Georgian Mathematical Journal
Volume 9 (2002), Number 1, 167–178

ON THE DIMENSION OF SOME SPACES OF GENERALIZED TERNARY THETA-SERIES

K. SHAVGULIDZE

Abstract. The upper bound of dimension of vector spaces of generalized theta-series corresponding to some ternary quadratic forms is established. In a number of cases, the dimension of vector spaces of generalized theta-series is established and bases of these spaces are constructed.

2000 Mathematics Subject Classification: 11E20, 11F27, 11F30.
Key words and phrases: Positive reduced ternary quadratic form, spherical polynomial with respect positive quadratic form, vector space of generalized theta-series, integral automorph of a quadratic form.

1. Introduction

Let

$$Q(X) = Q(x_1, \dots, x_r) = \sum_{1\le i\le j\le r} b_{ij}x_ix_j$$

be an integral positive definite quadratic form in an even number r of variables. That is, $b_{ij} \in \mathbb{Z}$ and $Q(X) > 0$ if $X \neq 0$. To $Q(X)$ we associate the even integral symmetric $r \times r$ matrix A defined by $a_{ii} = 2b_{ii}$ and $a_{ij} = a_{ji} = b_{ij}$, where $i < j$. If $X = [x_1, \dots, x_r]'$ denotes a column vector where $'$ denotes the transposition, then we have $Q(X) = \frac{1}{2}X'AX$. Let A_{ij} denote the algebraic adjunct of the element a_{ij} in $D = \det A$ and a^*_{ij} the corresponding element of A^{-1}.

Below we shall use the notions, notation and some results from [2].

A homogeneous polynomial $P(X) = P(x_1, \dots, x_r)$ of degree ν with complex coefficients, satisfying the condition

$$\sum_{1\le i,j\le r} a^*_{ij}\Big(\frac{\partial^2 P}{\partial x_i \partial x_j}\Big) = 0 \qquad (1)$$

is called a spherical polynomial of order ν with recpect to $Q(X)$ (see [1]), and

$$\vartheta(\tau, P, Q) = \sum_{n\in\mathbb{Z}^{\setminus}} P(n)z^{Q(n)}, \quad z = e^{2\pi i\tau}, \quad \tau \in \mathbb{C}, \quad \operatorname{Im}\tau > \nvdash, \qquad (2)$$

is the corresponding generalized r-fold theta-series.

Let $\mathcal{R}(\nu, \mathcal{Q})$ denote the vector space over $\mathbb{C}$ of spherical polynomials $P(X)$ of even order ν with respect to $Q(X)$. Hecke [2] calculated the dimension of the space $\mathcal{R}(\nu, \mathcal{Q})$ and showed that

$$\dim \mathcal{R}(\nu, \mathcal{Q}) = \frac{(\nu + \nabla - \ni)!}{(\nabla - \in)!\nu!}\,(\nabla + \in\nu - \in). \qquad (3)$$

ISSN 1072-947X / $8.00 /

Let $T(\nu, Q)$ denote the vector space over $\mathbb{C}$ of generalized multiple theta-series, i.e.,

$$T(\nu, Q) = \{\vartheta(\tau, P, Q) \,:\, P \in \mathcal{R}(\nu, \mathcal{Q})\}.$$

Gooding [1] calculated the dimension of the vector space $T(\nu, Q)$ for reduced binary quadratic forms Q.

In this paper the upper bound is established for dimension of the vector space $T(\nu, Q)$ for some positive reduced ternary quadratic forms. In a number of cases the dimension of vector spaces of theta-series is also calculated and the bases of this spaces are constructed.

In the sequel we use the following definition and results:

An integral $r \times r$ matrix U is called an integral automorph of the quadratic form $Q(X)$ in r variables if the condition

$$U'AU = A \tag{4}$$

is satisfied.

Lemma 1 ([1], p. 37). *Let $Q(X) = Q(x_1, \dots, x_r)$ be a positive definite quadratic form in r variables and $P(X) = P(x_1, \dots, x_r) \in \mathcal{R}(\nu, \mathcal{Q})$. Let G be the set of all integral automorphs of Q. Suppose*

$$\sum_{i=1}^{t} P(U_i X) = 0 \quad \text{for some} \quad U_1, \dots, U_t \subseteq G,$$

then $\vartheta(\tau, P, Q) = 0$.

Lemma 2 ([2], p. 853). *Among homogeneous quadratic polynomials in r variables*

$$\varphi_{ij} = x_i x_j - \frac{A_{ij}}{rD} 2Q(X) \quad (i, j = 1, \dots, r), \tag{5}$$

exactly $\frac{r(r+1)}{2} - 1$ ones are linearly independent and form the basis of the space of spherical polynomials of second order with respect to $Q(X)$.

