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SOLUTION OF A LINEAR INTEGRAL EQUATION OF THIRD KIND

D. SHULAIA

Abstract. The aim of this paper is to study, in the class of Hölder functions, a nonhomogeneous linear integral equation with coefficient $\cos x$. Necessary and sufficient conditions for the solvability of this equation are given under some assumptions on its kernel. The solution is constructed analytically, using the Fredholm theory and the theory of singular integral equations.

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1. Introduction

In this paper, a method is described for solving an integral equation which frequently occurs when studying many important problems of mathematical physics, in particular, in problems of the multidimensional transfer theory. An equation is written in the form

$$\cos x\varphi(x) = \int_0^{2\pi} K(x,y)\varphi(y)dy + f(x), \quad x \in (0,2\pi). \tag{1}$$

Such equations are frequently called equation of third kind. After the early works of Hilbert [1] and Picard [2], in the course of the last century there appeared a lot of papers on equations of third kind though attention given to them can hardly be compared with attention given to equations of first and second kind. Most of standard studies of integral equations touch very briefly upon equations of third kind. The investigation should certainly begin by the question what we must mean by a solution of equation (1). Various answers are suggested in literature. Following the classical definition of a solution which is natural and typical of some frequently occurring in the applications of third kind equations, we assume that the kernel $K \in H$ (i.e., satisfies the Hölder condition H), while a free term belongs to the Muskhelishvili class H^* (see [3]). Therefore we look for a solution φ of equation (1) which belongs to the class H^*.

Our investigation is based on the ideas of the theory of spectral expansion and the approach proposed by Hilbert and Schmidt for an integral equation of second

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kind. This paper is the continuation of [4]. The paper is organized as follows. In Section 2, using the initial equation, we introduce integral operators and their corresponding integral equations which depend on the parameter. Some of their properties, which will play an important role in further considerations, are introduced. In Section 3, singular integral operators, which are closely connected with the introduced operators and equations, are defined and studied. The main result states that an arbitrary function from H^* can be represented through singular operators and the eigenfunctions of the parameter-dependent operator. In the last section, a solution of the considered problem is given as an application of the results obtained in Section 2.

2. Notation and the Background

Let Ω_z denote the linear integral operator defined by the formula

$$\Omega_z(g(z,\cdot))(x) := g(z,x) + \int_0^{2\pi} \frac{K(x,y)}{z-\cos y} g(z,y)dy, \quad x \in [0,2\pi], \quad z \notin [-1,1], \tag{2}$$

where the real-valued function K satisfies the condition H with respect to both variables.

Recall that a function $\varphi(t)$ is said to satisfy the condition H (Hölder condition) on a certain smooth line Γ, if for any two points t_1, t_2 of Γ

$$|\varphi(t_1) - \varphi(t_2)| \leq A|t_1 - t_2|^\gamma,$$

where A and γ are some positive constants (see, e.g., [3]).

If a function $\varphi(t)$, given on Γ satisfies the condition H in each closed part of Γ not containing the ends, and near any ends c, $\varphi(t)$ behaves as

$$\varphi(t) = \frac{\varphi^*(t)}{|t-c|^\delta}, \quad 0 \leq \delta < 1,$$

where $\varphi^*(t)$ belongs to the class H, then it is said $\varphi(t)$ to belong to the class H^* on Γ.

Recall also that [3] if $\Phi(z)$ is a function holomorphic in each finite domain not containing points of a certain smooth line Γ and the function $\Phi(z)$ is continuous both from the left and from the right on Γ, with a possible exception for the ends, but near the ends satisfies the condition

$$|\Phi(z)| \leq \frac{C}{|z-c|^\alpha},$$

where c is the corresponding end, C and α are certain real constants, $\alpha < 1$, then $\Phi(z)$ is called piecewise holomorphic function with the line of discontinuity Γ.

The operator $\Omega_z(g(z,\cdot))(x)$, operating on any function $g(z,x)$ piecewise holomorphic in z with a cut on the real axis $[-1,1]$ and satisfying the condition H in x, will gives a piecewise holomorphic function with a cut on the $[-1,1]$.

Using the Plemelj–Sokhotskii's formulas (see, e.g.,[5], p. 86), we can calculate the boundary values of $\Omega_z(g)$ as

$$\Omega^{\pm}_{\zeta}(g^{\pm}(\zeta,\cdot))(x) := g^{\pm}(\zeta,x) + \int_0^{2\pi} \frac{K(x,y)}{\zeta - \cos y} g^{\pm}(\zeta,y)dy$$

$$\pm \frac{i\pi}{\sqrt{1-\zeta^2}}(K(x,\zeta_a)g^{\pm}(\zeta,\zeta_a) + K(x,\bar{\zeta}_a)g^{\pm}(\zeta,\bar{\zeta}_a)), \quad \zeta \in (-1,+1), \qquad (3)$$

where $\zeta_a = \arccos\zeta$, $\bar{\zeta}_a = 2\pi - \arccos\zeta$. Here the integration with respect to y is understood in the Cauchy principal value sense.

Theorem (Tamarkin). *Suppose the kernel $K(x,\xi,\lambda)$ of the integral equation*

$$u(x) = f(x) + \int_a^b K(x,\xi,\lambda)u(\xi)d\xi$$

is analytic in λ on an open domain A of the λ-plane and for almost all points (x,ξ) of the square

$$a \le x \le b; \quad a \le \xi \le b.$$

Suppose that the integrals

$$\int_a^b |K(x,\xi,\lambda)|^2 d\xi, \qquad \int_a^b |K(x,\xi,\lambda)|^2 dx$$

exist for almost all values of x and ξ, respectively, and that, on every closed subdomain A_0 of A we have

$$\int_a^b |K(x,\xi,\lambda)|^2 d\xi \le F_0(x), \qquad \int_a^b |K(x,\xi,\lambda)|^2 dx \le F_0(\xi),$$

where $F_0(x)$ is a positive function which depends only on A_0 and which is integrable on (a,b). Then the reciprocal $\Omega(x,\xi,\lambda)$ of the kernel $K(x,\xi,\lambda)$ either is meromorphic on A for almost all values of x,ξ or does not exist at all.

