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ON SINGULAR BOUNDARY VALUE PROBLEMS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS OF HIGHER ORDER

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Dedicated to the memory of Professor N. Muskhelishvili on the occasion of his 110th birthday

Abstract. Sufficient conditions are established for the solvability of the boundary value problem

$$x^{(n)}(t) = f(x)(t), \quad h_i(x) = 0 \ \ (i = 1, \dots, n),$$

where f is an operator (h_i $(i = 1, \dots, n)$ are operators) acting from some subspace of the space of $(n-1)$-times differentiable on the interval $]a, b[$ m-dimensional vector functions into the space of locally integrable on $]a, b[$ m-dimensional vector functions (into the space $\mathbb{R}^m$).

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1. Formulation of the Main Results

1.1. Formulation of the problem and a brief survey of literature. Consider the functional differential equation of n-th order

$$x^{(n)}(t) = f(x)(t) \tag{1.1}$$

with the boundary conditions

$$h_i(x) = 0 \ \ (i = 1, \dots, n). \tag{1.2}$$

When the operators $f : C^{n-1}([a, b]; \mathbb{R}^m) \to L([a, b]; \mathbb{R}^m)$ and $h_i : C^{n-1}([a, b]; \mathbb{R}^m) \to \mathbb{R}^m$ $(i = 1, \dots, n)$ are continuous, problem (1.1), (1.2) is called *regular*. If the operator f (operators h_i $(i = 1, \dots, n)$) acts from some subspace of the space $C^{n-1}(]a, b[\,; \mathbb{R}^m)$ into the space $L_{loc}(]a, b[\,; \mathbb{R}^m)$ (into the space $\mathbb{R}^m$), problem (1.1), (1.2) is called *singular*.

The basic principles of the theory of a wide enough class of regular problems of form (1.1), (1.2) are constructed in the monographs [4], [5], [43]. Optimal sufficient conditions for such problems to be solvable and uniquely solvable are given in [7], [8], [10]–[12], [22], [24], [26]–[28], [39].

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As to singular problems of form (1.1), (1.2), they have been studied with sufficient completeness in the case with the operator f having the form

$$f(x)(t) = g\Big(t, x(t), \dots, x^{(n-1)}(t)\Big)$$

(see [1], [2], [14]–[21], [32]–[35], [37], [45] and the references cited therein). For the singular functional differential equation (1.1), the weighted initial problem is studied in [30], [31], two-point problems in [3], [6], [23], [36], [38], [40]–[42], whereas the multi-point Vallée-Poussin problem in [25]. In the general case the singular problem (1.1), (1.2) remains studied but little. An attempt is made in this paper to fill up this gap to some extent.

Throughout the paper the following notation will be used.

$\mathbb{R} =]-\infty, +\infty[\,$, $\mathbb{R}_+ = [0, +\infty[\,$.

$\mathbb{R}^m$ is the space of m-dimensional column vectors $x = (x_i)_{i=1}^m$ with the components $x_i \in \mathbb{R}$ $(i = 1, \dots, m)$ and the norm

$$\|x\| = \sum_{i=1}^m |x_i|.$$

$\mathbb{R}^m_+ = \{x = (x_i)_{i=1}^m : \; x_i \in \mathbb{R}_+ \; (i = 1, \dots, m)\}$.

$\mathbb{R}^{m\times m}$ is the space of $m \times m$ matrices $X = (x_{ik})_{i,k=1}^m$ with the components $x_{ik} \in \mathbb{R}$ $(i, k = 1, \dots, m)$ and the norm

$$\|X\| = \sum_{i,k=1}^m |x_{ik}|.$$

If $x = (x_i)_{i=1}^m \in \mathbb{R}^m$ and $X = (x_{ik})_{i,k=1}^m \in \mathbb{R}^{m\times m}$, then

$$|x| = (|x_i|)_{i=1}^m \;\text{ and }\; |X| = (|x_{ik}|)_{i,k=1}^m.$$

$\mathbb{R}^{m\times m}_+ = \{X = (x_{ik})_{i,k=1}^m : \; x_{ik} \in \mathbb{R}_+ \; (i, k = 1, \dots, m)\}$.

$r(X)$ is the spectral radius of the matrix $X \in \mathbb{R}^{m\times m}$.

Inequalities between matrices and vectors are understood componentwise, i.e., for $x = (x_i)_{i=1}^m$, $y = (y_i)_{i=1}^m$, $X = (x_{ik})_{i,k=1}^m$ and $Y = (y_{ik})_{i,k=1}^m$ we have

$$x \le y \iff x_i \le y_i \;\; (i = 1, \dots, m)$$

and

$$X \le Y \iff x_{ik} \le y_{ik} \;\; (i, k = 1, \dots, m).$$

If k is a natural number and $\varepsilon \in]0, 1[\,$, then

$$(k - \varepsilon)! = \prod_{i=1}^k (i - \varepsilon).$$

If m and n are natural numbers, $-\infty < a < b < +\infty$, $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$, then $C^{n-1}_{\alpha,\beta}(]a, b[\,; \mathbb{R}^m)$ is the Banach space of $(n-1)$-times continuously differentiable vector functions $x :]a, b[\to \mathbb{R}^m$ having limits

$$\lim_{t\to a} (t - a)^{\alpha_i} x^{(i-1)}(t), \quad \lim_{t\to b} (b - t)^{\beta_i} x^{(i-1)}(t) \;\; (i = 1, \dots, n), \tag{1.3}$$

where

$$\alpha_i = \frac{\alpha + i - n + |\alpha + i - n|}{2}, \quad \beta_i = \frac{\beta + i - n + |\beta + i - n|}{2} \tag{1.4}$$
$$(i = 1, \dots, n).$$

The norm of an arbitrary element x of this space is defined by the equality

$$\|x\|_{C^{n-1}_{\alpha,\beta}} = \sup\Big\{ \sum_{k=1}^{n} (t-a)^{\alpha_i}(b-t)^{\beta_i} \|x^{(i-1)}(t)\| : \; a < t < b \Big\}.$$

$\widetilde{C}^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)$ is the set of $x \in C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)$ for which $x^{(n-1)}$ is locally absolutely continuous on $]a,b[$, i.e., absolutely continuous on $[a+\varepsilon, b-\varepsilon]$ for arbitrarily small positive ε.

$L_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)$ and $L_{\alpha,\beta}(]a,b[\,;\mathbb{R}^{m\times m})$ are respectively the Banach space of vector functions $y :]a,b[\to \mathbb{R}^m$ and the Banach space of matrix functions $Y :]a,b[\to \mathbb{R}^{m\times m}$ whose components are summable with weight $(t-a)^\alpha (b-t)^\beta$. The norms in these spaces are defined by the equalities

$$\|y\|_{L_{\alpha,\beta}} = \int_a^b (t-a)^\alpha (b-t)^\beta \|y(t)\|\, dt, \quad \|Y\|_{L_{\alpha,\beta}} = \int_a^b (t-a)^\alpha (b-t)^\beta \|Y(t)\|\, dt.$$

$L_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m_+) = \{y \in L_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m) : \; y(t) \in \mathbb{R}^m_+ \text{ for } t \in]a,b[\,\}$.

$L_{\alpha,\beta}(]a,b[\,;\mathbb{R}^{m\times m}_+) = \{Y \in L_{\alpha,\beta}(]a,b[\,;\mathbb{R}^{m\times m}) : \; Y(t) \in \mathbb{R}^{m\times m}_+ \text{ for } t \in]a,b[\,\}$.

In the sequel it will always be assumed that $-\infty < a < b < +\infty$,

$$\alpha \in [0, n-1], \quad \beta \in [0, n-1], \tag{1.5}$$

whereas $f : C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m) \to L_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)$ and $h_i : C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m) \to \mathbb{R}^m$ $(i = 1, \dots, n)$ are continuous operators which, for each $\rho \in]0, +\infty[$, satisfy the conditions

$$\sup\Big\{ \|f(x)(\cdot)\| : \; \|x\|_{C^{n-1}_{\alpha,\beta}} \le \rho \Big\} \in L_{\alpha,\beta}(]a,b[\,;\mathbb{R}_+), \tag{1.6}$$

$$\sup\Big\{ \|h_i(x)\| : \; \|x\|_{C^{n-1}_{\alpha,\beta}} \le \rho \Big\} < +\infty \;\; (i = 1, \dots, n). \tag{1.7}$$

By a solution of the functional differential equation (1.1) is understood a vector function $x \in \widetilde{C}^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)$ satisfying (1.1) almost everywhere on $]a,b[$. A solution of (1.1) satisfying (1.2) is called a solution of problem (1.1), (1.2).