Lemma 3 ([3], p. 533). *Among homogeneous polynomials of fourth degree in r variables*

$$\begin{aligned}\varphi_{ijkl} = x_i x_j x_k x_l &- \frac{1}{(r+4)D} (A_{ij} x_k x_l + A_{ik} x_j x_l + A_{il} x_j x_k + A_{jk} x_i x_l \\ &+ A_{jl} x_i x_k + A_{kl} x_i x_j) \cdot 2Q + \frac{1}{(r+2)(r+4)D^2} \cdot (A_{ij} A_{kl} + A_{ik} A_{jl} \\ &+ A_{il} A_{jk}) \cdot (2Q)^2 \quad (i, j, k, l = 1, \dots, r),\end{aligned} \tag{6}$$

exactly $\frac{1}{24} r(r^2-1)(r+6)$ ones are linearly independent and form the basis of the space of spherical polynomials of fourth order with respect to $Q(X)$.

Lemma 4 ([4], p. 28–30). *For a matrix of integers*

$$U = \|u_{ij}\|_{r \times r}$$

to be an integral automorph of the quadratic form

$$Q(X) = \sum_{1 \le i \le j \le r} b_{ij} x_i x_j$$

it is sufficient

a) *to find a representation of integers* $b_{11}, b_{22}, \dots, b_{rr}$ *by the quadratic form* $Q(X)$*, assuming that*

$$Q(u_{1i}, u_{2i}, \dots, u_{ri}) = b_{ii} \quad (1 \le i \le r), \tag{7}$$

and

b) *to verify that the* $\frac{r^2-r}{2}$ *conditions*

$$\sum_{t=1}^{r} \sum_{k=1}^{r} a_{tk} u_{ti} u_{kj} = b_{ij} \quad (i < j; \quad i, j = 1, \dots, r) \tag{8}$$

are satisfied.

2. The Basis of the Space $\mathcal{R}(\nu, \mathcal{Q})$

Let

$$P(x) = P(x_1, x_2, x_3) = \sum_{k=0}^{\nu} \sum_{i=0}^{k} a_{ki} x_1^i x_2^{k-i} x_3^{\nu-k} \tag{9}$$

be a spherical function of order ν with respect to the positive ternary quadratic form $Q(x_1, x_2, x_3)$. Hence, according to definition (1), the condition

$$\begin{aligned} \frac{A_{11}}{|A|} \frac{\partial^2 P}{\partial x_1^2} + 2\frac{A_{12}}{|A|} \frac{\partial^2 P}{\partial x_1 \partial x_2} + 2\frac{A_{13}}{|A|} \frac{\partial^2 P}{\partial x_1 \partial x_3} + \frac{A_{22}}{|A|} \frac{\partial^2 P}{\partial x_2^2} \\ +2\frac{A_{23}}{|A|} \frac{\partial^2 P}{\partial x_2 \partial x_3} + \frac{A_{33}}{|A|} \frac{\partial^2 P}{\partial x_3^2} = 0 \end{aligned} \tag{10}$$

is satisfied. Since

$$\begin{aligned} \frac{\partial^2 P}{\partial x_1^2} &= \sum_{k=0}^{\nu} \sum_{i=0}^{k} i(i-1) a_{ki} x_1^{i-2} x_2^{k-i} x_3^{\nu-k} \\ &= \sum_{k=1}^{\nu-1} \sum_{i=0}^{k-1} (i+1)(i+2) a_{k+1,i+2} x_1^i x_2^{k-i-1} x_3^{\nu-k-1} \end{aligned}$$

and one can easily obtain similar formulas for other second partial derivatives, condition (10) takes the form

$$\begin{aligned} \frac{1}{|A|} \sum_{k=1}^{\nu-1} \sum_{i=0}^{k-1} \big(A_{11}(i+1)(i+2) a_{k+1,i+2} + 2A_{12}(k-i)(i+1) a_{k+1,i+1} \\ +2A_{13}(\nu-k)(i+1) a_{k,i+1} + A_{22}(k-i)(k-i+1) a_{k+1,i} \\ +2A_{23}(\nu-k)(k-i) a_{ki} + A_{33}(\nu-k)(\nu-k+1) a_{k-1,i} \big) x_1^i x_2^{k-i-1} x_3^{\nu-k-1} = 0. \end{aligned}$$

Thus, for $0 \le i < k \le \nu - 1$, we obtain

$$\begin{aligned} &A_{11}(i+1)(i+2)a_{k+1,i+2} + 2A_{12}(k-i)(i+1)a_{k+1,i+1} \\ &+2A_{13}(\nu-k)(i+1)a_{k,i+1} + A_{22}(k-i)(k-i+1)a_{k+1,i} \\ &+2A_{23}(\nu-k)(k-i)a_{ki} + A_{33}(\nu-k)(\nu-k+1)a_{k-1,i} = 0. \end{aligned} \tag{11}$$

Let

$$L = [a_{00}, a_{10}, a_{11}, a_{20}, a_{21}, a_{22}, \dots, a_{\nu\nu}]'$$

be the column vector, where a_{ki} $(0 \le i \le k\nu)$ are the coefficients of polynomial (9).

