 0$ on S (if a vector of rigid displacements vanishes at three points which do not belong to the same straight line, then it is identically zero in $\mathbb{R}^3$, that is, the vectors a and b in (3.15) vanish). This implies $c_j(\varphi) = 0, \; j = \overline{1,6}$.

Step 2. Let us introduce a new norm in $H^{\frac{1}{2}}(S)$:

$$|||\varphi||| = \|\varphi\|_* + \|\varphi\|_{**}, \;\; \|\varphi\|_* = \|\varphi^{(1)}; H^{\frac{1}{2}}(S)\|, \;\; \|\varphi\|_{**} = \sum_{j=1}^{6} |l_j(\varphi)|, \tag{4.18}$$

where $\varphi^{(1)}$ and $l_j(\varphi)$ are given by (4.13) and (4.17),

Note that $\|\cdot\|_*$ and $\|\cdot\|_{**}$ represent semi-norms in $H^{\frac{1}{2}}(S)$, which admits the following estimates:

$$\begin{aligned} \|\varphi\|_* &= \|\varphi^{(1)}; H^{\frac{1}{2}}(S)\| = \|\varphi - \sum_{j=1}^{6} c_j(\varphi)\chi^{(j)}; H^{\frac{1}{2}}(S)\| \\ &\le \|\varphi; H^{\frac{1}{2}}(S)\| + \sum_{j=1}^{6} c_j(\varphi)\|\chi^{(j)}; H^{\frac{1}{2}}(S)\| \\ &\le M_1 \|\varphi; H^{\frac{1}{2}}(S)\|, \end{aligned} \tag{4.19}$$

$$M_1 = 1 + \sum_{j=1}^{6} \kappa_j \|\chi^{(j)}; H^{\frac{1}{2}}(S)\|, \quad \forall \varphi \in H^{\frac{1}{2}}(S),$$

$$\begin{aligned} \|\varphi\|_{**} &= \sum_{j=1}^{6} \left| \int\limits_{S_2} \varphi \cdot \chi^{(j)} dS \right| \le \sum_{j=1}^{6} \|\varphi; H^0(S)\| \, \|\chi^{(j)}; H^0(S)\| \\ &= 6 \, \|\varphi; H^0(S)\| \le 6 \, \|\varphi; H^{\frac{1}{2}}(S)\|, \quad \forall \varphi \in H^{\frac{1}{2}}(S), \end{aligned} \tag{4.20}$$

where the positive constant M_1 does not depend on φ (see (4.16)).

Further, we show that the seminorm $\|\cdot\|_{**}$ is a norm in $X(S)$, i.e., if $\varphi(x) = \sum\limits_{k=1}^{6} c_k\chi^{(k)}(x)$ and $\|\varphi\|_{**} = 0$, then $\varphi = 0$ on S. Indeed, these conditions yield

$$\|\varphi\|_{**} = \sum_{j=1}^{6}\left|\int\limits_{S_2}\sum_{k=1}^{6} c_k\chi^{(k)}(x)\cdot\chi^{(j)}(x)dS\right| = 0,$$

i.e.,

$$\int\limits_{S_2}\sum_{k=1}^{6} c_k\chi^{(k)}(x)\cdot\chi^{(j)}(x)dS = 0.$$

Hence

$$\int\limits_{S_2}\left(\sum_{k=1}^{6} c_k\,\chi^{(k)}\right)\cdot\left(\sum_{j=1}^{6} c_j\,\chi^{(j)}\right)dS = \int\limits_{S_2}|\varphi|^2 dS = 0,$$

and, consequently, $\varphi = 0$ on S (that is, $c_k = 0,\quad k = \overline{1,6}$). This proves that $\|\cdot\|_{**}$ is a norm in $X(S)$.

Since $X(S)$ is a six-dimensional space, we have the estimate

$$m_0\,\|\varphi; H^{\frac{1}{2}}(S)\| \le \|\varphi\|_{**}, \quad \forall\varphi\in X(S), \tag{4.21}$$

with some constant $m_0 > 0$ independent of φ, due to the equivalence of all norms in finite-dimensional spaces.