1.2. Theorem on the Fredholm property of a linear boundary value problem. We begin by introducing

Definition 1.1. A linear operator $p : C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m) \to \mathbb{R}^m$ is called strongly bounded if there exists $\zeta \in L_{\alpha,\beta}(]a,b[\,;\mathbb{R}_+)$ such that

$$\|p(x)(t)\| \le \zeta(t) \|x\|_{C^{n-1}_{\alpha,\beta}} \;\text{ for }\; a < t < b, \;\; x \in C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m). \tag{1.8}$$

Consider the boundary value problem

$$x^{(n)}(t) = p(x)(t) + q(t), \tag{1.9}$$

$$\ell_i(x) = c_{0i} \quad (i = 1, \dots, n), \tag{1.10}$$

where $p : C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m) \to L_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)$ is a linear, strongly bounded operator, $\ell_i : C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m) \to \mathbb{R}^m$ $(i = 1, \dots, m)$ are linear bounded operators,

$$q \in L_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m), \quad c_{0i} \in \mathbb{R}^m \quad (i = 1, \dots, m).$$

Theorem 1.1. *For problem* (1.9), (1.10) *to be uniquely solvable it is necessary and sufficient that the corresponding homogeneous problem*

$$x^{(n)}(t) = p(x)(t), \tag{1.9_0}$$

$$\ell_i(x) = 0 \quad (i = 1, \dots, n) \tag{1.10_0}$$

have only a trivial solution. Moreover, if problem (1.9_0) (1.10_0) *has only a trivial solution, then there exists a positive constant* γ *such that for any* $q \in L_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)$ *and* $c_{0i} \in \mathbb{R}^m$ $(i = 1, \dots, m)$*, a solution* x *of problem* (1.9), (1.10) *admits the estimate*

$$\|x\|_{C^{n-1}_{\alpha,\beta}} \le \gamma \Big(\sum_{i=1}^{n} \|c_{0i}\| + \|q\|_{L_{\alpha,\beta}} \Big). \tag{1.11}$$

The vector differential equation with deviating arguments

$$x^{(n)}(t) = \sum_{i=1}^{n} \mathcal{P}_i(t) x^{(i-1)}(\tau_i(t)) + q(t), \tag{1.12}$$

where $\tau_i : [a,b] \to [a,b]$ $(i = 1, \dots, n)$ are measurable functions, $\mathcal{P}_i :]a,b[\to \mathbb{R}^{m\times m}$ $(i = 1, \dots, n)$ are matrix functions with measurable components and $q \in L_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)$, is a particular case of equation (1.9). Along with (1.12), consider the corresponding homogeneous equation

$$x^{(n)}(t) = \sum_{i=1}^{n} \mathcal{P}_i(t) x^{(i-1)}(\tau_i(t)). \tag{1.12_0}$$

From Theorem 1.1 follows

Corollary 1.1. *Let almost everywhere on* $]a,b[$ *the inequalities*

$$\tau_i(t) > a \ \text{ for } \ i > n - \alpha, \quad \tau_j(t) < b \ \text{ for } \ j > n - \beta \tag{1.13}$$

be fulfilled. Moreover,

$$\int_a^b (t-a)^{\alpha} (b-t)^{\beta} \Big(\tau_i(t) - a\Big)^{-\alpha_i} \Big(b - \tau_i(t)\Big)^{-\beta_i} \|\mathcal{P}_i(t)\|\, dt$$
$$< +\infty \quad (i = 1, \dots, n).^{*)} \tag{1.14}$$

$^{*)}$ Here and in the sequel it will be assumed that if $\alpha_i = 0$ $(\beta_i = 0)$, then $(\tau_i(t) - a)^{-\alpha_i} \equiv 1$ $((\tau_i(t) - b)^{-\beta_i} \equiv 1)$.

Then for problem (1.12), (1.2) *to be uniquely solvable, it is necessary and sufficient that the corresponding homogeneous problem* (1.12_0), (1.2_0) *have only a trivial solution. Moreover, if problem* (1.12_0) (1.2_0) *has only a trivial solution, then there exists a positive constant* γ *such that for any* $q \in L_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)$ *and* $c_{0i} \in \mathbb{R}^m$ $(i = 1,\dots,m)$*, a solution* x *of problem* (1.12), (1.2) *admits estimate* (1.11)*.*

1.3. A priori boundedness principle for the nonlinear problem (1.1), (1.2). To formulate this principle we have to introduce

Definition 1.2. Let γ be a positive number. The pair $(p,(\ell_i)_{i=1}^n)$ of continuous operators $p : C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m) \times C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m) \to L_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)$ and $(\ell_i)_{i=1}^n : C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m) \times C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m) \to \mathbb{R}^{mn}$ is said to be γ-consistent if:

(i) the operators $p(x,\cdot) : C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m) \to L_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)$ and $\ell_i(x,\cdot) : C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m) \to \mathbb{R}^m$ are linear for any fixed $x \in C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)$ and $i \in \{1,\dots,n\}$;

(ii) for any x and $y \in C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)$ and for almost all $t \in]a,b[$ we have inequalities

$$\|p(x,y)(t)\| \le \delta\Big(t, \|x\|_{C^{n-1}_{\alpha,\beta}}\Big)\|y\|_{C^{n-1}_{\alpha,\beta}}, \quad \sum_{i=1}^n \|\ell_i(x,y)\| \le \delta_0\Big(\|x\|_{C^{n-1}_{\alpha,\beta}}\Big)\|y\|_{C^{n-1}_{\alpha,\beta}},$$

where $\delta_0 : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing, $\delta(\cdot,\rho) \in L_{\alpha,\beta}(]a,b[\,;\mathbb{R}_+)$ for every $\rho \in \mathbb{R}_+$, and $\delta(t,\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing for every $t \in]a,b[$;

(iii) for any $x \in C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)$, $q \in L_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)$ and $c_i \in \mathbb{R}^m$ $(i = 1,\dots,n)$, an arbitrary solution y of the boundary value problem

$$y^{(n)}(t) = p(x,y)(t) + q(t), \quad \ell_i(x,y) = c_i \ \ (i = 1,\dots,n) \tag{1.15}$$

admits the estimate

$$\|y\|_{C^{n-1}_{\alpha,\beta}} \le \gamma\Big(\sum_{i=1}^n \|c_i\| + \|q\|_{L_{\alpha,\beta}}\Big). \tag{1.16}$$

Definition 1.2′. The pair $(p,(\ell_i)_{i=1}^n)$ of continuous operators $p : C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m) \times C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m) \to L_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)$ and $(\ell_i)_{i=1}^n : C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m) \times C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m) \to \mathbb{R}^{mn}$ is said to be consistent if there exists $\gamma > 0$ such that this pair is γ-consistent.

Theorem 1.2. *Let there exist a positive number* ρ_0 *and a consistent pair* $(p,(\ell_i)_{i=1}^n)$ *of continuous operators* $p : C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m) \times C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m) \to \mathbb{R}^{mn}$ *and* $(\ell_i)_{i=1}^n : C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m) \times C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m) \to \mathbb{R}^{mn}$ *such that for any* $\lambda \in]0,1[$ *an arbitrary solution of the problem*

$$x^{(n)}(t) = (1-\lambda)p(x,x)(t) + \lambda f(x)(t), \tag{1.17}$$

$$(\lambda - 1)\ell_i(x,x) = \lambda h_i(x) \ \ (i = 1,\dots,n) \tag{1.18}$$

admits the estimate

$$\|x\|_{C^{n-1}_{\alpha,\beta}} \le \rho_0. \tag{1.19}$$

Then problem (1.1), (1.2) *is solvable.*

For $n = 1$ and $\alpha = \beta = 0$, Theorem 1.2 implies Theorem 1 from [27].