Conditions (11) in matrix notation have the form

$$S \cdot L = 0, \tag{12}$$

where the matrix S (the elements of this matrix are defined from conditions (11)) has the form

$$\begin{Vmatrix} A_{33}(\nu-1)\nu & 2A_{23}(\nu-1) & 2A_{13}(\nu-1) & 2A_{22} & \dots & 0 & 0 \\ 0 & A_{33}(\nu-2)(\nu-1) & 0 & 4A_{23}(\nu-2) & \dots & 0 & 0 \\ 0 & 0 & A_{33}(\nu-2)(\nu-1) & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 2A_{12}(\nu-1) & A_{11}(\nu-1)\nu \end{Vmatrix}.$$

The number of rows of the matrix S is equal to the number of conditions (11), i.e., to the number of pairs (i,k) with $0 \le i < k \le \nu - 1$. Hence the number of rows is equal to $\binom{\nu}{2} = \nu(\nu-1)/2$. As for the number of columns of the matrix S it is equal to the number of coefficients a_{ki} of polynomial (9), i.e., to the number of pairs (i,k) with $0 \le i \le k \le \nu$. Hence it is equal to $\binom{\nu+2}{2}$.

We partition the matrix S into two matrices S_1 and S_2. S_1 is the left square nondegenerate $\binom{\nu}{2} \times \binom{\nu}{2}$ matrix, it consists of the first $\binom{\nu}{2}$ columns of the matrix S; S_2 is the right $\binom{\nu}{2} \times (2\nu+1)$ matrix, it consists of the last $2\nu+1$ columns of the matrix S.

Similarly, we partition the matrix L into two matrices L_1 and L_2. L_1 is the $\binom{\nu}{2} \times 1$ matrix, it consists of the upper $\binom{\nu}{2}$ elements of the matrix L; L_2 is the $(2\nu+1) \times 1$ matrix, it consists of the lower $2\nu+1$ elements of the matrix L.

According to the new notation, the matrix equality (12) has the form:

$$S_1 L_1 + S_2 L_2 = 0,$$

i.e.,

$$L_1 = -S_1^{-1} S_2 L_2. \tag{13}$$

It follows from this equality that the matrix L_1 is expressed through the matrix L_2, i.e., the first $\binom{\nu}{2}$ elements of the matrix L are expressed through its other $2\nu+1$ elements. Since the matrix L consists of the coefficients of the spherical polynomial $P(X)$, its first $\binom{\nu}{2}$ coefficients can be expressed through the last $2\nu+1$ coefficients.

Let $Q(X) = Q(x_1, x_2, x_3)$ be a ternary quadratic form. According to the formula (3), $\dim \mathcal{R}(\nu, \mathcal{Q}) = \in\nu + \infty$.

Let us show that the polynomials[1]

$$\begin{aligned}
&P_1(a_{00}^{(1)}, a_{10}^{(1)}, \dots, a_{\nu-2,\nu-2}^{(1)}, 1, 0, 0, \dots, 0),\\
&P_2(a_{00}^{(2)}, a_{10}^{(2)}, \dots, a_{\nu-2,\nu-2}^{(2)}, 0, 1, 0, 0, \dots, 0),\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&P_{2\nu+1}(a_{00}^{(2\nu+1)}, a_{10}^{(2\nu+1)}, \dots, a_{\nu-2,\nu-2}^{(2\nu+1)}, 0, 0, 0, \dots, 1),
\end{aligned} \tag{14}$$

where in view of (13) the first $\binom{\nu}{2}$ coefficients from a_{00} to $a_{\nu-2,\nu-2}$ are calculated through other $2\nu+1$ coefficients and form the basis of the space $\mathcal{R}(\nu, \mathcal{Q})$.

Indeed, these polynomials satisfy condition (13), i.e., condition (1), hence polynomials (14) are spherical functions with respect to $Q(X)$. They are linearly independent and altogether are $2\nu+1$, as it is stated above that the dimension of the space $\mathcal{R}(\nu, \mathcal{Q})$ is equal to $2\nu+1$. Thus polynomials (14) form the basis of the space $\mathcal{R}(\nu, \mathcal{Q})$.

3. The Upper Bound of Dimension of $T(\nu, Q)$ for Some Ternary Forms

Consider the quadratic form

$$Q_1 = b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2,$$

where $0 < b_{11} < b_{22} < b_{33}$.