Step 3. Here we show the equivalence of the norms $|||\cdot|||$ and $\|\cdot\,; H^{\frac{1}{2}}(S)\|$ in $H^{\frac{1}{2}}(S)$. On the one hand, by virtue of (4.18), (4.19), and (4.20)

$$|||\varphi||| \le M\,\|\varphi; H^{\frac{1}{2}}(S)\|, \quad \varphi\in H^{\frac{1}{2}}(S)$$

with $M = 6 + M_1$.

On the other hand, with the help of (4.18), (4.20) and (4.21) we derive

$$\begin{aligned}
|||\varphi||| &= \|\varphi\|_* + \|\varphi\|_{**} \ge \|\varphi^{(1)}; H^{\frac{1}{2}}(S)\| + \frac{1}{12}\,\|\varphi^{(1)} + \varphi^{(0)}\|_{**}\\
&\ge \|\varphi^{(1)}; H^{\frac{1}{2}}(S)\| + \frac{1}{12}\,\|\varphi^{(0)}\|_{**} - \frac{1}{12}\,\|\varphi^{(1)}\|_{**} \ge \frac{1}{2}\,\|\varphi^{(1)}; H^{\frac{1}{2}}(S)\|\\
&+\frac{m_0}{12}\,\|\varphi^{(0)}; H^{\frac{1}{2}}(S)\| \ge m\left\{\|\varphi^{(1)}; H^{\frac{1}{2}}(S)\| + \|\varphi^{(0)}; H^{\frac{1}{2}}(S)\|\right\}\\
&= m\,\|\varphi; H^{\frac{1}{2}}(S)\|, \quad \forall\varphi\in H^{\frac{1}{2}}(S),
\end{aligned}$$

where the constant $m = \min\left\{\frac{1}{2}, \frac{m_0}{12}\right\} > 0$ is independent of φ.

Thus there exists positive constants m and M such that

$$m\,\|\varphi; H^{\frac{1}{2}}(S)\| \le |||\varphi||| \le M\|\varphi; H^{\frac{1}{2}}(S)\|, \quad \forall\varphi\in H^{\frac{1}{2}}(S). \tag{4.22}$$

Step 4. Here we complete the proof of the lemma.

Let $\forall\varphi \in \widetilde{H}^{\frac{1}{2}}(S\backslash\overline{S_2}) \subset H^{\frac{1}{2}}(S)$. Applying the self-adjointness of the operator $\mathcal{M}$ together with Corollary 3.11 and relations (4.15) and (4.22) we proceed as follows:

$$\begin{aligned}
&\langle\mathcal{M}\varphi,\varphi\rangle_S = \langle\mathcal{M}\left(\varphi^{(1)}+\varphi^{(0)}\right),\varphi^{(1)}+\varphi^{(0)}\rangle_S = \langle\mathcal{M}\varphi^{(1)},\varphi^{(1)}+\varphi^{(0)}\rangle_S \\
&= \langle\mathcal{M}\varphi^{(1)},\varphi^{(1)}\rangle_S + \langle\mathcal{M}\varphi^{(1)},\varphi^{(0)}\rangle_S = \langle\mathcal{M}\varphi^{(1)},\varphi^{(1)}\rangle_S + \langle\varphi^{(1)},\mathcal{M}\varphi^{(0)}\rangle_S \\
&= \langle\mathcal{M}\varphi^{(1)},\varphi^{(1)}\rangle_S \geq c_3'\,\|\varphi^{(1)};H^{\frac{1}{2}}(S)\|^2 = c_3'\left\{\|\varphi^{(1)}\|_* + \|\varphi\|_{**}\right\}^2 \\
&= c_3'\,|||\varphi|||^2 \geq \kappa_2\,\|\varphi;H^{\frac{1}{2}}(S)\|^2, \quad \kappa_2 = c_3' m^2,
\end{aligned}$$

with the constant $\kappa_2 > 0$ independent of φ.

The boundedness of the bilinear form $\langle\mathcal{M}\varphi,\psi\rangle_S$ is a trivial consequence of Corollary 3.11. $\square$

Now, let us recall the well-known theorem concerning an abstract variational inequality in a Hilbert space (see, e.g., [13], Ch.1, Theorems 2.1 and 2.2).