Corollary 1.2. *Let there exist a positive number γ, a γ-consistent pair $(p, (\ell_i)_{i=1}^n)$ of continuous operators $p : C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m) \times C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m) \to L_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)$, $(\ell_i)_{i=1}^n : C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m) \times C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m) \to \mathbb{R}^{mn}$ and functions $\eta :]a,b[\, \times \mathbb{R}_+ \to \mathbb{R}_+$ and $\eta_0 : \mathbb{R}_+ \to \mathbb{R}_+$ such that the inequalities*

$$\Big\| f(x)(t) - p(x,x)(t) \Big\| \le \eta\Big(t, \|x\|_{C^{n-1}_{\alpha,\beta}}\Big), \tag{1.20}$$

$$\sum_{i=1}^n \Big\| h_i(x) - \ell_i(x,x) \Big\| \le \eta_0\Big(\|x\|_{C^{n-1}_{\alpha,\beta}}\Big) \tag{1.21}$$

are fulfilled for any $x \in C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)$ and almost all $t \in]a,b[$. Moreover, $\eta(\cdot,\rho) \in L_{\alpha,\beta}(]a,b[\,;\mathbb{R}_+)$ for $\rho \in \mathbb{R}_+$ and

$$\limsup_{\rho \to +\infty} \left(\frac{\eta_0(\rho)}{\rho} + \frac{1}{\rho} \int_a^b (s-a)^\alpha (b-s)^\beta \eta(s,\rho)\, ds \right) < \frac{1}{\gamma}\,. \tag{1.22}$$

Then problem (1.1), (1.2) *is solvable.*

As an example, in $C^{n-1}_{\alpha,0}(]a,b[\,;\mathbb{R}^m)$ consider the boundary value problem

$$x^{(n)}(t) = g\Big(t, x(\tau_1(t)), \dots, x^{(n-1)}(\tau_n(t))\Big), \tag{1.23}$$

$$\begin{aligned} &\lim_{t \to a} x^{(i-1)}(t) = c_i(x) \quad (i = 1, \dots, k), \\ &\lim_{t \to b} x^{(i-1)}(t) = c_i(x) \quad (i = k+1, \dots, n). \end{aligned} \tag{1.24}$$

Here $k \in \{1, \dots, n-1\}$, $\alpha \in [0, n-k]$, $\tau_i : [a,b] \to [a,b]$ $(i = 1, \dots, n)$ are measurable functions, $c_i : \widetilde{C}^{n-1}_{\alpha,0}(]a,b[\,;\mathbb{R}^m) \to \mathbb{R}^m$ $(i = 1, \dots, m)$ are continuous operators, and $g :]a,b[\, \times \mathbb{R}^{mn} \to \mathbb{R}^m$ is a vector function such that $g(\cdot, x_1, \dots, x_n) :]a,b[\to \mathbb{R}^m$ is measurable for any $x_i \in \mathbb{R}^m$ $(i = 1, \dots, n)$ and $g(t, \cdot, \dots, \cdot) : \mathbb{R}^{mn} \to \mathbb{R}^m$ is continuous for almost all $t \in]a,b[$. We will also suppose that for $i > n - \alpha$ the inequality

$$\tau_i(t) > a$$

holds almost everywhere on $]a,b[$.

The following statement is valid.

Corollary 1.3. *Let there exist $\eta_0 : \mathbb{R}_+ \to \mathbb{R}_+$, $\mathcal{P}_i \in L_{\alpha,0}(]a,b[\,;\mathbb{R}^{mn}_+)$ $(i = 1, \dots, n)$ and $q :]a,b[\, \times \mathbb{R}_+ \to \mathbb{R}^m_+$ such that*

$$\sum_{i=1}^n \|c_i(x)\| \le \eta_0\Big(\|x\|_{C^{n-1}_{\alpha,0}}\Big) \ \ \text{for } x \in C^{n-1}_{\alpha,0}(]a,b[\,;\mathbb{R}^m) \tag{1.25}$$

and on $]a,b[\,\times\mathbb{R}^{mn}$ *the inequality*

$$\Big|g(t,x_1,\dots,x_n)\Big| \leq \sum_{i=1}^{n}\Big(\tau_i(t)-a\Big)^{\alpha_i}\mathcal{P}_i(t)|x_i| + q\Big(t,\sum_{i=1}^{n}\Big(\tau_i(t)-a\Big)^{\alpha_i}\|x_i\|\Big) \tag{1.26}$$

holds. Let, moreover, $q(\cdot,\rho)\in L_{\alpha,0}(]a,b[\,;\mathbb{R}_+^m)$ *for every* $\rho\in\mathbb{R}_+$*, the components of* $q(t,\rho)$ *are nondecreasing with respect to* ρ*,*

$$\lim_{\rho\to+\infty}\Big(\frac{\eta_0(\rho)}{\rho}+\frac{1}{\rho}\int_a^b (s-a)^{\alpha}\|q(s,\rho)\|\,ds\Big)=0 \tag{1.27}$$

and

$$r(\mathcal{P})<1, \tag{1.28}$$

where

$$\mathcal{P}=\sum_{i=1}^{k}\frac{(b-a)^{n-k-1-\alpha+\alpha_{k+1}}}{(n-k-1)!(k+1-i-\alpha_{k+1})!}\int_a^b (s-a)^{\alpha}\Big(\tau_i(s)-a\Big)^{k+1-i-\alpha_{k+1}}\mathcal{P}_i(s)\,ds + \sum_{i=k+1}^{n}\frac{(b-a)^{n-i-\alpha+\alpha_i}}{(n-i)!}\int_a^b (s-a)^{\alpha}\mathcal{P}_i(s)\,ds.$$

Then problem (1.23), (1.24) *is solvable.*

Before passing to the formulation of the next corollary we introduce

Definition 1.3. An operator $p: C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)\to L_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)$ (an operator $\ell: C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)\to\mathbb{R}^m$) is called positive homogeneous if the equality

$$p(\lambda x)(t)=\lambda p(x)(t)\quad\Big(\ell(\lambda x)=\lambda\ell(x)\Big)$$

is fulfilled for all $x\in C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)$, $\lambda\in\mathbb{R}_+$ and almost all $t\in\,]a,b[$.

Definition 1.4. A positive homogeneous operator $p: C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)\to L_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)$ (a positive homogeneous operator $\ell: C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)\to\mathbb{R}^m$) is called strongly bounded (bounded) if there exists a function $\zeta\in L_{\alpha,\beta}(]a,b[\,;\mathbb{R}_+)$ (a positive number ζ_0) such that the inequality

$$\|p(x)(t)\|\leq\zeta(t)\|x\|_{C^{n-1}_{\alpha,\beta}}\quad\Big(\|\ell(x)\|\leq\zeta_0\|x\|_{C^{n-1}_{\alpha,\beta}}\Big)$$

holds for all $x\in C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)$ and almost all $t\in\,]a,b[$.

Corollary 1.4. *Let there exist a linear, strongly bounded operator* $p: C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)\to L_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)$*, a positive homogeneous, continuous, strongly bounded operator* $\overline{p}: C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)\to L_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)$*, linear bounded* *operators* $\ell_i : C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m) \to \mathbb{R}^m$ $(i=1,\dots,n)$, *positive homogeneous, continuous, bounded operators* $\overline{\ell}_i : C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m) \to \mathbb{R}^m$ $(i=1,\dots,m)$, *and functions* $\eta :]a,b[\,\times\mathbb{R}_+$ *and* $\eta_0 : \mathbb{R}_+ \to \mathbb{R}_+$ *such that the inequalities*

$$\Big\|f(x)(t) - p(x)(t) - \overline{p}(x)(t)\Big\| \le \eta\Big(t, \|x\|_{C^{n-1}_{\alpha,\beta}}\Big), \tag{1.29}$$

$$\sum_{i=1}^{n} \Big\|h_i(x) - \ell_i(x) - \overline{\ell}_i(x)\Big\| \le \eta_0\Big(\|x\|_{C^{n-1}_{\alpha,\beta}}\Big) \tag{1.30}$$

hold for any $x \in C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)$ *and for almost all* $t \in]a,b[$. *Moreover,* $\eta(\cdot,\rho) \in L_{\alpha,\beta}(]a,b[\,;\mathbb{R}_+)$ *for any* $\rho \in \mathbb{R}_+$,

$$\lim_{\rho\to+\infty}\left(\frac{\eta_0(\rho)}{\rho} + \frac{1}{\rho}\int_a^b (s-a)^\alpha (b-s)^\beta \eta(s,\rho)\,ds\right) = 0 \tag{1.31}$$