Construct the integral automorphs U of the quadratic form Q_1. According to the definition (4) and Lemma 4,

$$\begin{aligned}
b_{11} &= Q(\pm 1, 0, 0),\\
b_{22} &= Q(0, \pm 1, 0),\\
b_{33} &= Q(0, 0, \pm 1).
\end{aligned}$$

Hence it is easy to verify that the integral automorphs of the quadratic form Q_1 are only

$$U = \begin{Vmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{Vmatrix} \quad (e_i = \pm 1, \quad i = 1, 2, 3). \tag{15}$$

From these 8 automorphs U we use only

$$U_1 = \begin{Vmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix} \quad \text{and} \quad U_2 = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{Vmatrix}. \tag{16}$$

Consider all possible polynomials $P_h(U_jX)$ $(h = 1, 2, \dots, 2\nu+1)$, where $P_h \in \mathcal{R}(\nu, \mathcal{Q}_\infty)$ are spherical polynomials of order ν with recpect to $Q_1(X)$, and $U_j \in G$ are integral automorphs of the quadratic form $Q_1(X)$.

[1]In the brackets of (14) the coefficients of polynomial P_i of form (9) are shown.

From (9) and (16) it follows that

$$P_h(U_1X)=\sum_{k=0}^{\nu}\sum_{i=0}^{k}a_{ki}^{(h)}(-x_1)^i x_2^{k-i}x_3^{\nu-k}=\sum_{k=0}^{\nu}\sum_{i=0}^{k}(-1)^i a_{ki}^{(h)}x_1^i x_2^{k-i}x_3^{\nu-k},$$

$$P_h(U_2X)=\sum_{k=0}^{\nu}\sum_{i=0}^{k}a_{ki}^{(h)}x_1^i x_2^{k-i}(-x_3)^{\nu-k}=\sum_{k=0}^{\nu}\sum_{i=0}^{k}(-1)^k a_{ki}^{(h)}x_1^i x_2^{k-i}x_3^{\nu-k}.$$

Further, we have

$$P_h(X)+P_h(U_1X)=\sum_{k=0}^{\nu}\sum_{i=0}^{k}(1+(-1)^i)a_{ki}^{(h)}x_1^i x_2^{k-i}x_3^{\nu-k},$$

$$P_h(X)+P_h(U_2X)=\sum_{k=0}^{\nu}\sum_{i=0}^{k}(1+(-1)^k)a_{ki}^{(h)}x_1^i x_2^{k-i}x_3^{\nu-k}.$$

We shall find out for which P_h, of two equalities

$$P_h(X)+P_h(U_1X)=0 \tag{17}$$

or

$$P_h(X)+P_h(U_2X)=0, \tag{18}$$

at least one takes place. Equality (17) takes place if and only if the coefficients

$$(1+(-1)^i)a_{ki}^{(h)}=0 \tag{19}$$

for all k,i. Keeping in mind the construction of a basis of the space of spherical functions, it is sufficient to show that (19) is true for the last $2\nu+1$ coefficients from $a_{\nu-1,0}^{(h)}$ to $a_{\nu\nu}^{(h)}$, i.e., when $k=\nu-1,\nu$; $i=0,1,\dots,k$. All these last coefficients $a_{ki}^{(h)}$ are zero, except one which is equal to one. Suppose $a_{ki}^{(h)}=1$ for a certain pair k,i with $\nu-1\le k\le\nu$ and $0\le i\le k$; if i is odd, then $(1+(-1)^i)a_{ki}^{(h)}=0$. Hence P_h satisfies equality (17) if among the last $2\nu+1$ coefficients, the index i – of the coefficient equal to one is odd. Similarly, it follows that P_h satisfies equality (18) if among the last $2\nu+1$ coefficients, the index k – of the coefficient equal to one is odd.

Calculate how many polynomials P_h satisfy at least one of two equalities (17) or (18), i.e., calculate how many coefficients a_{ki} with $\nu-1\le k\le\nu$ and $0\le i\le k$ have among the indices k or i, at least one odd. We have the following cases:

a) If $2\nmid k$, then $k=\nu-1$. The total number of coefficients with $k=\nu-1$ is equal to ν.

b) If $2\mid k$, but $2\nmid i$, then $k=\nu$, $0\le i\le\nu$, $2\nmid i$. The total number of coefficients with such indices is equal to $\nu/2$.

Thus, altogether we have $\nu+\nu/2$ polynomials P_h which satisfy at least one of equalities (17) or (18), but for such polynomials, according to Lemma 1,

$$\vartheta(\tau,P_h,Q_1)=0. \tag{20}$$

We have thus shown that among $2\nu+1$ theta-series, corresponding to linearly independent spherical polynomials, $\nu+\nu/2$ theta-series are zero. Hence the maximal number of linearly independent theta-series, i.e.,

$$\dim T(\nu, Q_1) \leq 2\nu + 1 - \Big(\nu + \frac{\nu}{2}\Big) = \frac{\nu}{2} + 1. \tag{21}$$

Now consider the quadratic form

$$Q_2 = b_{11}(x_1^2 + x_2^2) + b_{33}x_3^2,$$

where $0 < b_{11} < b_{33}$.