Theorem 4.3. *Let V_0 be a closed convex subset of a Hilbert space V, F be a linear bounded functional on V, and $B(\cdot,\cdot)$ be a coercive bilinear form on $V\times V$. Then the problem: Find $u \in V_0$ such that*

$$B(u,v-u) \geq F(v-u), \qquad \forall v \in V_0,$$

possesses a unique solution.

From this theorem along with Lemma 4.2 we get the following assertion.

Theorem 4.4. *The BVI* (4.11) *possesses a unique solution φ satisfying the estimate*

$$\|\varphi;H^{\frac{1}{2}}(S)\| \leq \kappa_2^{-1}\|g;L_2(S_1)\| \tag{4.23}$$

with the same κ_2 as in Lemma 4.2.

Proof. The first part of the theorem immediately follows from Theorem 4.3, since the cone $\widetilde{\mathbf{K}}(S)$ is a closed convex set of the Hilbert space $H_*^{\frac{1}{2}}(S\backslash\overline{S_2})$ and the linear functional defined by the right-hand side expression in (4.23) is bounded on $\widetilde{H}_*^{\frac{1}{2}}(S\backslash\overline{S_2})$:

$$\begin{aligned}
\left|\int\limits_{S_1} g\cdot\psi\,dS\right| &\leq \|g;L_2(S_1)\|\,\|\psi;H^{\frac{1}{2}}(S_1)\| \\
&\leq \|g;L_2(S_1)\|\,\|\psi;H^{\frac{1}{2}}(S)\| \quad \forall\psi \in H_*^{\frac{1}{2}}(S\backslash\overline{S_2}).
\end{aligned} \tag{4.24}$$

To prove (4.23), we proceed as follows. We put $\psi = 2\varphi$ and $\psi = 0$ in (4.11) to obtain the equality

$$\langle\mathcal{M}\varphi,\varphi\rangle_S = \int\limits_{S_1} g\cdot\varphi\,dS.$$

Further, applying the coercivity of the operator $\mathcal{M}$ (see Lemma 4.2) and relation (4.24) we arrive at inequality (4.23). $\square$

Remark 4.5. Note that we can extend the domain of the definition of the linear functional in the right-hand side of (4.11) with respect to g. In fact, instead of (4.11) we can consider the variational inequality

$$\langle \mathcal{M}\varphi, \psi-\varphi\rangle_S \geq \langle g, \psi-\varphi\rangle_{S_1}, \quad \forall \psi \in \widetilde{\mathbf{K}}(S), \tag{4.25}$$

where $\langle\cdot,\cdot\rangle_{S_1}$ is the duality pairing between either the spaces $H^{-\frac{1}{2}}(S_1)$ and $\widetilde{H}^{\frac{1}{2}}(S_1)$ if $\partial S_1 \cap \partial S_3 = \varnothing$, or $\widetilde{H}^{-\frac{1}{2}}(S_1)$ and $H^{\frac{1}{2}}(S_1)$ if $\partial S_1 \cap \partial S_3 \neq \varnothing$.

In this case a theorem similar to Theorem 4.4 holds with the corresponding estimate (instead of (4.23))

$$\|\varphi; H^{\frac{1}{2}}(S)\| \leq \begin{cases} \kappa_3 \|g; H^{-\frac{1}{2}}(S_1)\| & \text{for} \quad g \in H^{-\frac{1}{2}}(S_1), \\ \kappa_3 \|g; \widetilde{H}^{-\frac{1}{2}}(S_1)\| & \text{for} \quad g \in \widetilde{H}^{-\frac{1}{2}}(S_1), \end{cases}$$

where κ_3 is a positive constant independent of φ and g.

5. Galerkin Approximation of the BVI

In this section we treat the problem of numerical approximation of a solution to the BVI (4.11) by Galerkin's method.

Suppose that $\widetilde{H}^{\frac{1}{2}}_{(h)}(S\backslash\overline{S_2})$ is a finite dimensional subspace of $\widetilde{H}^{\frac{1}{2}}(S\backslash\overline{S_2})$ and let

$$\widetilde{\mathbf{K}}_h(S) = \{\psi_h \in \widetilde{H}^{\frac{1}{2}}_{(h)}(S\backslash\overline{S_2}) : \ \psi_h \cdot n \leq 0 \ \text{ on } \ S_3\} \tag{5.1}$$

be a convex closed nonempty subset of $\widetilde{H}^{\frac{1}{2}}(S\backslash\overline{S_2})$. Clearly, $\widetilde{\mathbf{K}}_h(S) \subset \widetilde{\mathbf{K}}(S)$.