and for any $\lambda \in [0,1]$ *the problem*

$$x^{(n)}(t) = p(x)(t) + \lambda\overline{p}(x)(t), \quad \ell_i(x) + \lambda\overline{\ell}_i(x) = 0 \ \ (i=1,\dots,n) \tag{1.32}$$

has only a trivial solution. Then problem (1.1), (1.2) *is solvable.*

As an example, for the second order singular half-linear differential equation

$$u''(t) = p_1(t)|u(t)|^\mu |u'(t)|^{1-\mu}\operatorname{sgn} u(t) + p_2(t)u'(t) + p_0(t) \tag{1.33}$$

let us consider the two-point boundary value problems

$$\lim_{t\to a} u(t) = c_1, \quad \lim_{t\to b} u(t) = c_2 \tag{1.34$_1$}$$

and

$$\lim_{t\to a} u(t) = c_1, \quad \lim_{t\to b} u'(t) = c_2. \tag{1.34$_2$}$$

We are interested in the case where $\mu \in [0,1]$ and $p_i :]a,b[\, \to \mathbb{R}$ $(i=0,1,2)$ are measurable functions satisfying either the conditions

$$\int_a^b (t-a)(b-t)|p_i(t)|\,dt < +\infty \ \ (i=0,1), \quad \int_a^b |p_2(t)|\,dt < +\infty, \tag{1.35$_1$}$$

$$p_1(t) \ge -\lambda_1[\sigma(t)]^{1+\mu}, \quad \left[p_2(t) - \frac{\sigma'(t)}{\sigma(t)}\right]\operatorname{sgn}(t_0 - t) \ge -\lambda_2\sigma(t) \quad \text{for } a<t<b, \tag{1.36$_1$}$$

or the conditions

$$\int_a^b (t-a)|p_0(t)|\,dt < +\infty \ \ (i=0,1), \quad \int_a^b |p_2(t)|\,dt < +\infty, \tag{1.35$_2$}$$

$$p_1(t) \ge -\lambda_1[\sigma(t)]^{1+\mu}, \quad p_2(t) - \frac{\sigma'(t)}{\sigma(t)} \ge -\lambda_2\sigma(t) \ \text{ for } a<t<b. \tag{1.36$_2$}$$

Here $t_0 \in]a,b[$, $\lambda_i \in \mathbb{R}_+$ $(i=1,2)$, and $\sigma :]a,b[\to \mathbb{R}_+$ is a locally absolutely continuous function such that either

$$\sigma'(t)\operatorname{sgn}(t_0-t) \le 0 \ \text{ for } \ a<t<b, \quad \int\limits_0^{+\infty} \frac{ds}{\lambda_1+\lambda_2 s+s^{(1+\mu)/\mu}}$$
$$> \frac{\mu}{2}\Bigg[\int\limits_a^b \sigma(s)\,ds + \Bigg|\int\limits_a^{t_0}\sigma(s)\,ds - \int\limits_{t_0}^b \sigma(s)\,ds\Bigg|\Bigg], \tag{1.37$_1$}$$

or

$$\int\limits_0^{+\infty} \frac{ds}{\lambda_1+\lambda_2 s+s^{(1+\mu)/\mu}} > \mu\int\limits_a^b \sigma(s)\,ds. \tag{1.37$_2$}$$

By virtue of Theorems 3.1 and 3.2 from [9] Corollary 1.4 implies

Corollary 1.5. *Let conditions* (1.35_i), (1.36_i) *and* (1.37_i) *be fulfilled for some* $i \in \{1,2\}$. *Then problem* $(1.33), (1.34_i)$ *has at least one solution.*

This corollary is a generalization of the classical result of Ch. de la Vallée-Poussin [44] for equation (1.33).

2. Auxiliary Propositions

Lemma 2.1. *Let* $\rho>0$, $\eta \in L_{\alpha,\beta}(]a,b[\,;\mathbb{R}_+)$, $t_0 \in]a,b[$, *and* S *be the set of* $(n-1)$*-times continuously differentiable vector functions* $x :]a,b[\to \mathbb{R}^m$ *satisfying the conditions*

$$\Big\|x^{(i-1)}(t_0)\Big\| \le \rho \ \ (i=1,\dots,n), \tag{2.1}$$
$$\Big\|x^{(n-1)}(t)-x^{(n-1)}(s)\Big\| \le \int\limits_s^t \eta(\xi)\,d\xi \ \ \textit{for} \ \ a<s\le t<b. \tag{2.2}$$

Then $S \subset \widetilde{C}^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)$ *and* S *is a compact set of the space* $C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)$.

Proof. Let x be an arbitrary element of the set S. Then by (2.2) the function $x^{(n-1)}$ is locally absolutely continuous on $]a,b[$ and

$$\|x^{(n)}(t)\| \le \eta(t) \ \text{ for almost all } \ t \in]a,b[\,. \tag{2.3}$$

Therefore

$$x^{(n)} \in L_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m), \tag{2.4}$$
$$x^{(i-1)}(t) = \sum_{j=i}^n \frac{(t-t_0)^{j-i}}{(j-i)!}\,x^{(j-1)}(t_0)$$
$$+\frac{1}{(n-i)!}\int\limits_{t_0}^t (t-s)^{n-i}x^{(n)}(s)\,ds \ \text{ for } \ a<t<b \ \ (i=1,\dots,n), \tag{2.5}$$

and

$$\left\|x^{(i-1)}(t)\right\| \leq \varepsilon_i(t) \ \text{ for } \ a<t<b \ \ (i=1,\dots,n), \tag{2.6}$$

where

$$\varepsilon_i(t)=\rho\sum_{j=i}^{n}\frac{(b-a)^{j-i}}{(j-i)!}+\frac{1}{(n-i)!}\left|\int_{t_0}^{t}(t-s)^{n-i}\eta(s)\,ds\right| \ \ (i=1,\dots,n). \tag{2.7}$$

Let

$$i_1=\max\{i:\ \alpha_i=0\}, \quad i_2=\max\{i:\ \beta_i=0\}.$$

Then

$$n-i\geq\alpha, \quad \alpha_i=0 \ \text{ for } \ i\leq i_1, \quad \alpha_i=\alpha+i-n>0 \ \text{ for } \ i>i_1, \tag{2.8$_1$}$$
$$n-i\geq\beta, \quad \beta_i=0 \ \text{ for } \ i\leq i_2, \quad \beta_i=\beta+i-n>0 \ \text{ for } \ i>i_2. \tag{2.8$_2$}$$

Therefore

$$\varepsilon_i(t)\leq\varepsilon_i(a+)<+\infty \ \text{ for } \ i\leq i_1, \ \ a<t\leq t_0, \tag{2.9}$$
$$\int_a^{t_0}\varepsilon_{1+i_1}(s)\,ds<+\infty \ \text{ if } \ i_1<n-1, \tag{2.10}$$
$$\varepsilon_i(t)\leq\varepsilon_i(b-)<+\infty \ \text{ for } \ i\leq i_2, \ \ t_0\leq t<b, \tag{2.11}$$
$$\int_{t_0}^{b}\varepsilon_{1+i_2}(s)\,ds<+\infty \ \text{ if } \ i_2<n-1. \tag{2.12}$$

If $i>i_1$, then, with (2.7) and (2.8$_1$) taken into account, for any $\delta\in\,]0,t_0-a[$ we find

$$\limsup_{t\to a}\Big[(t-a)^{\alpha_i}\varepsilon_i(t)\Big]=\limsup_{t\to a}\left[\frac{(t-a)^{\alpha+i-n}}{(n-i)!}\int_t^{a+\delta}(s-t)^{n-i}\eta(s)\,ds\right]$$
$$\leq\frac{1}{(n-i)!}\int_a^{a+\delta}(s-a)^{\alpha}\eta(s)\,ds.$$

Hence, because of the arbitrariness of δ, it follows that

$$\lim_{t\to a}\Big[(t-a)^{\alpha_i}\varepsilon_i(t)\Big]=0 \ \text{ for } \ i>i_1. \tag{2.13}$$

Analogously, it can be shown that

$$\lim_{t\to b}\Big[(b-t)^{\beta_i}\varepsilon_i(t)\Big]=0 \ \text{ for } \ i>i_2. \tag{2.14}$$

If $i \le i_1$ (if $i \le i_2$), then by virtue of conditions (2.3) and (2.8_1) (conditions (2.3) and (2.8_2)) we have

$$\int_a^{t_0} (s-a)^{n-i}\|x^{(n)}(s)\|\,ds < +\infty \quad \Big(\int_{t_0}^{b} (b-s)^{n-i}\|x^{(n)}(s)\|\,ds < +\infty \Big).$$

Hence (2.5) implies the existence of the limit

$$\lim_{t\to a} x^{(i-1)}(t) \quad \Big(\lim_{t\to b} x^{(i-1)}(t) \Big).$$

If however $i > i_1$ ($i > i_2$), then from (2.6) and (2.13) (from (2.6) and (2.14)) we have

$$\lim_{t\to a}(t-a)^{\alpha_i} x^{(i-1)}(t) = 0 \quad \Big(\lim_{t\to b}(b-t)^{\beta_i} x^{(i-1)}(t) = 0 \Big).$$

We have thereby proved the existence of limit (1.3). Therefore $S \subset \widetilde{C}^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)$.