We construct the integral automorphs U of the quadratic form Q_2. Since

$$b_{11} = Q_2(\pm 1, 0, 0) = Q_2(0, \pm 1, 0), \quad b_{33} = Q_2(0, 0, \pm 1),$$

it is easy to verify that the integral automorphs of the quadratic form Q_2 are automorphs (15) and, moreover, the automorphs

$$\begin{Vmatrix} 0 & e_1 & 0 \\ e_2 & 0 & 0 \\ 0 & 0 & e_3 \end{Vmatrix} \quad (e_i = \pm 1, \quad i = 1, 2, 3). \tag{22}$$

From automorphs (22) we use only

$$U_3 = \begin{Vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{Vmatrix}. \tag{23}$$

The automorphs of the quadratic form Q_1 are also automorphs of the quadratic form Q_2, therefore

$$\dim T(\nu, Q_2) \leq \frac{\nu}{2} + 1.$$

We improve this estimation. From (9) and (23) it follows that

$$P_h(U_3X) = \sum_{k=0}^{\nu} \sum_{i=0}^{k} a_{ki}^{(h)} x_2^i x_1^{k-i} x_3^{\nu-k} = \sum_{k=0}^{\nu} \sum_{i=0}^{k} a_{ki}^{(h)} x_1^{k-i} x_2^{k-(k-i)} x_3^{\nu-k}.$$

From here it follows that if all last $2\nu+1$ coefficients of the basis polynomial $P_h(X)$ are equal to zero, except one $a_{ki}^{(h)} = 1$, then all last $2\nu+1$ coefficients of polynomial $P_h(U_3X)$ are equal to zero, except one $a_{k,k-i} = 1$. Hence $P_h(U_3X)$ is a basis polynomial of the space $\mathcal{R}(\nu, \mathcal{Q}_\in)$. Further, it is known ([1], p. 38) that

$$\vartheta(\tau, P_h(X), Q_2) = \vartheta(\tau, P_h(U_3X, Q_2);$$

thus the theta-series $\vartheta(\tau, P_h(X), Q_2)$ and $\vartheta(\tau, P_h(U_3X), Q_2)$, corresponding to different basis polynomials $P_h(X)$ and $P_h(U_3X)$, are linearly dependent.

Calculate how many such linearly dependent theta-series we have. Let k and i be even (otherwise, it can be shown similarly to (20) that $\vartheta(\tau, P_h, Q_2) = 0$), i.e., $k = \nu$, and i takes $\frac{\nu}{2} + 1$ even values. Thus altogether we have $[\frac{1}{2}(\frac{\nu}{2} + 1)]$ linearly dependent theta-series.

But
$$\Big[\frac{1}{2}\Big(\frac{\nu}{2}+1\Big)\Big]=\begin{cases}\frac{\nu}{4} & \text{if } \nu\equiv 0 \pmod 4,\\ \frac{\nu+2}{4} & \text{if } \nu\equiv 2 \pmod 4.\end{cases}$$
Hence
$$\begin{aligned}&\dim T(\nu,Q_2)\le\frac{\nu}{2}+1-\frac{\nu}{4}=\frac{\nu}{4}+1 \quad \text{if } \nu\equiv 0 \pmod 4,\\ &\dim T(\nu,Q_2)\le\frac{\nu}{2}+1-\frac{\nu+2}{4}=\frac{\nu+2}{4} \quad \text{if } \nu\equiv 2 \pmod 4.\end{aligned} \tag{24}$$
Consider now the quadratic form
$$Q_3=b_{11}(x_1^2+x_2^2)+b_{33}x_3^2+b_{12}x_1x_2,$$
where $0<|b_{12}|<b_{11}<b_{33}$.

It is easy to verify that the integral automorphs of the quadratic form Q_3 are
$$\begin{Vmatrix}e_1&0&0\\0&e_1&0\\0&0&e_2\end{Vmatrix} \quad\text{and}\quad \begin{Vmatrix}0&e_1&0\\e_1&0&0\\0&0&e_2\end{Vmatrix} \quad (e_i=\pm1,\ \ i=1,2). \tag{25}$$
From automorphs (25) we use
$$U_4=\begin{Vmatrix}1&0&0\\0&1&0\\0&0&-1\end{Vmatrix} \quad\text{and}\quad U_5=\begin{Vmatrix}0&1&0\\1&0&0\\0&0&1\end{Vmatrix}. \tag{26}$$
From (9) and (26) it follows that
$$\begin{aligned}P_h(U_4X)&=\sum_{k=0}^{\nu}\sum_{i=0}^{k}a_{ki}^{(h)}x_1^ix_2^{k-i}(-x_3)^{\nu-k}=\sum_{k=0}^{\nu}\sum_{i=0}^{k}(-1)^{\nu-k}a_{ki}^{(h)}x_1^ix_2^{k-i}x_3^{\nu-k},\\ P_h(U_5X)&=\sum_{k=0}^{\nu}\sum_{i=0}^{k}a_{ki}^{(h)}x_2^ix_1^{k-i}x_3^{\nu-k}=\sum_{k=0}^{\nu}\sum_{i=0}^{k}a_{ki}^{(h)}x_1^{k-i}x_2^{k-(k-i)}x_3^{\nu-k}.\end{aligned} \tag{27}$$
We have
$$P_h(X)+P_h(U_4X)=\sum_{k=0}^{\nu}\sum_{i=0}^{k}(1+(-1)^k)a_{ki}^{(h)}x_1^ix_2^{k-i}x_3^{\nu-k}. \tag{28}$$
Calculate how many polynomials P_h satisfy the equality
$$P_h(X)+P_h(U_4X)=0. \tag{29}$$