An element $\varphi_h \in \widetilde{\mathbf{K}}_h(S)$ is said to be an approximate solution of the BVI (4.11) if

$$\langle \mathcal{M}\varphi_h, \psi_h-\varphi_h\rangle_S \geq \int_{S_1} g\cdot(\psi_h-\varphi_h)\,dS \quad \forall \psi_h \in \widetilde{\mathbf{K}}_h(S). \tag{5.2}$$

The existence and uniqueness theorems for the solution of the BVI (5.2) follow immediately from Theorem 4.3. Furthermore, we have

Theorem 5.1. *Let $\varphi_h \in \widetilde{\mathbf{K}}_h(S)$ be a solution of BVI* (4.11) *and $\varphi_h \in \widetilde{\mathbf{K}}_h(S)$ be an approximate solution of the BVI* (5.2).

Then the following abstract error estimate holds:

$$\begin{aligned}\|\varphi-\varphi_h; H^{\frac{1}{2}}(S)\|^2 \leq c^* \inf_{\psi_h\in\widetilde{\mathbf{K}}_h(S)} \Big\{ & \|\varphi-\psi_h; H^{\frac{1}{2}}(S)\|^2 \\ & + \Big|\langle \mathcal{M}\varphi, \psi_h-\varphi\rangle_S - \int_{S_1} g\cdot(\psi_h-\varphi)dS\Big| \Big\}\end{aligned} \tag{5.3}$$

with some positive constant c^* independent of g, f, and φ_h.

Proof. Due to the coerciveness and boundedness of the operator $\mathcal{M}$ (see Lemma 4.2) we derive

$$\begin{aligned}
\|\varphi-\varphi_h;H^{\frac12}(S)\|^2 &\le \frac{1}{\kappa_2}\langle\mathcal{M}(\varphi-\varphi_h),\varphi-\varphi_h\rangle_S\\
&= \frac{1}{\kappa_2}\left\{\langle\mathcal{M}(\varphi-\varphi_h),\varphi-\psi_h\rangle_S+\langle\mathcal{M}(\varphi-\varphi_h),\psi_h-\varphi_h\rangle_S\right\}\\
&\le \frac{\kappa_1}{\kappa_2}\|\varphi-\varphi_h;H^{\frac12}(S)\|\,\|\varphi-\psi_h;H^{\frac12}(S)\|+\frac{1}{\kappa_2}\left\{\langle\mathcal{M}\varphi,\psi_h-\varphi_h\rangle_S\right.\\
&\left.-\langle\mathcal{M}\varphi_h,\psi_h-\varphi_h\rangle_S\right\}\le\frac12\|\varphi-\varphi_h;H^{\frac12}(S)\|^2\\
&+\frac{\kappa_1^2}{2\kappa_2^2}\|\varphi-\psi_h;H^{\frac12}(S)\|^2+\frac{1}{\kappa_2}\left\{\langle\mathcal{M}\varphi,\psi_h-\varphi\rangle_S\right.\\
&\left.-\langle\mathcal{M}\varphi,\varphi_h-\varphi\rangle_S-\langle\mathcal{M}\varphi_h,\psi_h-\varphi_h\rangle_S\right\},
\end{aligned}$$

where $\psi_h\in\widetilde{\mathbf{K}}_h(S)$, and κ_1 and κ_2 are as in Lemma 4.2.