By the Arzela–Ascoli lemma, from estimates (2.3), (2.6) and conditions (2.9)–(2.14) it follows that S is a compact set of the space $C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)$. $\square$

Let $(p,(\ell_i)_{i=1}^n)$ be a γ-consistent pair of continuous operators $p : C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m) \times C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m) \to L_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)$ and $(\ell_i)_{i=1}^n : C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m) \times C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m) \to \mathbb{R}^{mn}$, and $q : C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m) \to L_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)$, $c_{0i} : C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m) \to \mathbb{R}^m$ ($i=1,\dots,n$) be continuous operators. For any $x \in C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)$, consider the linear boundary value problem

$$y^{(n)}(t) = p(x,y)(t) + q(x)(t), \quad \ell_i(x,y) = c_{0i}(x) \ \ (i=1,\dots,n). \tag{2.15}$$

By condition (iii) of Definition 1.2, the homogeneous problem

$$y^{(n)}(t) = p(x,y)(t), \quad \ell_i(x,y) = 0 \ \ (i=1,\dots,n) \tag{2.15_0}$$

has only a trivial solution. By Theorem 1.1 this fact guarantees the existence of a unique solution y of problem (2.15). We write

$$u(x)(t) = y(t).$$

Lemma 2.2. *$u : C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m) \to C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)$ is a continuous operator.*

Proof. Let

$$x_i \in C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m), \quad y_i(t) = u(x_i)(t) \ \ (i=1,2)$$

and

$$y(t) = y_2(t) - y_1(t).$$

Then

$$y^{(n)}(t) = p_2(x_2,y)(t) + q_0(x_1,x_2)(t),$$
$$\ell_i(x_2,y) = c_i(x_1,x_2) \ \ (i=1,\dots,n),$$

where

$$q_0(x_1,x_2)(t)=p(x_1,y_1)(t)-p(x_2,y_1)(t)+q(x_2)(t)-q(x_1)(t),$$
$$c_i(x_1,x_2)=\ell_i(x_1,x_2)-\ell_i(x_2,y_1)+c_{0i}(x_2)-c_{0i}(x_1)\ \ (i=1,\dots,n).$$

Hence, by condition (iii) of Definition 1.2 we have

$$\Big\|u(x_2)-u(x_1)\Big\|_{C^{n-1}_{\alpha,\beta}}\le\gamma\Big(\sum_{i=1}^n\|c_i(x_1,x_2)\|+\|q_0(x_1,x_2)\|_{L_{\alpha,\beta}}\Big).$$

Since the operators p, q, ℓ_i and c_{0i} $(i=1,\dots,n)$ are continuous, this estimate implies the continuity of the operator u. □

Lemma 2.3. *Let $k\in\{1,\dots,n-1\}$, $\alpha\in[0,n-k]$, and $x\in C^{n-1}_{\alpha,0}(]a,b[\,;\mathbb{R}^m)$ be a vector function satisfying conditions* (1.24). *Then on $]a,b[$ the following inequalities are fulfilled:*

$$|x^{(i-1)}(t)|\le\sum_{j=i}^n(b-a)^{j-i}|c_j(x)|$$
$$+\frac{1}{(n-i)!}\,(b-a)^{n-i-\alpha+\alpha_i}(t-a)^{-\alpha_i}y(x)\ \ (i=k+1,\dots,n),\tag{2.16}$$
$$|x^{(i-1)}(t)|\le\sum_{j=i}^n(b-a)^{j-i}|c_j(x)|$$
$$+\frac{(b-a)^{n-k-1-\alpha+\alpha_{k+1}}}{(n-k-1)!(k+1-i-\alpha_{k+1})!}\,(t-a)^{k+1-i-\alpha_{k+1}}y(x)\ \ (i=1,\dots,k),\tag{2.17}$$

where

$$y(x)=\int_a^b(s-a)^\alpha|x^{(n)}(s)|\,ds.\tag{2.18}$$

Proof. Let $x_0(t)$ be a polynomial of degree not higher than $n-1$ satisfying the conditions

$$x_0^{(i-1)}(a)=c_i(x)\ \ (i=1,\dots,k),\quad x_0^{(i-1)}(b)=c_i(x)\ \ (i=k+1,\dots,n).$$

Then

$$|x_0^{(i-1)}(t)|\le\sum_{j=i}^n(b-a)^{j-i}|c_j(x)|\ \text{ for }\ a\le t\le b\ \ (i=1,\dots,n).\tag{2.19}$$

On the other hand,

$$x^{(i-1)}(t)=x_0^{(i-1)}(t)-\frac{(-1)^{n-i}}{(n-i)!}\int_t^b(s-t)^{n-i}x^{(n)}(s)\,ds\tag{2.20}$$
$$(i=k+1,\dots,n),$$

$$x^{(i-1)}(t)=c_i(x)+\int_a^t x^{(i)}(s)\,ds\ \ (i=1,\dots,k).\tag{2.21}$$

By (1.4)

$$n-i-\alpha-\alpha_i \geq 0 \quad (i=1,\dots,n).$$

Therefore

$$\begin{aligned}(s-t)^{n-i} &\leq (s-a)^{n-i-\alpha+\alpha_i}(s-a)^{-\alpha_i}(s-a)^{\alpha}\\ &\leq (b-a)^{n-i-\alpha+\alpha_i}(t-a)^{-\alpha_i}(s-a)^{\alpha} \ \text{ for } \ t\leq s<b \quad (i=1,\dots,n).\end{aligned}$$

If along with this we take into account inequality (2.19), then from (2.20) we obtain estimates (2.16).

It is clear that

$$\alpha_{k+1}\leq 1,$$

since $\alpha \leq n-k$. If $\alpha_{k+1}<1$, then by virtue of (2.16) and (2.19), from (2.21) follow estimates (2.17).

To complete the proof of the lemma it remains to consider the case where $\alpha_{k+1}=1$. Then $\alpha=n-k$ and thus from (2.19)–(2.21) we find

$$\begin{aligned}|x^{(k-1)}(t)| &\leq \sum_{j=k}^{n}(b-a)^{j-k}|c_j(x)|\\ &+\frac{1}{(n-k-1)!}\int_a^t\left(\int_\tau^b (s-\tau)^{n-k-1}|x^{(n)}(s)|\,ds\right)d\tau\\ &=\sum_{j=k}^{n}(b-a)^{j-k}|c_j(x)|\\ &+\frac{1}{(n-k-1)!}\left[(t-a)\int_t^b (s-a)^{n-k-1}|x^{(n)}(s)|\,ds+\int_a^t (s-a)^{n-k}|x^{(n)}(s)|\,ds\right]\\ &\leq \sum_{j=k}^{n}(b-a)^{j-k}|c_j(x)|+\frac{1}{(n-k-1)!}\,y(x)\end{aligned}$$

and

$$|x^{(i-1)}(t)| \leq \sum_{j=i}^{n}(b-a)^{j-i}|c_j(x)|+\frac{1}{(n-k-1)!(k-i)!}\,(t-a)^{k-i}y(x) \qquad (i=1,\dots,k).$$