According to (28), we must calculate how many coefficients a_{ki}, with $\nu-1\le k\le\nu$ and $0\le i\le k$, have the index k odd, i.e., $k=\nu-1$. Thus altogether we have ν polynomials P_h which satisfy equality (29). But for such polynomials, according to the Lemma 1,
$$\vartheta(\tau,P_h,Q_3)=0. \tag{30}$$
From (14) and (27) it follows that if all last $2\nu+1$ coefficients of the basis polynomial $P_h(X)$ are equal to zero, except one $a_{ki}^{(h)}=1$, then all last $2\nu+1$ coefficients of polynomial $P_h(U_5X)$ are equal to zero, except one $a_{k,k-i}=1$.

Hence $P_h(U_5X)$ is a basis polynomial of the space $\mathcal{R}(\nu, \mathcal{Q}_\ni)$. Further, it is known ([1], p. 38) that

$$\vartheta(\tau, P_h(X), Q_3) = \vartheta(\tau, U_5(X), Q_3);$$

thus the theta-series $\vartheta(\tau, P_h(X), Q_3)$ and $\vartheta(\tau, P_h(U_5X), Q_3)$, corresponding to the different basis polynomials $P_h(X)$ and $P_h(U_5X)$, are linearly dependent.

We count the number of linearly dependent theta-series. Let k be even (otherwise (30) holds true) i.e., $k = \nu, i$ takes $\nu+1$ values. Thus we have $[\frac{1}{2}(\nu+1)] = \frac{\nu}{2}$ linearly dependent theta-series.

We have thus shown that among $2\nu+1$ theta-series, corresponding to linearly independent spherical polynomials, ν theta-series are zero and $\nu/2$ are linearly dependent theta-series. Hence the maximal number of linearly independent theta-series

$$\dim T(\nu, Q_3) \leq 2\nu + 1 - \nu - \frac{\nu}{2} = \frac{\nu}{2} + 1. \tag{31}$$

Consider the quadratic forms:

$$\begin{aligned}
Q_4 &= b_{11}(x_1^2 + x_2^2) + b_{33}x_3^2 + b_{23}x_2x_3, \\
Q_5 &= b_{11}(x_1^2 + x_2^2) + b_{33}x_3^2 + b_{13}x_1x_3, \\
Q_6 &= b_{11}(x_1^2 + x_2^2) + b_{33}x_3^2 + b_{13}(x_1x_3 + x_2x_3), \\
Q_7 &= b_{11}(x_1^2 + x_2^2) + b_{33}x_3^2 + b_{13}(x_1x_3 + x_2x_3) + b_{12}x_1x_2,
\end{aligned}$$

where $0 < |b_{13}| < b_{11} < b_{33}$, $|b_{23}| < b_{22}$. For each quadratic form Q_i ($i = 4, 5, 6, 7$) we construct the corresponding integral automorphs; consider all possible polynomials $P_h(U_jX)$ ($h = 1, 2, \ldots, 2\nu + 1$), where $P_h \in \mathcal{R}(\nu, \mathcal{Q}_\rangle)$ are spherical basis polynomials of order ν with respect to Q_i (see (14)), $U_j \in G$ are integral automorphs of the quadratic form Q_i. We investigate the sets $\{\vartheta(\tau, P_h(U_j), Q_i\}_{h,j}$. From these sets we remove the zero theta-series and leave those through which other theta-series are expressed. Thus we obtain:

$$\dim T(\nu, Q_4), \dim T(\nu, Q_5), \dim T(\nu, Q_6), \dim T(\nu, Q_7) \leq \nu + 1.$$