For a proof, see [6].

Let $\varkappa$ denote the set of all values of z when the homogeneous equation

$$\Omega_z(g) = 0 \qquad (4)$$

admits non-zero solutions. Such values of z are called eigenvalues of $K(x,y)$ or of Ω_z. Since the kernel of the operator Ω_z is a function piecewise analytic in z, satisfying the condition H and vanishing as $z \to \infty$, the Tamarkin theorem tells us $\varkappa$ is at most countable in the plane z with a cut on $[-1,1]$. Obviously, if g is a solution of the latter equation, then the function

$$\varphi_{z_k}(x) = \frac{g(z_k,x)}{\cos x - z_k},$$

where $z_k \in \varkappa$, is also a solution of the equation

$$(\cos x - z)\varphi_z(x) = \int_0^{2\pi} K(x,y)\varphi_z(y)dy, \quad x \in [0, 2\pi],$$

when $z = z_k$ and vice versa. Moreover, sets of the eigenvalues of the operators Ω_z and

$$\Omega_z^*(h(z,\cdot))(x) := h(z,x) + \int_0^{2\pi} \frac{K(y,x)}{z - \cos y} h(z,y)dy, \quad x \in [0, 2\pi],$$

are identical. Clearly, if $z \neq z'$, then

$$\int_0^{2\pi} \varphi_z^*(x)\varphi_{z'}(x)dx = 0, \tag{5}$$

where φ_z^* is a solution of the equation

$$(\cos x - z)\varphi_z^*(x) = \int_0^{2\pi} K(y,x)\varphi_z^*(y)dy, \quad x \in [0, 2\pi]. \tag{6}$$

It is not difficult to show that if $K(x,y) = K(y,x)$, then all eigenvalues, if they exist, are real.

The functions $\varphi_{z_k}(x)$ and $\varphi_{z_k}^*(x)$, $z_k \in \varkappa$, are called the eigenfunctions of the kernel $K(x,y)$ and $K(y,x)$, respectively. Moreover, in the sequel the operators and functions determined by $K(y,x)$ just in the same way as by $K(x,y)$, will be provided with the superscript *.

Define

$$\alpha(t,y) := (1 - \sigma(t))(1 - \sigma(y)) + \sigma(t)\sigma(y), \quad t, y \in [0, 2\pi],$$

where $\sigma(t)$ is the characteristic function of the segment $[0, \pi]$ and consider the following integral equation of second kind

$$M_0(t,x) = \int_0^{2\pi} \widetilde{K}(t,x,y)M_0(t,y)dy, \quad x \in [0, 2\pi], \tag{7}$$

where

$$\widetilde{K}(t,x,y) = \frac{K(x,y) - K(x,t)}{\cos y - \cos t}\alpha(t,y) + \frac{K(x,y) - K(x,\bar{t})}{\cos y - \cos t}\alpha(\bar{t},y)$$

and $\bar{t} = 2\pi - t$. Here $t \in [0, \pi]$ is a parameter. We see that the kernel $\widetilde{K}$ of this equation does not belong to the type which is usually called regular. In spite of this to the equation (7) we may apply the basic Fredholm theorems. Indeed, it is possible to reduce such equations to the Fredholm equation with bounded kernel (see, e.g., [3], Section 111 and [7], Section 16).

Using this fact we have

Theorem 2.1. *Let for some value of the parameter* $t = t' \in [0, \pi]$ *the homogeneous equation* (7) *have only a trivial solution. Then* $z' = \cos t' \notin \varkappa$.

Proof. The proof in a completely analogous to that of Theorem 1 from [4]. □

In order to eliminate additional arguments, in the sequel we shall assume that:

R_1. $\varkappa$ is a finite set.

R_2. The homogeneous equations (7) and

$$M_0^*(t,x) = \int_0^{2\pi} \widetilde{K}^*(t,x,y) M_0^*(t,y)dy, \quad x \in [0,\pi],$$

where

$$\widetilde{K}^*(t,x,y) = \frac{K(y,x) - K(t,x)}{\cos y - \cos t}\alpha(t,y) + \frac{K(y,x) - K(\bar{t},x)}{\cos y - \cos t}\alpha(\bar{t},y),$$

admit only trivial solutions for any values of the parameter $t \in [0, \pi]$.

Note that these is an important restriction, although it is satisfied for a sufficiently wide class of kernels $K(x,y)$, in particular, for kernels from the problems of the transport theory [8]–[11].

In view of Theorem 2.1 the assumption R_2 implies:

A_1. $\varkappa \cap [-1,+1] = \varnothing$.

A_2. The nonhomogeneous integral equations

$$M(t,x) = \int_0^{2\pi} \widetilde{K}(t,x,y) M(t,y)dy + |\sin t| K(x,t), \quad t, x \in [0, 2\pi], \tag{8}$$

and

$$M^*(t,x) = \int_0^{2\pi} \widetilde{K}^*(t,x,y) M^*(t,y)dy + |\sin t| K(t,x), \quad t, x \in [0, 2\pi], \tag{9}$$

admit only unique solutions satisfying the H condition with respect to t and x.