Therefore estimates (2.17) are valid. □

3. Proof of the Main Results

Proof of Theorem 1.1. Let $B=C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)\times\mathbb{R}^{mn}$ be a Banach space with elements $u=(x;c_1,\dots,c_n)$, where $x\in C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)$, $c_i\in\mathbb{R}^m$ $(i=1,\dots,n)$, and the norm

$$\|u\|_B=\|u\|_{C^{n-1}_{\alpha,\beta}}+\sum_{i=1}^{n}\|c_i\|.$$

Fix arbitrarily $t_0 \in]a,b[$ and, for any $u = (x; c_1, \dots, c_n)$, set

$$\widetilde{p}(u)(t) = \Bigg(\sum_{i=1}^{n} \frac{(t-t_0)^{i-1}}{(i-1)!} \Big(c_i + x^{(i-1)}(t_0) \Big)$$

$$+ \frac{1}{(n-1)!} \int_{t_0}^{t} (t-s)^{n-1} p(x)(s)\, ds; c_1 - \ell_1(x), \dots, c_n - \ell_n(x) \Bigg),$$

$$\widetilde{q}(t) = \Bigg(\frac{1}{(n-1)!} \int_{t_0}^{t} (t-s)^{n-1} q(s)\, ds; c_{01}, \dots, c_{0n} \Bigg).$$

Problem (1.9), (1.10) is equivalent to the operator equation

$$u = \widetilde{p}(u) + \widetilde{q} \tag{3.1}$$

in the space B since $u = (x; c_1, \dots, c_n)$ is a solution of equation (3.1) if and only if $c_i = 0$ $(i = 1, \dots, n)$ and x is a solution of problem (1.9), (1.10). As for the homogeneous equation

$$u = \widetilde{p}(u) \tag{3.1$_0$}$$

it is equivalent to the homogeneous problem $(1.9_0), (1.10_0)$.

From condition (1.8) and Lemma 2.1 it immediately follows that the linear operator $\widetilde{p} : B \to \widetilde{B}$ is compact. By this fact and the Fredholm alternative for operator equations ([13], Ch. XIII, § 5, Theorem 1), equation (3.1) is uniquely solvable if and only if equation (3.1_0) has only a trivial solution. Moreover, if equation (3.1_0) has only a trivial solution, then the operator $I - \widetilde{p}$ is invertible and $(I - \widetilde{p})^{-1} : B \to B$ is a linear bounded operator, where $I : B \to B$ is an identical operator. Therefore there exists $\gamma_0 > 0$ such that for any $\widetilde{q} \in B$ the solution u of equation (3.1) admits the estimate

$$\|u\|_B \le \gamma_0 \|\widetilde{q}\|_B.$$

However,

$$\|\widetilde{q}\|_B \le \sum_{i=1}^{n} \|c_{0i}\| + \gamma_1 \|q\|_{L_{\alpha,\beta}},$$

where $\gamma_1 > 1$ is a constant depending only on α, β, a, b, t_0 and n. Hence

$$\|u\|_B \le \gamma \Big(\sum_{i=1}^{n} \|c_{0i}\| + \|q\|_{L_{\alpha,\beta}} \Big), \tag{3.2}$$

where $\gamma = \gamma_0 \gamma_1$.

Since problem (1.9), (1.10) is equivalent to equation (3.1), it is clear that problem (1.9), (1.10) is uniquely solvable if and only if problem $(1.9_0), (1.10_0)$ has only a trivial solution. Moreover, if $(1.9_0), (1.10_0)$ has only a trivial solution, then by virtue of (3.2) the solution x of problem (1.9), (1.10) admits estimate (1.11). □

Proof of Corollary 1.1. We set

$$p(x)(t)=\sum_{i=1}^{n}\mathcal{P}_i(t)x^{(i-1)}(\tau_i(t))$$

for any $x\in C^{n-1}_{\alpha,\beta}(]a,b[;\mathbb{R}^m)$. Then equations (1.12) and (1.12_0) take respectively forms (1.9) and (1.9_0). On the other hand, in view of (1.13) and (1.14)

$$p: C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)\to L_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)$$

is a strongly bounded linear operator. Therefore the conditions of Theorem 1.1 are fulfilled. □

Proof of Theorem 1.2. Let δ, δ_0 and γ be the functions and numbers appearing in Definitions 1.2 and 1.2′. We set

$$\eta(t)=2\rho_0\delta(t,2\rho_0)+\sup\Big\{\|f(x)(t)\|:\ \ \|x\|_{C^{n-1}_{\alpha,\beta}}\le 2\rho_0\Big\},$$

$$\eta_0=2\rho_0\delta_0(2\rho_0)+\sum_{i=1}^{n}\sup\Big\{\|h_i(x)\|:\ \ \|x\|_{C^{n-1}_{\alpha,\beta}}\le 2\rho_0\Big\},$$

$$\rho_1=\gamma\Big(\eta_0+\|\eta\|_{L_{\alpha,\beta}}\Big),\quad \eta^*(t)=\delta(t,\rho_1)\rho_0+\eta(t), \tag{3.3}$$

$$B_0=\Big\{x\in C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m):\ \ \|x\|_{C^{n-1}_{\alpha,\beta}}\le\rho_1\Big\}, \tag{3.4}$$

$$\chi(s)=\begin{cases}1 & \text{for }\ 0\le s\le\rho_0\\ 2-s/\rho_0 & \text{for }\ \rho_0<s<2\rho_0\ ,\\ 0 & \text{for }\ s\ge 2\rho_0\end{cases} \tag{3.5}$$

$$q(x)(t)=\chi\Big(\|x\|_{C^{n-1}_{\alpha,\beta}}\Big)\Big[f(x)(t)-p(x,x)(t)\Big], \tag{3.6}$$

$$c_{0i}(x)=\chi\Big(\|x\|_{C^{n-1}_{\alpha,\beta}}\Big)\Big[\ell_i(x,x)-h_i(x)\Big]\ \ (i=1,\dots,n). \tag{3.7}$$

By (1.6) and (1.7)

$$\eta_0<+\infty,\quad \eta\in L_{\alpha,\beta}(]a,b[\,;\mathbb{R}_+),\quad \eta^*\in L_{\alpha,\beta}(]a,b[\,;\mathbb{R}_+)$$

and for every $x\in C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)$ and almost all $t\in\,]a,b[$ we have the inequalities

$$\|q(x)(t)\|\le\eta(t),\quad \sum_{i=1}^{n}\|c_{0i}(x)\|\le\eta_0. \tag{3.8}$$

Let $u: C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)\to C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)$ be an operator which to every $x\in C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)$ assigns the solution y of problem (2.15). By Lemma 2.1, u is a continuous operator. On the other hand, by conditions (ii) and (iii) of Definition 1.2, notations (3.3), (3.4) and inequalities (3.8), the vector function $y=u(x)$ satisfies, for each $x\in B_0$, the conditions

$$\|y\|_{C^{n-1}_{\alpha,\beta}}\le\rho_1,\quad \Big\|y^{(n-1)}(t)-y^{(n-1)}(s)\Big\|\le\int_s^t\eta^*(\xi)\,d\xi\ \ \text{for}\ \ a<s\le t<b.$$

By Lemma 2.2 this implies that the operator u maps the ball B_0 into its own compact subset. Therefore, owing to Schauder's principle, there exists $x \in B_0$ such that

$$x(t) = u(x)(t) \ \text{ for } \ a < t < b.$$

By notations (3.6), (3.7) the function x is a solution of problem (1.17), (1.18), where

$$\lambda = \chi\Big(\|x\|_{C^{n-1}_{\alpha,\beta}}\Big). \tag{3.9}$$

Let us show that x admits estimate (1.19). Assume the contrary. Then either

$$\rho_0 < \|x\|_{C^{n-1}_{\alpha,\beta}} < 2\rho_0, \tag{3.10}$$

or

$$\|x\|_{C^{n-1}_{\alpha,\beta}} \geq 2\rho_0. \tag{3.11}$$

If condition (3.10) is fulfilled, then by virtue of (3.5) and (3.9)

$$\lambda \in \,]0,1[\,,$$

which, by one of the conditions of the theorem, guarantees the validity of estimate (1.19). But this contradicts condition (3.10).

Assume now that inequality (3.11) is fulfilled. Then by virtue of (3.5) and (3.9)

$$\lambda = 0$$

and therefore x is a solution of problem (2.15_0). Thus $x(t) \equiv 0$ since problem (2.15_0) has only a trivial solution. But this contradicts inequality (3.11). The contradiction obtained proves the validity of estimate (1.19).