By virtue of (4.11) and (5.2) we conclude that

$$\begin{aligned}
\|\varphi-\varphi_h;H^{\frac12}(S)\|^2 &\le \frac{\kappa_1^2}{\kappa_2^2}\|\varphi-\psi_h;H^{\frac12}(S)\|^2\\
&+\frac{2}{\kappa_2}\left\{\langle\mathcal{M}\varphi,\psi_h-\varphi\rangle_S-\int_{S_1}g\cdot(\varphi_h-\varphi)dS-\int_{S_1}g\cdot(\psi_h-\varphi_h)dS\right\}\\
&=\frac{\kappa_1^2}{\kappa_2^2}\|\varphi-\psi_h;H^{\frac12}(S)\|^2+\frac{2}{\kappa_2}\left\{\langle\mathcal{M}\varphi,\psi_h-\varphi\rangle_S-\int_{S_1}g\cdot(\psi_h-\varphi)\,dS\right\}
\end{aligned}$$

for all $\psi_h\in\widetilde{\mathbf{K}}_h(S)$, whence (5.3) follows with $c^*=\max\left\{\dfrac{2}{\kappa_2},\dfrac{\kappa_1^2}{\kappa_2^2}\right\}$. $\square$

Remark 5.2. Note that

$$|\langle\mathcal{M}\varphi,\psi_h-\varphi\rangle_S|\le\kappa_1\|\varphi;H^{\frac12}(S)\|\,\|\psi_h-\varphi;H^{\frac12}(S)\|,$$

$$\left|\int_{S_1}g\cdot(\psi_h-\varphi)\,dS\right|\le\|g;L_2(S_1)\|\,\|\psi_h-\varphi;L_2(S_1)\|,$$

where κ_1 is independent of φ and ψ_h. Therefore (5.3) implies the inequality

$$\begin{aligned}
\|\varphi-\varphi_h;H^{\frac12}(S)\|^2 &\le c^{**}\inf_{\psi_h\in\widetilde{\mathbf{K}}_h(S)}\Big\{\|\varphi-\psi_h;H^{\frac12}(S)\|^2\\
&+\|\varphi;H^{\frac12}(S)\|\,\|\psi_h-\varphi;H^{\frac12}(S)\|+\|g;L_2(S_1)\|\,\|\psi_h-\varphi;L_2(S_1)\|\Big\}
\end{aligned}$$

with $c^{**}=\max\left\{\dfrac{2}{\kappa_2},\dfrac{\kappa_1^2}{\kappa_2^2},\dfrac{2\kappa_1}{\kappa_2}\right\}$.

Remark 5.3. Observe that the expression under modulus in the second term in the right-hand side of (5.3) is, actually, supported on the sub-manifold $\overline{S_3}$ since $\mathcal{M}\varphi = g$ on S_1, and ψ_h and φ_h vanish on S_2.

References

1. M. Agranovich, Spectral properties of diffraction problems. (Russian) *In: N. Voitovich, B. Kazenelenbaum, A. Sivov, Generalized Method of Eigenoscillation in the Diffraction Theory*, 279–281, *Nauka, Moscow*, 1977.
2. T. V. Burchuladze and T. G. Gegelia, Development of the potential methods in the theory of elasticity. (Russian) *Metsniereba, Tbilisi,* 1985.
3. M. Costabel, Boundary integral operators on Lipschitz domains: elementary results. *SIAM J. Math. Anal.* **19**(1988), 613–626.
4. R. Dautray and J. L. Lions, Mathematical analysis and numerical methods for science and technology. Vol. 4, *Integral Equations and Numerical Methods, Springer-Verlag, Berlin–Heidelberg,* 1990.
5. R. Duduchava, D. Natroshvili, and E. Shargorodsky, Boundary value problems of the mathematical theory of cracks. *Proc. I. Vekua Inst. Appl. Math.* **39**(1990), 68–84.
6. R. Duduchava, D. Natroshvili, and E. Shargorodsky, Basic boundary value problems of thermoelasticity for anisotropic bodies with cuts. I, II. *Georgian Math. J.* **2**(1995), No. 2, 123–140, No. 3, 259–276.
7. G. Duvaut and J. L. Lions, Les inéquations en méchanique et en physique. *Dunod, Paris,* 1972.
8. C. Eck and W. L. Wendland, A residual-based error estimator for BEM-discretizations of contact problems. (To appear)
9. C. Eck, O. Steinbach, and W. L. Wendland, A symmetric boundary element method for contact problems with friction. *Math. Comp. Simulation* **50**(1999), 1–4, 43–61.
10. G. Eskin, Boundary value problems for elliptic pseudodifferential equations. *Translation of Math. Monographs, Amer. Math. Soc.* 52, *Providence, Rhode Island,* 1981.
11. G. Fichera, Problemi elastostatici con vincoli unilaterali: il problema di Signorini com ambigue condicioni al contorno. *Accad. Naz. Lincei* **8**(1963–1964), 91–140.
12. G. Fichera, Existence theorems in elasticity. *Handb. der Physik, Bd.* 6/2, *Springer-Verlag, Heidelberg,* 1973.
13. R. Glowinski, J. L. Lions, and R. Tremolieres, Numerical analysis of variational inequalities. *North–Holand, Amsterdam,* 1981.
14. G. Grubb, Pseudodifferential boundary problems in L_p–spaces. *Comm. Part. Diff. Equations* **15**(1990), 289–340.
15. H. Han and G. C. Hsiao, The boundary element method for a contact problem. *Theory and Applications of Boundary Element Methods, Proc. 2nd China–Jap. Symp., Beijing* 1988, 33–38, 1990.
16. H. Han, A boundary element method for Signorini problem in linear elasticity. *Numer. Math. J. Chinese Univ.* **1**(1992), No. 1, 66–74.
17. H. Han, A boundary element procedure for the Signorini problem in three–dimensional elasticity. *Numer. Math. J. Chinese Univ.* **3**(1994), No. 1, 104–117.
18. G. C. Hsiao and W. L. Wendland, A finite element method for some integral equations of the first kind. *J. Math. Anal. Appl.* **58**(1977), 449–481.