Remark 2.2. If a kernel $K \in H$ admits an expansion into a uniformly convergent series of the form

$$K(x,y) = \sum_n (2n+1) g_n |\sin x| P_n(\cos x) P_n(\cos y),$$

where g_n is a real number and P_n is the nth order Legendre polynomial, then equations (8) and (9) have unique solutions as functions expressed by uniformly convergent series of the form

$$M(t,x) = \sum_n (2n+1) g_n |\sin x| P_n(\cos x) h_n(t)$$

and

$$M^*(t,x) = \sum_n (2n+1) g_n |\sin t| P_n(\cos x) h_n(t),$$

respectively, where h_n is defined from the recurrent relation

$$(n+1)h_{n+1}(t) + nh_{n-1}(t) = (2n+1)(\cos t - 4g_n)h_n(t),$$
$$h_0(t) = |\sin t|, \qquad (n = 0, 1, \cdots).$$

In our further consideration we will also need the following identity, which immediately can be obtained by using the Bertrand–Poincaré formula [5]:

$$\int_0^{2\pi} \frac{M^*(t_0,x)}{\cos x - \cos t_0} dx \int_0^{2\pi} \frac{M(t,x)}{\cos x - \cos t} u(t)dt$$
$$= \int_0^{2\pi} \frac{u(t)}{\cos t_0 - \cos t} dt \Big(\int_0^{2\pi} \frac{M^*(t_0,x)M(t,x)}{\cos x - \cos t_0} dx - \int_0^{2\pi} \frac{M^*(t_0,x)M(t,x)}{\cos x - \cos t} dx \Big)$$
$$\pi \Big((\bar{M}^*(t_0,t_0)\bar{M}(t_0,t_0) + \bar{M}^*(t_0,\bar{t}_0)\bar{M}(t_0,\bar{t}_0) \Big) u(t_0)$$
$$+ \Big(\bar{M}^*(t_0,t_0)\bar{M}(\bar{t}_0,t_0) + \bar{M}^*(t_0,\bar{t}_0)\bar{M}(\bar{t}_0,\bar{t}_0) \Big) u(\bar{t}_0), \quad t_0 \in (0,2\pi), \tag{10}$$

where $u \in H^*$, $\bar{M}(t,x)$ and $\bar{M}^*(t,x)$ are unique solutions of the nonhomogeneous integral equations

$$\bar{M}(t,x) = \int_0^{2\pi} \widetilde{K}(t,x,y)\bar{M}(t,y)dy + K(x,t), \quad t,\ x \in [0,2\pi],$$

and

$$\bar{M}^*(t,x) = \int_0^{2\pi} \widetilde{K}^*(t,x,y)\bar{M}^*(t,y)dy + K(t,x), \quad t,\ x \in [0,2\pi],$$

respectively. It is obvious that

$$M(t,x) = |\sin t|\bar{M}(t,x) \quad \text{and} \quad M^*(t,x) = |\sin t|\bar{M}^*(t,x).$$

3. Representation Theorem for a Function of the Class H^*

As is known that in the Hilbert–Schmidt theory for integral equations of second kind an important role is played by one property of eigenfunctions, which in our case corresponds to the following equality:

$$(\cos x - z)\varphi_{z_k}(x) - \int_0^{2\pi} K(x,y)\varphi_{z_k}(y)dy$$
$$= (z_k - z)\varphi_{z_k}(x), \quad x \in [0,2\pi], \quad z_k \in \varkappa. \tag{11}$$

But in contrast to this theory, in our case the set of eigenfunctions, is not a complete system even in the class of solutions of the initial equation. Therefore, based on the spectral expansion theory, for this aim we try to find such a singular operator which has a similar property.

Theorem 3.1. *Let $M \in H$ be a solution of* (8) *and let u be function such that*

$$L(u(\cdot))(x) := A(x)u(x) + B(\bar{x})u(\bar{x}) + \int_0^{2\pi} \frac{M(t,x)}{\cos x - \cos t} u(t)dt, \quad x \in (0, 2\pi), \tag{12}$$

where $\bar{x} = 2\pi - x$,

$$A(x) = |\sin x| + \int_0^{2\pi} \frac{\alpha(x,y)M(x,y)}{\cos x - \cos y} dy, \quad B(\bar{x}) = \int_0^{2\pi} \frac{\alpha(x,y)M(\bar{x},y)}{\cos x - \cos y} dy,$$

is meaningful and, moreover, integrable. Then the equality

$$\begin{aligned} &(\cos x - z)L(u(\cdot))(x) - \int_0^{2\pi} K(x,y)L(u(\cdot))(y)dy \\ &= L((\cos(\cdot) - z)u(\cdot))(x), \quad x \in (0, 2\pi), \end{aligned}$$

holds.

Proof. Indeed, in view of (8) and (12) straightforward calculations shows that

$$\int_0^{2\pi} K(x,y)L(u(\cdot))(y)dy = \int_0^{2\pi} M(t,x)u(t)dt.$$

By (8) we obtain

$$L((\cos x - \cos(\cdot))u(\cdot))(x) = \int_0^{2\pi} M(t,x)u(t)dt$$

and the result follows. □

The latter result gives a motive to study the operator L more thoroughly. Before beginning our systematic study we have to prove

Lemma 3.2. *For every $t \in (0, \pi)$ the system of algebraic equations*

$$\begin{aligned} A(t)v(t) + B(\bar{t})w(t) = 0, \\ B(t)v(t) + A(\bar{t})w(t) = 0 \end{aligned} \tag{13}$$

with respect to (v, w) admits only a trivial solution.