By (1.19), (3.5)–(3.7) and (3.9), it clearly follows from (1.17), (1.18) that $\lambda = 1$ and x is a solution of problem (1.1), (1.2). $\square$

Proof of Corollary 1.2. By (1.22) there is $\rho_0 > 0$ such that

$$\gamma\left(\eta_0(\rho) + \int\limits_a^b (s-a)^\alpha (b-s)^\beta \eta(s,\rho)\, ds\right) < \rho \ \text{ for } \ \rho > \rho_0. \tag{3.12}$$

Let x be a solution of problem (1.17), (1.18) for some $\lambda \in \,]0,1[\,$. Then $y = x$ is also a solution of problem (1.15) where

$$q(t) = \lambda\Big(f(x)(t) - p(x,x)(t)\Big),$$
$$c_{0i}(x) = \lambda\Big(\ell_i(x,x) - h_i(x)\Big) \ \ (i = 1, \dots, n).$$

Assume that

$$\rho = \|x\|_{C^{n-1}_{\alpha,\beta}}.$$

By the γ-consistency of the pair $(p,(\ell_i)_{i=1}^n)$ and inequalities (1.20), (1.21) we have

$$\rho \le \gamma\Big(\sum_{i=1}^n \|c_{0i}(x)\| + \|q\|_{L_{\alpha,\beta}}\Big)$$
$$\le \gamma\Big(\eta_0(\rho) + \int\limits_a^b (s-a)^\alpha (b-s)^\beta \eta(s,\rho)\,ds\Big).$$

Hence by (3.12) it follows that $\rho \le \rho_0$. Therefore estimate (1.19) is valid, which due to Theorem 1.2 guarantees the solvability of problem (1.1), (1.2). $\square$

Proof of Corollary 1.3. Problem (1.23), (1.24) is obtained from problem (1.1), (1.2) when

$$f(x)(t) \equiv g\Big(t, x(\tau_1(t)), \dots, x^{(n-1)}(\tau_n(t))\Big), \tag{3.13}$$

$$\begin{aligned} h_i(x) &= \lim_{t\to a} x^{(i-1)}(t) - c_i(x) \quad (i=1,\dots,k),\\ h_i(x) &= \lim_{t\to b} x^{(i-1)}(t) - c_i(x) \quad (i=k+1,\dots,n). \end{aligned} \tag{3.14}$$

By virtue of the restrictions imposed on g, τ_i, c_i $(i=1,\dots,n)$ and the inequality $\alpha \le n-k$ it is obvious that $f : C^{n-1}_{\alpha,0}(]a,b[\,;\mathbb{R}^m) \to L_{\alpha,0}(]a,b[\,;\mathbb{R}^m)$ and $h_i : C^{n-1}_{\alpha,0}(]a,b[\,;\mathbb{R}^m) \to \mathbb{R}^m$ $(i=1,\dots,m)$ are continuous operators satisfying conditions (1.6) and (1.7), where $\beta = 0$.

Assume for any x, $y \in C^{n-1}_{\alpha,0}(]a,b[\,;\mathbb{R}^m)$ and $t \in\,]a,b[$ that

$$\begin{aligned} p(x,y)(t) = 0, \quad \ell_i(x,y) &= \lim_{t\to a} y^{(i-1)}(t) \quad (i=1,\dots,k),\\ \ell_i(x,y) &= \lim_{t\to b} y^{(i-1)}(t) \quad (i=k+1,\dots,n). \end{aligned} \tag{3.15}$$

According to Definition 1.2$'$ and Theorem 1.2 the pair $(p,(\ell_i)_{i=1}^n)$ of continuous operators $p : C^{n-1}_{\alpha,0}(]a,b[\,;\mathbb{R}^m) \times C^{n-1}_{\alpha,0}(]a,b[\,;\mathbb{R}^m) \to L_{\alpha,0}(]a,b[\,;\mathbb{R}^m)$, $(\ell_i)_{i=1}^n :$ $C^{n-1}_{\alpha,0}(]a,b[\,;\mathbb{R}^m) \times C^{n-1}_{\alpha,0}(]a,b[\,;\mathbb{R}^m) \to \mathbb{R}^{mn}$, is consistent.

To prove Corollary 1.3, by Theorem 1.2 it is sufficient to show that for each $\lambda \in\,]0,1[$ an arbitrary solution x of problem (1.17), (1.18) admits the estimate

$$\|x\|_{C^{n-1}_{\alpha,0}} \le \rho_0, \tag{3.16}$$

where ρ_0 is a non-negative constant not depending on λ and x.

By virtue of (3.13)–(3.15) problem (1.17), (1.18) takes the form

$$x^{(n)}(t) = \lambda g\Big(t, x(\tau_1(t)), \dots, x^{(n-1)}(\tau_n(t))\Big), \tag{3.17}$$

$$\begin{aligned} &\lim_{t\to a} x^{(i-1)}(t) = \lambda c_i(x) \quad (i=1,\dots,k),\\ &\lim_{t\to b} x^{(i-1)}(t) = \lambda c_i(x) \quad (i=k+1,\dots,n). \end{aligned} \tag{3.18}$$

Let x be a solution of problem (3.17), (3.18) for some $\lambda \in\,]0,1[$. Then by virtue of Lemma 2.3 we conclude that estimates (2.16), (2.17), where $y(x)$ is the vector given by equality (2.18), are true. On the other hand, on account of (1.26) we have

$$y(x) \le \sum_{i=1}^{n} \int_a^b (s-a)^{\alpha} \Big(\tau_i(s)-a\Big)^{\alpha_i} \mathcal{P}_i(s) |x^{(i-1)}(s)|\, ds + \int_a^b (s-a)^{\alpha} q\Big(s, \|x\|_{C^{n-1}_{\alpha,0}}\Big)\, ds.$$

If, along with (2.16) and (2.17), we take into account that $\alpha_i = 0$ $(i = 1, \dots, k)$, then from the latter inequality we obtain

$$y(x) \le \mathcal{P} y(x) + y_0(x)$$

and therefore

$$(E - \mathcal{P}) y(x) \le y_0(x), \tag{3.19}$$

where E is the unique $m \times m$ matrix and

$$y_0(x) = \sum_{i=1}^{n} \bigg(\int_a^b \Big(\tau_i(s)-a\Big)^{\alpha_i} \mathcal{P}_i(s)\, ds \bigg) \sum_{j=i}^{n} (b-a)^{j-i} |c_j(x)| + \int_a^b (s-a)^{\alpha} q\Big(s, \|x\|_{C^{n-1}_{\alpha,0}}\Big)\, ds. \tag{3.20}$$

By the nonnegativeness of the matrix $\mathcal{P}$ and inequality (1.28), from (3.19) it follows that

$$y(x) \le (E - \mathcal{P})^{-1} y_0(x).$$

If along with this we take into account condition (1.25) and equality (3.20), then (2.16) and (2.17) imply that

$$\|x\|_{C^{n-1}_{\alpha,0}} \le \eta_1\Big(\|x\|_{C^{n-1}_{\alpha,0}}\Big), \tag{3.21}$$

where

$$\eta_1(\rho) = \mu \bigg(\eta_0(\rho) + \int_a^b (s-a)^{\alpha} \|q(s,\rho)\|\, ds \bigg),$$

and μ is a positive constant depending only on α_i, $\mathcal{P}_i$, τ_i $(i = 1, \dots, n)$, a and b. On the other hand, due to condition (1.27) we have

$$\lim_{\rho \to +\infty} \frac{\eta_1(\rho)}{\rho} = 0$$

and therefore

$$\eta_1(\rho) < \rho \ \text{ for } \ \rho > \rho_0,$$

where

$$\rho_0 = \inf \bigg\{ \rho > 0 : \ \frac{\eta(s)}{s} < 1 \ \text{ for } \ s \in [\rho, +\infty[\bigg\}.$$

Therefore from (3.21) we obtain estimate (3.16). On the other hand, it is obvious that the constant ρ_0 does not depend on λ and x. $\square$

Proof of Corollary 1.4. The strong boundedness of the operators p and $\overline{p}$ and the boundedness of the operators $(\ell_i)_{i=1}^n$ and $(\overline{\ell}_i)_{i=1}^n$ guarantee the existence of $\zeta \in L_{\alpha,\beta}(]a,b[\,;\mathbb{R}_+)$ and $\zeta_0 \in \mathbb{R}_+$ such that the inequalities

$$\begin{gathered} \|p(x)(t)\| + \|\overline{p}(x)(t)\| \le \zeta(t)\|x\|_{C^{n-1}_{\alpha,\beta}}, \\ \sum_{i=1}^n \Big(\|\ell_i(x)\| + \|\overline{\ell}_i(x)\|\Big) \le \zeta_0\|x\|_{C^{n-1}_{\alpha,\beta}} \end{gathered} \tag{3.22}$$

hold for each $x \in C^{n-1}_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)$ and almost all $t \in]a,b[$.