19. D. Kinderlehrer, Remarks about Signorini's problem in linear elasticity. *Ann. Scuola Norm. Sup. Pisa, Cl. Sci.* (4) vol. VIII, **4**(1981), 605–645.

20. V. D. Kupradze, T. G. Gegelia, M. O. Basheleishvili, and T. V. Burchuladze, Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity. (Russian) *Nauka, Moscow,* 1976; English translation: *North-Holland Series in Applied Mathematics and Mechanics* 25, *North-Holland Publishing Company, Amsterdam-New York-Oxford,* 1979.

21. J. L. Lions and G. Stampacchia, Variational inequalities. *Comm. Pure Appl. Math.* **20**(1976), 493–519.

22. J. L. Lions and E. Magenes, Problèmes aux limites non homogènes et applications. I. *Dunod, Paris,* 1968.

23. L. A. Lusternik and B. I. Sobolev, Elements of functional analysis. (Russian) *Nauka, Moscow,* 1965.

24. W. McLean, Strongly elliptic systems and boundary integral equations. *Cambridge University Press, Cambridge* , 2000.

25. D. Natroshvili, Investigation of boundary value and initial boundary value problems of the mathematical theory of anisotropic elasticity and thermoelasticity by means of potential methods. (Russian) *Doct. Thesis, A. Razmadze Math. Inst., Tbilisi,* 1–325, 1984.

26. D. Natroshvili, Mixed interface problems for anisotropic elastic bodies. *Georgian Math. J.* **2**(1995), No. 6, 631–652.

27. D. Natroshvili, Boundary integral equation method in the steady state oscillation problems for anisotropic bodies. *Math. Methods Appl. Sci.* **20**(1997), No. 2, 95–119.

28. D. Natroshvili, O. Chkadua, and E. Shargorodsky, Mixed boundary value problems of the anisotropic elasticity. (Russian) *Proc. I. Vekua Inst. Appl. Math. Tbilisi State University* **39** (1990), 133–181.

29. J. Nečas, Méthodes Directes en Théorie des Équations Élliptique. *Masson Éditeur, Paris,* 1967.

30. W. Rudin, Functional analysis. *McGraw-Hill Book Company, New York–Toronto,* 1973.

31. H. Triebel, Interpolation theory, function spaces, differential operators. *North–Holand, Amsterdam,* 1978.

32. H. Triebel, Theory of function spaces. *Birkhäuser Verlag, Basel–Boston–Stuttgart,* 1983.

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Authors' addresses:

A. Gachechiladze
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., Tbilisi 380093
Georgia

D. Natroshvili
Department of Mathematics
Georgian Technical University
77, Kostava St., Tbilisi 380075
Georgia
E-mail: natrosh@hotmail.com