Proof. Let us assume the contrary. Suppose for some $t = t' \in (0, \pi)$ this system has a non-zero solution $(v(t'), w(t'))$. Then by virtue of (8) and (13) the function

$$M(t', x)v(t') + M(\bar{t}', x)w(t')$$

is a non-zero solution of equation (4) when $z' = \cos t'$, which contradicts the assumption A_1. □

The basic result for L is the following

Theorem 3.3. *Let $f \in H^*$. Then provided that the conditions*

$$\int_0^{2\pi} f \varphi^*_{z_k} dx = 0, \quad z_k \in \varkappa, \tag{14}$$

are fulfilled, there exists a unique solution u of the class H^*, for which

$$L(u) = f. \tag{15}$$

Proof. Suppose that $u \in H^*$ satisfies (15) and introduce into consideration the function

$$\Psi(z,x) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{M(t,x)}{\cos t - z} u(t) dt, \quad x \in [0, 2\pi], \quad z \notin [-1,1].$$

This function possesses the following properties:

P_1. In the plane with a cut $[-1,+1]$ it is piecewise holomorphic with respect to the variable z, while for the variable x it satisfies the H condition.

P_2. As $z \to \infty$ it vanishes uniformly in x.

P_3. By the Plemelj–Sokhotskii formulas for boundary values, we have

$$\Psi^{\pm}(\zeta,x) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{M(t,x)}{\cos t - \zeta} u(t) dt \pm \pm \frac{1}{2\sqrt{1-\zeta^2}} (M(\zeta_a,x)u(\zeta_a) + M(\bar{\zeta}_a,x)u(\bar{\zeta}_a)), \quad \zeta \in (-1,+1), \quad x \in [0,2\pi].$$

Combining (3) with the latter equality we get

$$\begin{aligned} &\Omega^+_\zeta(\Psi^+(\zeta,\cdot))(x) - \Omega^-_\zeta(\Psi^-(\zeta,\cdot))(x) \\ &= \frac{1}{\sqrt{1-\zeta^2}} (M(\zeta_a,x)u(\zeta_a) + M(\bar{\zeta}_a,x)u(\bar{\zeta}_a)) \\ &+ \frac{1}{\sqrt{1-\zeta^2}} \int_0^{2\pi} \frac{K(x;y)}{\zeta - \cos y} (M(\zeta_a,y)u(\zeta_a) + M(\bar{\zeta}_a,y)u(\bar{\zeta}_a)) dy \\ &+ \frac{1}{\sqrt{1-\zeta^2}} \Big(K(x,\zeta_a) \int_0^{2\pi} \frac{M(t;\zeta_a)}{\zeta - \cos t} u(t) dt + K(x,\bar{\zeta}_a) \int_0^{2\pi} \frac{M(t;\bar{\zeta}_a)}{\zeta - \cos t} u(t) dt\Big), \\ &\zeta \in (-1,+1), \quad x \in [0,2\pi]. \end{aligned}$$

Recall that $M(t,x)$ satisfies (8) and then by straightforward calculation and using (15) we obtain

$$\Omega^+_\zeta(\Psi^+)(x) - \Omega^-_\zeta(\Psi^-)(x) = \frac{1}{\sqrt{1-\zeta^2}} (K(x,\zeta_a) f(\zeta_a) + K(x,\bar{\zeta}_a) f(\bar{\zeta}_a)), \quad \zeta \in (-1,+1), \quad x \in [0,2\pi].$$

Now it is evident that if we consider the function

$$F(z,x) = \Omega_z(\Psi)(x),$$

then we will see that it is also piecewise holomorphic in z, vanishing as $z \to \infty$ and its boundary values satisfy the condition

$$F^{+}(\zeta, x) - F^{-}(\zeta, x) = \frac{1}{\sqrt{1-\zeta^2}}(K(x, \zeta_a) f(\zeta_a) + K(x, \bar{\zeta}_a) f(\bar{\zeta}_a)),$$
$$\zeta \in (-1, +1), \quad x \in [0, 2\pi].$$

Consequently, using the Plemelj–Sokhotskii formulas we get

$$F(z, x) = \frac{1}{2\pi i} \int_{-1}^{+1} \frac{1}{\sqrt{1-\xi^2}} \frac{K(x, \xi_a) f(\xi_a) + K(x, \bar{\xi}_a) f(\bar{\xi}_a)}{\xi - z} d\xi. \tag{16}$$

Here $\xi_a = \arccos \xi$, $\bar{\xi}_a = 2\pi - \arccos \xi$. After some transformation of the right-hand side of (16) we conclude that the function $\Psi(z, x)$ satisfies the integral equation

$$\Omega_z(\Psi)(x) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{K(x, t)}{\cos t - z} f(t) dt, \quad x \in [0, 2\pi]. \tag{17}$$

However the condition of the solubility of the integral equation (17) is that its free term be orthogonal to the eigenfunctions of the kernel $K(y, x)$, that is,

$$\int_0^{2\pi} \varphi^*_{z_k}(x) \int_0^{2\pi} \frac{K(x, t)}{\cos t - z_k} f(t) dt\, dx = 0, \quad z_k \in \varkappa.$$

Now, using (6) we immediately come to conditions (14).

Now suppose that $f \in H^*$ satisfies conditions (14). Then by virtue of Tamarkin's Theorems it follows that there is a unique solution Ψ of (17) which is a piecewise holomorphic function in z. Moreover, using (3) from (17) we obtain

$$\widetilde{\Psi}(t_0, x) - \int_0^{2\pi} K(x, y) \widetilde{\Psi}(t_0, y) dy + K(x, t_0) \left(\int_0^{\pi} \frac{\widetilde{\Psi}(t, t_0)}{\cos t - \cos t_0} dt + f(t_0) \right)$$
$$+ K(x, \bar{t}_0) \left(\int_0^{\pi} \frac{\widetilde{\Psi}(t, \bar{t}_0)}{\cos t - \cos t_0} dt + f(\bar{t}_0) \right) = 0, \;\; t_0 \in (0, 2\pi), \;\; x \in [0, 2\pi], \tag{18}$$

where

$$\widetilde{\Psi}(t_0, x) = |\sin t_0| (\Psi^{+}(\cos t_0, x) - \Psi^{-}(\cos t_0, x)).$$

Now consider the nonhomogeneous system of equations:

$$\begin{aligned} A(t_0) u(t_0) + B(\bar{t}_0) v(t_0) &= \int_0^{\pi} \frac{\widetilde{\Psi}(t, t_0)}{\cos t - cost_0} dt + f(t_0), \\ B(t_0) u(t_0) + A(\bar{t}_0) v(t_0) &= \int_0^{\pi} \frac{\widetilde{\Psi}(t, \bar{t}_0)}{\cos t - cost_0} dt + f(\bar{t}_0), \;\; t_0 \in (0, \pi), \end{aligned} \tag{19}$$

with respect to (u,v). Lemma 3.2 shows us that there is a unique solution of (19) and, moreover, it is easy to see that $v(t_0)=u(\bar t_0)$.