By Theorem 1.1 and Definition 1.2$'$ the pair $(p,(\ell_i)_{i=1}^n)$ is consistent since for $\lambda = 0$ problem (1.32) has only a trivial solution.

Let us consider for arbitrary $\lambda \in [0,1]$, $q \in L_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m)$ and $c_{0i} \in \mathbb{R}^m$ $(i = 1,\dots,n)$ the boundary value problem

$$x^{(n)}(t) = p(x)(t) + \lambda\overline{p}(x)(t) + q(t), \tag{3.23}$$

$$\ell_i(x) + \lambda\overline{\ell}_i(x) = c_{0i} \quad (i = 1,\dots,n) \tag{3.24}$$

and prove that every solution x of this problem admits the estimate

$$\|x\|_{C^{n-1}_{\alpha,\beta}} \le \gamma\Big(\sum_{i=1}^n \|c_{0i}\| + \|q\|_{L_{\alpha,\beta}}\Big), \tag{3.25}$$

where γ is a positive constant not depending on λ, q, c_{0i} $(i = 1,\dots,n)$ and x. Assume the contrary that this is not so. Then for each natural k there are

$$\lambda_k \in [0,1], \quad q_k \in L_{\alpha,\beta}(]a,b[\,;\mathbb{R}^m), \quad c_{ki} \in \mathbb{R}^m \quad (i = 1,\dots,n)$$

such that the problem

$$\begin{gathered} x^{(n)}(t) = p(x)(t) + \lambda_k\overline{p}(x)(t) + q_k(t), \\ \ell_i(x) = \lambda_k\overline{\ell}_i(x) + c_{ki} \quad (i = 1,\dots,n) \end{gathered}$$

has a solution x_k admitting the estimate

$$\rho_k \overset{\text{def}}{=} \|x_k\|_{C^{n-1}_{\alpha,\beta}} > k\Big(\sum_{i=1}^n \|c_{ki}\| + \|q_k\|_{L_{\alpha,\beta}}\Big).$$

If we assume that

$$\overline{x}_k(t) = \rho_k^{-1}x_k(t), \quad \overline{q}_k(t) = \rho_k^{-1}q_k(t), \quad \overline{c}_{ki} = \rho_k^{-1}c_{ki} \quad (i = 1,\dots,n),$$

then we have

$$\|\overline{x}_k\|_{C^{n-1}_{\alpha,\beta}} = 1, \tag{3.26}$$

$$\|\overline{q}_k\|_{C^{n-1}_{\alpha,\beta}} < \frac{1}{k}\,, \quad \sum_{i=1}^n \|\overline{c}_{ki}\| < \frac{1}{k}\,, \tag{3.27}$$

$$\overline{x}_k^{(n)}(t) = p(\overline{x}_k)(t) + \lambda_k\overline{p}(\overline{x}_k)(t) + \overline{q}_k(t), \tag{3.28}$$

$$\ell_i(\overline{x}_k) = \lambda_k \overline{\ell}_i(\overline{x}_k) + \overline{c}_{ki} \quad (i = 1, \dots, n). \tag{3.29}$$

Let $t_0 = \frac{a+b}{2}$. Then (3.28) implies

$$\overline{x}_k(t) = y_k(t) + z_k(t), \tag{3.30}$$

where

$$\begin{aligned} y_k(t) &= \sum_{i=1}^{n} \frac{(t-t_0)^{i-1}}{(i-1)!}\, \overline{x}_k^{(i-1)}(t_0) \\ &\quad + \frac{1}{(n-1)!} \int_{t_0}^{t} (t-s)^{n-1} \Big(p(\overline{x}_k)(s) + \lambda_k \overline{p}(\overline{x}_k)(s) \Big)\, ds, \end{aligned} \tag{3.31}$$

$$z_k(t) = \frac{1}{(n-1)!} \int_{t_0}^{t} (t-s)^{n-1} \overline{q}_k(s)\, ds.$$

By (3.27) we have

$$\lim_{k \to +\infty} \|z_k\|_{C^{n-1}_{\alpha,\beta}} = 0, \quad \lim_{k \to +\infty} \overline{c}_{ki} = 0 \quad (i = 1, \dots, n). \tag{3.32}$$

On the other hand, with (3.22) and (3.26) taken into account, from (3.31) we find

$$\|y_k^{(i-1)}(t_0)\| \le \rho^* \quad (i = 1, \dots, n),$$

$$\Big\| y_k^{(n-1)}(t) - y_k^{(n-1)}(s) \Big\| \le \int_s^t \zeta(\xi)\, d\xi \quad \text{for } a < s < t < b,$$

where ρ^* is a positive constant not depending on k. By virtue of these inequalities and Lemma 1.1, we can assume without loss of generality that the sequence $(y_k)_{k=1}^{+\infty}$ is converging in the norm of the space $C^{n-1}_{\alpha,\beta}(]a, b[\,; \mathbb{R}^m)$. It can also be assumed without loss of generality that the sequence $(\lambda_k)_{k=1}^{+\infty}$ is converging. Assume that

$$\lambda = \lim_{k \to +\infty} \lambda_k, \quad x(t) = \lim_{k \to +\infty} y_k(t).$$

Then by (3.29)–(3.32) we have

$$\lim_{k \to +\infty} \|x_k - x\|_{C^{n-1}_{\alpha,\beta}} = \lim_{k \to +\infty} \|y_k - x\|_{C^{n-1}_{\alpha,\beta}} = 0 \tag{3.33}$$

and

$$\ell_i(x) = \lambda \overline{\ell}_i(x) \quad (i = 1, \dots, n),$$

$$x(t) = \sum_{i=1}^{n} \frac{(t-t_0)^{i-1}}{(i-1)!}\, x^{(i-1)}(t_0) + \frac{1}{(n-1)!} \int_{t_0}^{t} (t-s)^{n-1} \Big(p(x)(s) + \lambda \overline{p}(x)(s) \Big)\, ds.$$

Therefore x is a solution of problem (1.32). On the other hand, from (3.26) and (3.33) it clearly follows that

$$\|x\|_{C^{n-1}_{\alpha,\beta}} = 1.$$

But this is impossible because for each $\lambda \in [0,1]$ problem (1.32) has only a trivial solution. The contradiction obtained proves the existence of a positive number γ that possesses the above-mentioned property.

By condition (1.31) there is $\rho_0 > 0$ such that inequality (3.12) is fulfilled.

To prove Corollary 1.4, it is sufficient due to Theorem 1.2 to establish that for each $\lambda \in]0,1[$ an arbitrary solution x of the problem

$$x^{(n)}(t) = p(x)(t) + \lambda\Big[f(x)(t) - p(x)(t)\Big], \tag{3.34}$$

$$\ell_i(x) = \lambda\Big(\ell_i(x) - h_i(x)\Big) \ \ (i = 1,\dots,n) \tag{3.35}$$

admits estimate (1.19).

It is obvious that each solution x of problem (3.34), (3.35) is a solution of problem (3.23), (3.24), where

$$\begin{aligned} q(t) &= \lambda\Big(f(x)(t) - p(x)(t) - \overline{p}(x)(t)\Big), \\ c_{0i} &= \lambda\Big(\ell_i(x) + \overline{\ell}_i(x) - h_i(x)\Big) \ \ (i = 1,\dots,n). \end{aligned} \tag{3.36}$$

According to the above proof, x admits estimate (3.25) from which, with (1.29), (1.30) and (3.36) taken into account, we find

$$\|x\|_{C^{n-1}_{\alpha,\beta}} \le \gamma\left(\eta_0\Big(\|x\|_{C^{n-1}_{\alpha,\beta}}\Big) + \int_a^b (s-a)^\alpha (b-s)^\beta \eta\Big(s, \|x\|_{C^{n-1}_{\alpha,\beta}}\Big)\, ds\right).$$

Hence, by virtue of (3.12), we obtain estimate (1.19). □

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