In ([4], p. 31) we have shown that for the quadratic forms

$$
\begin{aligned}
&Q_8 = b_{11}x_1^2 + b_{22}(x_2^2 + x_3^2),\\
&Q_9 = b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 + b_{ij}x_ix_j \quad \text{with } 1 \le i < j \le 3,\\
&Q_{10} = b_{11}x_1^2 + b_{22}(x_2^2 + x_3^2) + b_{1j}x_1x_j \quad \text{with } j = 2, 3,\\
&Q_{11} = b_{11}x_1^2 + b_{22}(x_2^2 + x_3^2) + b_{12}(x_1x_2 + x_1x_3),\\
&Q_{12} = b_{11}x_1^2 + b_{22}(x_2^2 + x_3^2) + b_{12}(x_1x_2 + x_1x_3) + b_{23}x_2x_3,\\
&Q_{13} = b_{11}x_1^2 + b_{22}(x_2^2 + x_3^2) + b_{23}x_2x_3,\\
&\dim T(\nu, Q_8) \le \frac{\nu}{4} + 1 \quad \text{if } \nu \equiv 0 \pmod 4,\\
&\dim T(\nu, Q_8) \le \frac{\nu+2}{4} \quad \text{if } \nu \equiv 2 \pmod 4,\\
&\dim T(\nu, Q_9), \dim T(\nu, Q_{10}), \dim T(\nu, Q_{11}), \dim T(\nu, Q_{12}) \le \nu + 1,\\
&\dim T(\nu, Q_{13}) \le \frac{\nu}{2} + 1.
\end{aligned}
$$

4. Dimension and the Basis of the Space $T(\nu, Q)$ for Some Ternary Forms

From (24), for $\nu = 2$ and $\nu = 4$, it follows that $\dim(2, Q_2) \le 1$ and $\dim(4, Q_2) \le 2$, respectively. Let us show that $\dim(2, Q_2) = 1$ and $\dim(4, Q_2) = 2$.

For the quadratic form

$$Q_2 = b_{11}(x_1^2 + x_2^2) + b_{33}x_3^2,$$

where $0 < b_{11} < b_{33}$, we have $D = \det A = 8b_{11}^2 b_{33}$, $A_{11} = A_{22} = 4b_{11}b_{33}$, $A_{33} = 4b_{11}^2$. Hence, according to (5) and (6),

$$\varphi_{11} = x_1^2 - \frac{A_{11}}{3D}\, 2Q_2 = x_1^2 - \frac{1}{3b_{11}}\, Q_2, \tag{32}$$

$$\varphi_{1111} = x_1^4 - \frac{6}{7b_{11}}\, x_1^2 Q_2 + \frac{3}{35b_{11}^2}\, Q_2^2, \tag{33}$$

$$\varphi_{3333} = x_3^4 - \frac{6}{7b_{33}}\, x_3^2 Q_2 + \frac{3}{35b_{33}^2}\, Q_2^2. \tag{34}$$

Consider the equation

$$b_{11}(x_1^2 + x_2^2) + b_{33}x_3^2 = n. \tag{35}$$

1) When $n = b_{11}$, equation (35) has 4 solutions:

$$
\begin{aligned}
x_1 = \pm 1, \quad x_2 = x_3 = 0,\\
x_2 = \pm 1, \quad x_1 = x_3 = 0.
\end{aligned}
$$

2) When $n = b_{33}$,

a) if $b_{33} \ne b_{11}h^2$ and $b_{33} \ne b_{11}(u^2 + v^2)$, then (35) has 2 solutions:

$$x_1 = x_2 = 0, \quad x_3 = \pm 1.$$

b) if $b_{33} = b_{11}h^2$ and $b_{33} \neq b_{11}(u^2+v^2)$, then (35) has 6 solutions:

$$\begin{aligned} &x_1 = x_2 = 0, && x_3 = \pm 1;\\ &x_1 = \pm h, && x_2 = x_3 = 0;\\ &x_2 = \pm h, && x_1 = x_3 = 0. \end{aligned}$$

c) if $b_{33} \neq b_{11}h^2$ and $b_{33} = b_{11}(u^2+v^2)$, then (35) has 10 solutions if $u \neq v$ and 6 solutions if $u = v$:

$$\begin{aligned} &x_1 = x_2 = 0, && x_3 = \pm 1;\\ &x_1 = \pm u, && x_2 = \pm v, && x_3 = 0;\\ &x_1 \pm v, && x_2 = \pm u, && x_3 = 0;\\ &x_1 = \pm u, && x_2 = \mp v, && x_3 = 0;\\ &x_1 = \pm v, && x_2 = \mp u, && x_3 = 0. \end{aligned}$$

d) if $b_{33} = b_{11}h^2$ and $b_{33} = b_{11}(u^2+v^2)$, then (35) has 14 solutions:

$$\begin{aligned} &x_1 = x_2 = 0, && x_3 = \pm 1;\\ &x_1 = \pm h, && x_2 = x_3 = 0;\\ &x_2 = \pm h, && x_1 = x_3 = 0;\\ &x_1 = \pm u, && x_2 = \pm v, && x_3 = 0;\\ &x_1 = \pm v, && x_2 = \pm u, && x_3 = 0;\\ &x_1 = \pm u, && x_2 = \mp v, && x_3 = 0;\\ &x_1 = \pm v, && x_2 = \mp u, && x_3 = 0. \end{aligned}$$