Let

$$\widetilde{M}(t_0,x)=\widetilde{\Psi}(t_0,x)-M(t_0,x)u(t_0)-M(\bar t_0,x)u(\bar t_0).$$

A quick calculation using (18) and (8) shows that $\widetilde{M}(t_0,x)$ is a solution of the homogeneous equation (4) when $z=\cos t_0$. However, since K has no the eigenvalues on $[-1,+1]$, we obtain

$$\widetilde{\Psi}(t_0,x)=M(t_0,x)u(t_0)+M(\bar t_0,x)u(\bar t_0),\quad t_0\in(0,2\pi),\quad x\in[0,2\pi].$$

Combining this with (19), we have

$$A(t_0)u(t_0)+B(\bar t_0)u(\bar t_0)=f(t_0)+\int\limits_0^{\pi}\frac{M(t,t_0)}{\cos t-\cos t_0}u(t)dt \\ +\int\limits_{\pi}^{2\pi}\frac{M(t,t_0)}{\cos t-\cos t_0}u(t)dt,\quad t_0\in(0,2\pi),\quad x\in[0,2\pi].$$

Consequently (15) holds and the proof is completed. □

Now a major goal of ours is to invert the operator L. To do this, we need to introduce, in the class H^*, the integral operators

$$S(v(\cdot))(t):=A(t)v(t)+B(t)v(\bar t)+\int\limits_0^{2\pi}\frac{M(t,y)}{\cos y-\cos t}v(y)dy,\quad t\in(0,2\pi),\tag{20}$$

and S^* introduced similar to S according to the rule mentioned above. The operator (20) can be rewritten as

$$S(v(\cdot))(t):=|\sin t|v(t) \\ +\int\limits_0^{2\pi}\Big(\frac{v(y)-v(t)}{\cos y-\cos t}\alpha(t,y)+\frac{v(y)-v(\bar t)}{\cos y-\cos t}\alpha(\bar t,y)\Big)M(t,y)dy,\quad t\in(0,2\pi);$$

the operator S^* can be rewritten similarly.

It is easy to see that for any u and v from H^*

$$\int\limits_0^{2\pi}u(t)S(v(\cdot))(t)dt=\int\limits_0^{2\pi}v(x)L(u(\cdot))(x)dx.$$

Hence, if there exists u such that (15) is fulfilled, then it is necessary that

$$\int\limits_0^{2\pi}v(x)f(x)dx=0,\tag{21}$$

where v is an arbitrary solution of the homogeneous equation

$$S(v)=0.\tag{22}$$

The converse statement is also true. Before proving this, we want to give some properties of the operators S and S^*.

The following lemma is an immediate consequence of the construction of S, together with equation (8).

Lemma 3.4. *The equality*

$$S(K(x,\cdot))(t) = M(t,x), \qquad x,\ t \in [0,2\pi],$$

holds.

The same type of arguments leads to

Lemma 3.5. *The equality*

$$S^*(K(\cdot,x))(t) = M^*(t,x), \qquad x,\ t \in [0,2\pi],$$

holds.

Lemma 3.6. *The equality*

$$S^*(M(t_0,\cdot))(t) = S(M^*(t,\cdot))(t_0), \qquad t_0,\ t \in [0,2\pi], \tag{23}$$

holds.

Proof. In view of (8), by Lemma 3.5 we conclude that

$$\begin{gathered} S^*(M(t_0,\cdot))(t) = |\sin t_0| M^*(t,t_0) \\ + \int_0^{2\pi} \Big(\frac{M^*(t,y) - M^*(t,t_0)}{\cos y - \cos t_0} \alpha(t_0,y) + \frac{M^*(t,y) - M^*(t,\bar{t}_0)}{\cos y - \cos t_0} \alpha(\bar{t}_0,y) \Big) M(t_0,y)dy \\ = S(M^*(t,\cdot))(t_0). \qquad \square \end{gathered}$$

We can put these pieces together to show a very important property of our operators.

Let us start with some notation:

$$\begin{aligned} A_M(t_0) &= \widetilde{A}(t_0) - \pi^2(\bar{M}^*(t_0,t_0)\bar{M}(t_0,t_0) + \bar{M}^*(t_0,\bar{t}_0)\bar{M}(t_0,\bar{t}_0)), \\ B_M(\bar{t}_0) &= \widetilde{B}(\bar{t}_0) - \pi^2(\bar{M}^*(t_0,t_0)\bar{M}(\bar{t}_0,t_0) + \bar{M}^*(t_0,\bar{t}_0)\bar{M}(\bar{t}_0,\bar{t}_0)), \end{aligned} \tag{24}$$

where $\bar{t}_0 = 2\pi - t_0$ and

$$\begin{aligned} \widetilde{A}(t_0) &= A^*(t_0)A(t_0) + B^*(\bar{t}_0)B(\bar{t}_0), \\ \widetilde{B}(t_0) &= A^*(t_0)B(\bar{t}_0) + B^*(t_0)A(\bar{t}_0). \end{aligned}$$

Theorem 3.7. *The composition S^*L contains no singular parts and the equality*

$$S^*(L(u))(t_0) = A_M(t_0)u(t_0) + B_M(\bar{t}_0)u(\bar{t}_0), \quad t_0 \in (0,2\pi),$$

holds.