Using these solutions and performing easy calculations by (32) we obtain

$$\vartheta(\tau, \varphi_{11}, Q_2) = \sum_{n=1}^{\infty} \left(\sum_{Q_2 = n} x_1^2 - \frac{1}{3b_{11}}\, Q_2 \right) = z^n + \cdots + \frac{2}{3}\, z^{b_{11}} + \cdots .$$

Since $\vartheta(\tau, \varphi_{11}, Q_2) \in T(2, Q_2)$ and, as is shown above, $\dim T(2, Q_2) \leq 1$, we have proved

Theorem 1. $\dim T(2, Q_2) = 1$ *and the generalized ternary theta-series* $\vartheta(\tau, \varphi_{11}, Q_2)$ *is a basis of the space* $T(2, Q_2)$.

Performing easy calculations and using the solutions of equation (35), for $b_{33} \neq b_{11}h^2$ and $b_{33} \neq b_{11}(u^2+v^2)$ by (33) and (34) we obtain

$$\begin{aligned} \vartheta(\tau, \varphi_{1111}, Q_2) &= \sum_{n=1}^{\infty} \left(\sum_{Q_2 = n} x_1^4 - \frac{6}{7b_{11}}\, Q_2 + \frac{3}{35b_{11}^2}\, Q_2^2 \right) z^n \\ &= \cdots + \frac{22}{35}\, z^{b_{11}} + \cdots + \frac{6}{35}\, \frac{b_{33}^2}{b_{11}^2}\, z^{b_{33}} + \cdots , \\ \vartheta(\tau, \varphi_{3333}, Q_2) &= \sum_{n=1}^{\infty} \left(\sum_{Q_2 = n} x_3^4 - \frac{6}{7b_{33}}\, x_3^2 Q_2 + \frac{3}{35b_{33}^2}\, Q_2^2 \right) z^n \\ &= \cdots + \frac{12}{35}\, \frac{b_{11}^2}{b_{33}^2}\, z^{b_{11}} + \cdots + \frac{16}{35}\, z^{b_{33}} + \cdots . \end{aligned}$$

These generalized ternary theta-series are linearly independent since the determinant of second order constructed from the coefficients of these theta-series is not equal to zero. Similarly, in the remaining cases, the theta-series $\vartheta(\tau, \varphi_{1111}, Q_2)$ and $\vartheta(\tau, \varphi_{3333}, Q_2)$ are linearly independent. Above it is shown that $\dim T(4, Q_2) \leq 2$; hence $\dim T(4, Q_2) = 2$. Thus we have proved

Theorem 2. $\dim T(4, Q_2) = 2$ *and the generalized ternary theta-series* $\vartheta(\tau, \varphi_{1111}, Q_2)$ *and* $\vartheta(\tau, \varphi_{3333}, Q_2)$ *form a basis of the space* $T(4, Q_2)$.

The proof of the following theorem is similar.

Theorem 3. *Let*

$$Q_1 = b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2,$$

then $T(2, Q_1) = 2$ *and the generalized ternary theta-series*

$$\vartheta(\tau, \varphi_{11}, Q_2) = \sum_{n=1}^{\infty} \bigg(\sum_{Q_1 = n} x_1^2 - \frac{1}{3b_{11}} Q_1 \bigg) z^n = \frac{4}{3} z^{b_{11}} + \cdots - \frac{2}{3} \frac{b_{22}}{b_{11}} z^{b_{22}} + \cdots,$$

$$\vartheta(\tau, \varphi_{22}, Q_2) = \sum_{n=1}^{\infty} \bigg(\sum_{Q_1 = n} x_2^2 - \frac{1}{3b_{22}} Q_1 \bigg) z^n = -\frac{2}{3} \frac{b_{11}}{b_{22}} z^{b_{11}} + \cdots + \frac{4}{3} z^{b_{22}} + \cdots$$

form a basis of the space $T(2, Q_1)$.

References

[1] F. Gooding, Modular forms arising from spherical polynomials and positive definite quadratic forms. *J. Number Theory* **9**(1977), 36–47.

[2] E. Hecke, Mathematische Werke. Zweite Auflage. *Vandenhoeck und Ruprecht, Göttingen,* 1970.

[3] G. A. Lomadze, On the basis of the space of fourth order spherical functions with respect to a positive quadratic form. (Russian) *Bull. Acad. Sci. Georgian SSR* **69**(1973), No. 3, 533–536.

[4] K. Sh. Shavgulidze, To the dimension of spaces of generalized ternary theta-series. (Russian) *Trudy Tbiliss. Univ. Mat. Mekh. Astronom.* **26**(1989), 22–37.

(Received 11.01.2001)

Author's address:
Faculty of Mechanics and Mathematics
I. Javakhishvili Tbilisi State University
2, University St., Tbilisi 380043
Georgia
E-mail: svanadze@viam.hepi.edu.ge