Proof. Performing the operations indicated on the left-hand side of (24) and using identity (10), by the repeated integration we obtain

$$S^*(L(u))(t_0) = A_M(t_0)u(t_0) + B_M(\bar{t}_0)u(\bar{t}_0) + \int_0^{2\pi} \frac{u(t)}{\cos t_0 - \cos t} \Big(S^*(M(t,\cdot))(t_0) - S(M^*(t_0,\cdot)(t)\Big)dt.$$

Lemma 3.6 now completes the proof. □

The following is an immediate consequence of Lemma 3.6 if in (23) we successively take: a) $t_0 = t = s$, b) $t_0 = 2\pi - s$, $t = s$, c) $t_0 = s$, $t = 2\pi - s$, d) $t_0 = t = 2\pi - s$.

Corollary 3.8. *The equalities*

$$\begin{aligned}
&a) \quad A^*(s)M(s,s) + B^*(\bar{s})M(s,\bar{s}) = A(s)M^*(s,s) + B(\bar{s})M^*(s,\bar{s}),\\
&b) \quad A^*(\bar{s})M(s,\bar{s}) + B^*(s)M(s,s) = A(s)M^*(\bar{s},s) + B(\bar{s})M^*(\bar{s},\bar{s}),\\
&c) \quad A^*(s)M(\bar{s},s) + B^*(\bar{s})M(\bar{s},\bar{s}) = A(\bar{s})M^*(s,\bar{s}) + B(s)M^*(s,s),\\
&d) \quad A^*(\bar{s})M(\bar{s},\bar{s}) + B^*(\bar{s})M(\bar{s},s) = A(\bar{s})M^*(\bar{s},\bar{s}) + B(s)M^*(\bar{s},s),\\
&\qquad\qquad s \in (0,2\pi),
\end{aligned}$$

where $\bar{s} = 2\pi - s$, hold.

Our next task is to examine the system of equations

$$\begin{aligned}
A_M(t_0)v(t_0) + B_M(\bar{t}_0)w(t_0) = 0,\\
B_M(t_0)v(t_0) + A_M(\bar{t}_0)w(t_0) = 0.
\end{aligned} \tag{25}$$

Let

$$\mathbf{C}(t_0) = \begin{pmatrix} A(t_0) & B(\bar{t}_0) \\ B(t_0) & A(\bar{t}_0) \end{pmatrix},$$

$$\bar{\mathbf{M}}(t_0) = \begin{pmatrix} \bar{M}(t_0,t_0) & \bar{M}(\bar{t}_0,t_0) \\ \bar{M}(t_0,\bar{t}_0) & \bar{M}(\bar{t}_0,\bar{t}_0) \end{pmatrix},$$

$$\mathbf{C}_M(t_0) = \begin{pmatrix} A_M(t_0) & B_M(\bar{t}_0) \\ B_M(t_0) & A_M(\bar{t}_0) \end{pmatrix}.$$

Defines the operators $\mathbf{C}^*$ and $\bar{\mathbf{M}}^*$ similarly. By Corollary 3.8 a simple calculation shows that

$$\mathbf{C}^{*'}\bar{\mathbf{M}} = \bar{\mathbf{M}}^{*'}\mathbf{C},$$

where $\mathbf{C}^{*'}$ and $\bar{\mathbf{M}}^{*'}$ are the transposed operators of $\mathbf{C}^*$ and $\bar{\mathbf{M}}^*$, respectively.

As a result of this equality we have

Lemma 3.9. *There exists a factorization*

$$\mathbf{C}_M = (\mathbf{C}^{*'} + i\pi\bar{\mathbf{M}}^{*'})(\mathbf{C} - i\pi\bar{\mathbf{M}}).$$

The next lemma can be considered as a corollary of Lemmas 3.2 and 3.9.

Lemma 3.10. *For every $t_0 \in (0,\pi)$, the homogeneous system of algebraic equations* (25) *admits only a trivial solution.*

Denote

$$T(v(\cdot))(t_0) := \frac{1}{\Delta(t_0)}(A_M(\bar{t}_0)S^*(v(\cdot))(t_0) \\ - B_M(t_0)S^*(v(\cdot))(\bar{t}_0)), \quad t_0 \in (0,2\pi), \tag{26}$$

where

$$\Delta(t_0) = \det \mathbf{C}_M(t_0),$$

moreover, here $\Delta(\bar{t}_0) = \Delta(t_0)$.

By using the Theorem 3.7 we have

Theorem 3.11. *The equality*

$$T(L(u)) = u \tag{27}$$

holds.

We shall now consider a relationship between the eigenfunctions and the above introduced operators.

Theorem 3.12. *The eigenfunctions of the kernel $K(x,y)$ are solutions of the homogeneous equation* (22).

Proof. By (20) and (6) we obtain

$$S(\varphi^*_{z_k}(\cdot))(t) = A(t)\varphi^*_{z_k}(t) + B(t)\varphi^*_{z_k}(\bar{t}) \\ + \int_0^{2\pi} \frac{M(t,x)}{\cos x - \cos t} \frac{\int_0^{2\pi} K(y,x)\varphi^*_{z_k}(y)dy}{\cos x - z_k} dx, \quad t \in (0,2\pi).$$

It is easy to see that

$$\frac{1}{\cos x - \cos t}\,\frac{1}{\cos x - z_k} = \left(\frac{1}{\cos x - \cos t} - \frac{1}{\cos x - z_k}\right)\frac{1}{\cos t - z_k}.$$

Therefore

$$S(\varphi^*_{z_k}(\cdot))(t) = A(t)\varphi^*_{z_k}(t) + B(t)\varphi^*_{z_k}(\bar{t}) \\ + \int_0^{2\pi} \frac{M(t,x)}{\cos x - \cos t}\,\frac{1}{\cos t - z_k} dx \int_0^{2\pi} K(y,x)\varphi^*_{z_k}(y)dy - \int_0^{2\pi} \frac{M(t,x)}{\cos t - z_k}\varphi^*_{z_k}(x)dx.$$

But according to (8) we have

$$\int_0^{2\pi} \frac{\varphi^*_{z_k}(x)}{\cos t - z_k} M(t,x)dx = A(t)\varphi^*_{z_k}(t) + B(t)\varphi^*_{z_k}(\bar{t})$$

$$+\int_0^{2\pi} \frac{M(t,x)}{\cos x - \cos t} \frac{1}{\cos t - z_k} dx \int_0^{2\pi} K(y,x)\varphi_{z_k}^*(y)dy.$$

Hence we have proved the theorem. $\square$

Analogously we obtain

Theorem 3.13. *The equalities*

$$S^*(\varphi_{z_k}) = 0, \qquad z_k \in \varkappa,$$

hold.

Corollary 3.14. *The equalities*

$$T(\varphi_{z_k}) = 0, \qquad z_k \in \varkappa,$$

hold.

Now we shall prove one important property of the eigenfunctions

Theorem 3.15. *The systems of eigenvalues $\{\varphi_{z_k}\}$ and $\{\varphi_{z_k}^*\}$ are biorthogonal systems.*

Proof. Owing to equality (5) it remains to prove that the numbers

$$N_{z_k} = \int_0^{2\pi} \varphi_{z_k}\varphi_{z_k}^* dx, \quad z_k \in \varkappa,$$

are different from zero. Let us assume the contrary, i.e., that $N_{z_p} = 0$ holds for some z_p. Then it is obvious that φ_{z_p} satisfies the conditions of Theorem 3.3 and the integral equation

$$L(u) = \varphi_{z_p}$$

has a unique solution. It follows from Theorem 3.11 and Corollary 3.14 that $\varphi_{z_p} = 0$. Thus we get a contradiction and the theorem is proved. $\square$

Remark 3.16. This results implies:

(i) Only the functions $\varphi_{z_k}^*$, $z_k \in \varkappa$, and their linear combination are solutions of the homogeneous equation (22).

(ii) Condition (21) is also sufficient for the solvability of equation (15).

The main result of this section is summarized in the following theorem.

Theorem 3.17. *Let $\psi \in H^*$. Then*

$$\psi = \sum_k a_{z_k}\varphi_{z_k} + L(u), \tag{28}$$

where

$$a_{z_k} = \frac{1}{N_{z_k}} \int_0^{2\pi} \psi\varphi_{z_k}^* dx, \quad u = T(\psi).$$

Moreover, a_{z_k} and u are defined uniquely.

Proof. In view of Theorem 3.15 it is evident that the function

$$\psi_0 = \psi - \sum_k a_{z_k}\varphi_{z_k}$$

has the property

$$\int_0^{2\pi} \psi_0 \varphi^*_{z_k} dx = 0, \quad z_k \in \varkappa.$$

By Theorem 3.3 this yields (28). The uniqueness of a_{z_k} and u is also obvious. □

4. Main Results

The results obtained in the preceding sections can be successfully applied to solve of the nonhomogeneous integral equation

$$(\cos x - z)\widetilde{\varphi}_z(x) = \int_0^{2\pi} K(x;y)\widetilde{\varphi}_z(y)dy + f(x), \quad x \in (0,2\pi), \tag{29}$$

where $f \in H^*$.

Theorem 4.1. *Let $f \in H^*$ and let $z \notin [-1,+1] \cup \varkappa$. Then equation* (29) *has one and only one solution $\widetilde{\varphi}_z \in H^*$ expressed by the formula*

$$\widetilde{\varphi}_z(x) = \sum_k \frac{\varphi_{z_k}(x)}{z_k - z}\frac{1}{N_{z_k}}\int_0^{2\pi} f(y)\varphi^*_{z_k}(y)dy + L\Big(\frac{1}{\cos(\cdot) - z}T(f)(\cdot)\Big)(x). \tag{30}$$

Proof. Let $\widetilde{\varphi}_z \in H^*$ be a solution of equation (29). By virtue of Theorem 3.17 the function $\widetilde{\varphi}_z$ can be written in the form

$$\widetilde{\varphi}_z = \sum_k \widetilde{a}_{z_k}\varphi_{z_k} + L(\widetilde{u}). \tag{31}$$

To find the coefficients $\widetilde{a}_{z_k}$ and the function $\widetilde{u}$, we proceed in the following way. Putting (31) into equation (29) and using relation (11) and Theorem 3.1, we get

$$\sum_k \widetilde{a}_{z_k}(z_k - z)\varphi_{z_k} + L((\cos(\cdot) - z)\widetilde{u}(\cdot)) = f.$$

From this, by Theorem 3.17 we obtain

$$\widetilde{a}_{z_k}(z_k - z) = \frac{1}{N_{z_k}}\int_0^{2\pi} f\varphi^*_{z_k}dy, \qquad z_k \in \varkappa,$$

$$(\cos t - z)u(t) = T(f)(t), \qquad t \in (0,2\pi).$$

Now let we show that the function $\widetilde{\varphi}_z$ from (30) satisfies equation (29). Substituting (30) into (29) and using again the relation (11) and Theorem 3.1, we obtain

$$\sum_k \varphi_{z_k}\frac{1}{N_{z_k}}\int_0^{2\pi} f\varphi^*_{z_k}dy + L(T(f)) = f.$$

According to Theorem 3.17 last equality is true. □

Theorem 4.2. *Let $z = z_1$ be r-tiple eigenvalue of the kernel K. Then the solution of equation (29) exists if and only if the conditions*

$$\int_0^{2\pi} f\varphi^*_{z_k} dx = 0, \qquad k \le r \tag{32}$$

are fulfilled. In this case equation (29) has in the class H^* infinitely many solutions represented by the formula

$$\widetilde{\varphi}_{z_1} = \sum_{k\le r} c_k\varphi_{z_k} + \sum_{k>r} \frac{\varphi_{z_k}(x)}{z_k - z_1}\frac{1}{N_{z_k}} \int_0^{2\pi} f(y)\varphi^*_{z_k}(y)dy + L\Big(\frac{1}{\cos(\cdot) - z_1}T(f)(\cdot)\Big), \tag{33}$$

where c_k are arbitrary constant numbers.

Proof. Let $\widetilde{\varphi}_{z_1} \in H^*$ be a solution of (29). Then multiplying (29) by $\varphi^*_{z_k}$ and integrating with respect to x from 0 to 2π, and using, in addition, equality (11), we get

$$(z_k - z_1)\int_0^{2\pi} \widetilde{\varphi}_{z_1}(x)\varphi^*_{z_k}(x)dx = \int_0^{2\pi} f(x)\varphi^*_{z_k}(x)dx.$$

Since $z_k = z_1$ for $k \le r$,we obtain

$$\int_0^{2\pi} f(x)\varphi^*_{z_k}(x)dx = 0, \quad k \le r.$$

Thus we have r necessary conditions for the solvability of equation (29).

Before proving that the function represented by formula (33) satisfies equation (29), we show that this equation satisfies the function

$$\bar{\varphi}_{z_1} = L(\frac{1}{\cos(\cdot) - z_1}T(f)(\cdot)) + \sum_{k>r} \frac{\varphi_{z_k}}{z_k - z_1}\frac{1}{N_{z_k}} \int_0^{2\pi} f\varphi^*_{z_k} dx.$$

Just as in Theorem 4.1 we find

$$\sum_{k>r} \varphi_{z_k}\frac{1}{N_{z_k}} \int_0^{2\pi} f\varphi^*_{z_k} dx + L(T(f)) = f.$$

But by (32) and Theorem 3.17 we conclude that latter equality is true. Now it remains to observe that the function

$$\widetilde{\varphi}_{z_1} = \sum_{k\le r} c_k\varphi_{z_k} + \bar{\varphi}_{z_1}$$

satisfies equation (29). □

Theorem 4.3. *Let $z = \cos t_0$, where $t_0 \in (0, \pi)$. In order that equation* (29) *be solvable in the class H^*; it is necessary and sufficient that its free term satisfy the conditions*

$$T(f)(t_0) = T(f)(\bar{t}_0) = 0. \tag{34}$$

Then the unique solution $\bar{\varphi}_{t_0} \in H^*$ can be written by formula (30).

Proof. Just as in Theorem 4.1, if a solution of (29) is exists, then

$$(\cos t - \cos t_0)u(t) = T(f)(t)$$

whence for $t = t_0$ and $t = \bar{t}_0$ there follows equality (34). This in fact is the necessary condition which is satisfied by the function $f(x)$ in order that equation (29) would hold.

We can show that the function $\varphi_z \in H^*$, where $z = \cos t_0$, defined by formula (30) satisfies equation (29) the condition (34) is fulfilled. Thus the theorem is proved. □

Corollary 4.4. *Let $z \notin \varkappa$ and let*

$$f(x) = \sum_k a_{z_k} \varphi_{z_k}(x). \tag{35}$$

Then equation (29) *has one and only one solution which is expressed by the formula*

$$\widetilde{\varphi}_z(x) = \sum_k \frac{a_{z_k}}{z_k - z} \varphi_{z_k}(x).$$

Corollary 4.5. *Let $z = z_1$ be an r-tiple eigenvalue of the kernel K and let the function $f(x)$ has form* (35). *Then the solution of equation* (29) *exists if and only if the conditions $a_{z_k} = 0$, $k \leq r$, are fulfilled. In this case equation* (29) *has infinitely many solutions represented by the formula*

$$\widetilde{\varphi}_{z_1} = \sum_{k \leq r} c_k \varphi_{z_k} + \sum_{k > r} \frac{a_{z_k}}{z_k - z_1} \varphi_{z_k}(x),$$

where c_k are arbitrary constant numbers.

Obviously as a particular case of the above stated results we obtain the following theorem which answers the question posed for equation (1) in the Introduction. Note that in this connection we shall use (26).

Theorem 4.6. *Let $K \in H$ be such that the assumptions R_1 and R_2 are fulfilled. Then the solution of equation* (1) *exists in the class H^* if and only if $f(x) \in H^*$ satisfies the conditions*

$$A_M\Big(\frac{3\pi}{2}\Big) S^*(f)\Big(\frac{\pi}{2}\Big) = B_M\Big(\frac{\pi}{2}\Big) S^*(f))\Big(\frac{3\pi}{2}\Big),$$
$$A_M\Big(\frac{\pi}{2}\Big) S^*(f)\Big(\frac{3\pi}{2}\Big) = B_M\Big(\frac{3\pi}{2}\Big) S^*(f)\Big(\frac{\pi}{2}\Big),$$

where A_M and B_M are defined by (24). *Moreover, if the latter conditions are fulfilled, then the solution is unique and can be expressed by the formula*

$$\varphi(x)=\sum_k \varphi_{z_k}(x)\frac{z_k^{-1}}{N_{z_k}}\int_0^{2\pi} f(y)\varphi^*_{z_k}(y)dy+L(\frac{1}{\cos(\cdot)}T(f)(\cdot))(x).$$

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Author's address:
I.Vekua Institute of Applied Mathematics
I. Javakhishvili Tbilisi State University
2, University St., Tbilisi 380043
Georgia